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CLASSIFICATION OF FINITE SPACES OF ORDERINGS

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1. Introduction. A space of orderings will refer to what was called a "set of quasi-orderings" in [**5**]. That is, a *space of orderings* is a pair (X, G) where G is an elementary 2-group (i.e. $x^2 = 1$ for all $x \in G$) with a distinguished element $-1 \in G$, and X is a subset of the character group $\chi(G) = \text{Hom}(G, \{1, -1\})$ satisfying the following properties:

0_1: X is a closed subset of $\chi(G)$.

0_2: $\sigma(-1) = -1$ holds for all $\sigma \in X$.

0_3: $X^\perp = \{a \in G | \sigma a = 1 \text{ for all } \sigma \in X\} = 1$.

0_4: If f and g are forms over G and if $x \in D_{f \oplus g}$, then there exist $y \in D_f$ and $z \in D_g$ such that $x \in D_{\langle y, z\rangle}$.

Property 0_4 contains some undefined terms which are defined below:

Terminology. A *form* (of dimension n) over G is a symbol $f = \langle a_1, \ldots, a_n\rangle$ with $a_1, \ldots, a_n \in G$. The *signature* of such a form at $\sigma \in X$ is the integer σf defined by $\sigma f = \sum_1^n \sigma(a_i)$. Two forms f, g are said to be *congruent* (denoted $f \equiv g$, or $f \equiv g \pmod X$) if they have the same dimension and the same signature at each $\sigma \in X$. We say a form f *represents* $x \in G$ if there exist $x_2, \ldots, x_n \in G$ such that $f \equiv \langle x, x_2, \ldots, x_n\rangle$. D_f denotes the set of all elements of G represented by f. If $f = \langle a_1, \ldots, a_n\rangle$ and $g = \langle b_1, \ldots, b_m\rangle$, their *sum* and *product* are $f \oplus g = \langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle$ and

$$f \otimes g = \langle a_1b_1, \ldots, a_1b_m, \ldots, a_nb_1, \ldots, a_nb_m\rangle.$$

For $a \in G$, af will denote the form $\langle a\rangle \otimes f$, and $-a$ will denote the element of G defined by $-a = (-1)a$, where -1 is the distinguished element of G.

For S any subset of G, $S^\perp$ will denote the group of all characters of G satisfying $\sigma(s) = 1$ for all $s \in S$. Similarly, if $T \subseteq \chi(G)$,

$$T^\perp = \{a \in G | \sigma(a) = 1 \quad \text{for all } \sigma \in T\}.$$

Two spaces of orderings (X, G) and (X', G') are considered *equivalent* (denoted $(X, G) \backsim (X', G')$) if there exists a group isomorphism $\alpha: G \cong G'$ such that the dual isomorphism $\alpha^* : \chi(G') \to \chi(G)$ carries X' onto X.

Note. For more details on equivalence, see [**5**]. There, the concept of the *Witt ring* of a space of orderings is defined, and it is proved that two spaces of orderings are equivalent if and only if their corresponding Witt rings are isomorphic. (In fact 0_4 is not needed in the proof of this.)

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Example 1. Any formally real field F has a space of orderings associated to it, namely (X_F, G_F) where X_F is the set of all orderings of F and $G_F = F^\times/\Sigma F^{2\times}$. ($\Sigma F^{2\times}$ denotes the subgroup of $F^\times$ consisting of sums of squares.). This follows from classical results in [**6**]. The Witt ring in this case is the reduced Witt ring of F, i.e. the classical Witt ring of F [**7**], factored by its torsion ideal.

Example 2. Let A be a semi-local commutative ring, $A^\times$ the unit group of A, $\Sigma A^{2\times}$ the elements of $A^\times$ which are sums of squares in A. Assume no residue class field of A has only two elements. Let X_A denote the set of all signatures of A in the sense of [**4**], and let $G_A = A^\times/\Sigma A^{2\times}$. It follows from results in [**4**] together with the Transversality Theorem 2.7 in [**1**] that (X_A, G_A) is a space of orderings.

(This example was pointed out to me by M. Knebusch and E. Becker.)

A space of orderings is said to be *finite* if X is finite (or equivalently, if G is finite). The main Theorem (Theorem 4.11) in this paper can now be stated:

Every finite space of orderings is equivalent to the space of orderings (X_F, G_F) of some Pythagorian Field F.

Remarks. 1. This theorem is already known [**2**, **3**] for finite spaces of orderings of type (X_K, G_K), K formally real.

2. The proof of this theorem is somewhat indirect. Finite spaces of orderings coming from Pythagorian fields have already been classified (see [**2**], and [**3**]). The technique here is essentially to classify finite spaces of orderings in general, and then compare the two classifications.

3. The theorem may be interpreted as giving a simple system of axioms for the reduced theory of quadratic forms (at least in the finite case).

4. It is perhaps worthwhile to dispose of one trivial case immediately. One verifies that for a space of orderings (X, G) the following are equivalent:

i) $X = \emptyset$

ii) $-1 = 1 \in G$

iii) $G = \{1\}$

For such a space we have $(X, G) \backsim (X_F, G_F)$ where F is any field satisfying $F^2 = F$. This case is excluded in the remainder of our considerations.

2. Subspaces. Throughout we assume that (X, G) is a finite space of orderings. For $\sigma_1, \ldots, \sigma_m \in X$ we can consider all linear combinations

$$(1) \quad \sigma = \sigma_1^{\epsilon_1} \ldots \sigma_m^{\epsilon_m}, \ \epsilon_1, \ldots, \epsilon_m \in \{0, 1\}$$

in $\chi(G)$. We are more interested in linear combinations which are in X. By 0_2 a necessary condition for a linear combination (1) to be in X is

$$\epsilon_1 + \ldots + \epsilon_m \equiv 1 \pmod 2.$$

Let $\sigma_1, \ldots, \sigma_m$ be arbitrary elements of X, and define Y, Δ by:

$$Y = \{\sigma \in X | \sigma \text{ is a linear combination of } \sigma_1, \ldots, \sigma_m\},$$
$$\Delta = \{a \in G | \sigma_i(a) = 1 \text{ for all } i = 1, 2, \ldots, m\}.$$

Then Y, Δ satisfy the duality condition

(2) $\Delta = Y^\perp$, $Y = \Delta^\perp \cap X$.

The system $(Y, G/\Delta)$ is referred to as *the subspace of* (X, G) *generated by* $\sigma_1, \ldots, \sigma_m$. More generally, a *subspace* of (X, G) is any system $(Y, G/\Delta)$ where $Y \subseteq X$, $\Delta \subseteq G$ satisfy the duality condition (2).

Conversely suppose we begin with $a_1, \ldots, a_m \in G$, let f denote the *Pfister form* $\langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_m\rangle$, and let $X(a_1, \ldots, a_m)$ denote the *Harrison Basic set*

$$\{\sigma \in X | \sigma(a_i) = 1 \text{ for all } i = 1, 2, \ldots, m\}.$$

LEMMA 2.1. *$Y = X(a_1, \ldots, a_m)$ and $\Delta = D_f$ satisfy the duality condition* (2).

Proof. Let $a \in D_f$, $\sigma \in X(a_1, \ldots, a_m)$. Then $\sigma f = 2^m = \dim f$, so $\sigma(a) = 1$. Thus $D_f \subseteq X(a_1, \ldots, a_m)^\perp$, and $X(a_1, \ldots, a_m) \subseteq D_f^\perp \cap X$. Since $a_1, \ldots, a_m \in D_f$, it is clear that $D_f^\perp \cap X = X(a_1, \ldots, a_m)$. Now let $a \in G$ satisfy $\sigma(a) = 1$ for all $\sigma \in X(a_1, \ldots, a_m)$. Consider the forms af and f. f represents 1, so af represents a. Comparing signatures at $\sigma \in X$, we see that

$$\sigma(af) = \sigma(f) = \begin{cases} 2^m & \text{if } \sigma \in X(a_1, \ldots, a_m) \\ 0 & \text{if } \sigma \notin X(a_1, \ldots, a_m). \end{cases}$$

Thus $f \equiv af$, so f represents a. Thus $X(a_1, \ldots, a_m)^\perp \subseteq D_f$.

THEOREM 2.2. *Let $(Y, G/\Delta)$ be any subspace of (X, G). Then $(Y, G/\Delta)$ is also a space of orderings.*

Proof. There exist $a_1, \ldots, a_m \in G$ such that $Y = X(a_1, \ldots, a_m)$, and by Lemma 2.1 $\Delta = D_f$ where f is the Pfister form associated to $a_1, \ldots, a_m$. Everything is clear except 0_4. This follows from the following:

LEMMA 2.3. *A form g over G represents $x \in G$ modulo $X(a_1, \ldots, a_m)$ if and only if $f \otimes g$ represents x modulo X.*

Assuming this lemma, we complete the proof of the theorem. Suppose g, h are forms over G such that $g \oplus h$ represents $x \in G$ modulo Y. Thus, by Lemma 2.3 the form $f \otimes (g \oplus h) = (f \otimes g) \oplus (f \otimes h)$ represents x modulo X. Since (X, G) satisfies 0_4, there exist y, z represented by $f \otimes g$ and $f \otimes h$ respectively (mod X) such that $\langle y, z\rangle$ represents x (mod X). Thus, by the lemma, y and z are represented by g and h respectively (mod Y), and clearly $\langle y, z\rangle$ represents x (mod Y).

Proof of Lemma 2.3. Suppose $g \equiv h$ mod $X(a_1, \ldots, a_m)$ where h has x appearing in its diagonal representation. Since f has 1 appearing in its diagonal representation, it follows that x appears in the diagonal representation of $f \otimes h$. Now $\sigma g = \sigma h$ holds for all $\sigma \in X(a_1, \ldots, a_m)$. Also $\sigma f = 0$ holds, for at $\sigma \in X, \sigma \notin X(a_1, \ldots, a_m)$. It follows that $\sigma(f \otimes g) = \sigma f \sigma g = \sigma f \sigma h = \sigma(f \otimes h)$ holds for all $\sigma \in X$. Thus $f \otimes g \equiv f \otimes h \pmod X$. Thus $f \otimes g$ represents $x \pmod X$.

Conversely, suppose $f \otimes g$ represents $x \pmod X$. Write $g = \langle y_1, \ldots, y_k \rangle$. Thus $f \otimes g \equiv y_1 f \oplus \ldots \oplus y_k f \pmod X$, so by 0_4, there exist $s_1, \ldots, s_k \in D_f$ such that $\langle y_1 s_1, \ldots, y_k s_k \rangle$ represents $x \pmod X$. But $\langle y_1 s_1, \ldots, y_k s_k \rangle \equiv \langle y_1, \ldots, y_k \rangle \pmod{X(a_1, \ldots, a_m)}$. It follows that g represents x mod $X(a_1, \ldots, a_m)$.

3. Decomposition into connected components. By 0_3 there is a basis $\sigma_1, \ldots, \sigma_n$ of $\chi(G)$ consisting entirely of elements of X. We will refer to such a basis as *a basis for X*. n will be referred to as the *rank* (or *dimension*) *of X*. Clearly if $\sigma_1, \ldots, \sigma_n$ is a basis of X, then any element of X is the product of an odd number of $\{\sigma_1, \ldots, \sigma_n\}$.

BASIC LEMMA 3.1. *Suppose X consists of n independent orders (n odd, $n \geqq 5$) together with their product $\sigma_1\sigma_2 \ldots \sigma_n$ together with some (possibly empty) subset of $\sigma_1\sigma_3\sigma_4$, $\sigma_1\sigma_3\sigma_5$, $\sigma_2\sigma_3\sigma_4$, $\sigma_2\sigma_3\sigma_5$. Then X is not a space of orderings.*

Proof. Let $a_1, \ldots, a_n$ be the dual basis of G, i.e.

$$\sigma_i(a_j) = \begin{cases} 1 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}, \quad i, j = 1, \ldots, n.$$

Consider the forms

$$f = \langle 1, a_1a_2a_3a_4, a_1a_2a_3a_5 \rangle \quad \text{and}$$
$$g = \langle a_1a_2a_4a_5, a_2a_3, a_3a_1 \rangle.$$

A straightforward verification shows that $\sigma f = \sigma g$ holds for all $\sigma \in X$, so $f \equiv g$. Now if X were a space of orderings then by 0_4 there would exist an element b represented by $\langle a_1a_2a_3a_4, a_1a_2a_3a_5 \rangle \equiv a_1a_2a_3a_4\langle 1, a_4a_5 \rangle$ such that $\langle 1, b \rangle$ represents $a_1a_2a_4a_5$. Now a_4a_5 is positive at $\sigma_1, \sigma_2, \sigma_3, \sigma_6, \ldots, \sigma_n$ and $\sigma_1\sigma_2 \ldots \sigma_n$ ($n - 1$ independent orders). It follows from Lemma 2.1 that the Pfister form $\langle 1, a_4a_5 \rangle$ only represents 1 and a_4a_5, so $\langle a_1a_2a_3a_4, a_1a_2a_3a_5 \rangle$ only represents $a_1a_2a_3a_4$ and $a_1a_2a_3a_5$. Thus b would be either $a_1a_2a_3a_4$ or $a_1a_2a_3a_5$.

Now σ_5 is positive at $a_1a_2a_3a_4$ but negative at $a_1a_2a_4a_5$, so $\langle 1, a_1a_2a_3a_4 \rangle$ cannot represent $a_1a_2a_4a_5$. Similarly σ_4 is positive at $a_1a_2a_3a_5$ but negative at $a_1a_2a_4a_5$, so $\langle 1, a_1a_2a_3a_5 \rangle$ cannot represent $a_1a_2a_4a_5$. Thus we see that no such b exists, so X is not a space of orderings.

We will say two orders $\sigma, \sigma' \in X$ are *simply connected* in X (denoted $\sigma \backsim_S \sigma'$) if there exist orders $\tau, \tau' \in X$, $\{\sigma, \sigma'\} \neq \{\tau, \tau'\}$ such that $\sigma\sigma' = \tau\tau'$. (We say two orders $\sigma, \sigma' \in X$ are *connected* in X (denoted $\sigma \backsim \sigma'$) if either $\sigma = \sigma'$ or if there exists a sequence of orders $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ in X such that $\sigma_{i-1} \backsim_S \sigma_i$ for $i = 1, \ldots, k$.)

Notes (1). If Y is a subspace of X and $\sigma, \sigma' \in Y$, then it is concievable that σ, σ' could be connected in X without being connected in Y.

(2) We will see later that $\sigma \backsim \sigma'$ implies either $\sigma \backsim_S \sigma'$ or $\sigma = \sigma'$.

LEMMA 3.2. *If $\sigma_1, \ldots, \sigma_m \in X$ are independent and $\sigma = \sigma_1 \ldots \sigma_m \in X$ then $\sigma \backsim \sigma_i$ holds for each $i = 1, \ldots, m$.*

Proof. It is enough to show $\sigma \backsim \sigma_1$. The proof is by induction on m. If $m = 3$, there is nothing to show. Assume $m \geqq 5$, Consider the subspace generated by $\sigma_1, \ldots, \sigma_m$. This is a space of orderings, so by the Basic Lemma it must consist of more than $\sigma_1, \ldots, \sigma_m, \sigma$. Thus there exists an order σ' which is the product of at least 3 and at most $m - 2$ of $\{\sigma_1, \ldots, \sigma_m\}$. There are two cases to consider:

(a) σ_1 appears in σ'. Then without loss of generality $\sigma' = \sigma_1\sigma_2 \ldots \sigma_l$, $3 \leqq l \leqq m - 2$. By induction $\sigma' \backsim \sigma_1$. Also $\sigma = \sigma'\sigma_{l+1} \ldots \sigma_m$. so again by induction, $\sigma \backsim \sigma'$. Thus, by transitivity of $\backsim$, $\sigma \backsim \sigma_1$.

(b) σ_1 does not appear in σ'. Then without loss of generality, $\sigma' = \sigma_2 \ldots \sigma_l$, $4 \leqq l \leqq m - 1$. In this case, $\sigma = \sigma_1\sigma'\sigma_{l+1} \ldots \sigma_m$ so by induction, $\sigma \backsim \sigma_1$.

Let $X = X_1 \cup \ldots \cup X_k$ denote the decomposition of X determined by the equivalence relation $\backsim$. The classes X_i, $i = 1, \ldots, k$ will be referred to as the *connected components* of X. If X has only one component, we will say X is *connected*. An ordering $\sigma \in X$ will be called *Archimedian* (in X) if $\{\sigma\}$ is a component of X.

THEOREM 3.3 (Decomposition Theorem). *Suppose $X_1, \ldots, X_k$ are the connected components of X. Then each X_i is a subspace of X and*

$$\text{rank } X = \textstyle\sum_1^k \text{rank } X_i.$$

Proof. Let $\Delta_i = X_i^\perp$. Then clearly X_i generates $\Delta_i^\perp \cap X$. To show X_i is a subspace we must show that $X_i = \Delta_i^\perp \cap X$. This is clear by Lemma 3.2.

Let σ_{ij}, $j = 1, \ldots, l_i$ be a basis for each X_i, $i = 1, \ldots, k$. We wish to show that the complete set $\{\sigma_{ij} | i = 1, \ldots, k; j = 1, \ldots, l_i\}$ is a basis for X. It is clear this set spans X. If these elements were independent we could find a relation

$$\textstyle\prod_{i,j} \sigma_{ij}^{\epsilon_{ij}} = 1 \quad \epsilon_{ij} \in \{0, 1\}$$

with not all $\epsilon_{ij} = 0$. Of all such relations pick the one with the minimal number of non-zero ϵ_{ij}. By Lemma 3.2 each σ_{ij} appearing with a non-zero exponent is equivalent to every other such σ_{ij}. Thus, all such σ_{ij} lie in the same component.

Thus, there exists i such that $\epsilon_{rj} = 0$ for $r \neq i$. Thus our assumed relation has the form

$$\textstyle\prod_j \sigma_{ij}^{\epsilon_{ij}} = 1.$$

This would contradict the independence of $\sigma_{i1}, \ldots, \sigma_{il_i}$.

Remarks. 1. Let $G_i = G/\Delta_i$ where $\Delta_i = X_i^\perp$, $i = 1, \ldots, k$. By the above theorem, the natural injection of G into $G_1 \times G_2 \times \ldots \times G_k$ is an isomorphism.

If we identify G with $G_1 \times \ldots \times G_k$ via this isomorphism, we see that Δ_i is identified with $\prod_{j \neq i} G_j$ for $i = 1, \ldots, k$. Also X is identified with $\bigcup_1^k Y_i$ where Y_i is obtained from X_i by extending each element by the identity character on $\prod_{j \neq i} G_j$. Thus, the structure of (X, G) is completely determined by the structure of the subspaces (X_i, G_i), $i = 1, \ldots, k$. We will express this by writing

$$(X, G) = \bigoplus_{i=1}^{k} (X_i, G_i)$$

and will refer to (X, G) as the *direct sum* of the spaces (X_i, G_i), $1 \leqq i \leqq k$.

2. Let W, $W_i, \ldots, W_k$ be the Witt rings associated to $X, X_1, \ldots, X_k$ respectively. Then there is a canonical ring embedding of W into the product ring $W_1 \times \ldots \times W_k$. The image consists of those k-triples $(f_1, \ldots, f_k)$ whose entries have the same dimension modulo 2.

4. The connected case. The results of the previous section reduce the study of finite spaces of orderings to finite connected spaces of orderings. In this section we study the structure of such spaces.

THEOREM 4.1. *Suppose σ is a character of G, $\sigma \neq 1$, satisfying the following property*:

$$x \in \text{kern } \sigma \Rightarrow D_{\langle 1,x\rangle} \subseteq \text{kern } \sigma.$$

Then $\sigma \in X$.

Proof. It is enough to show that for each finite subset $a_1, \ldots, a_k \in \text{kern } \sigma$, there exists $\sigma' \in X$ such that $a_1, \ldots, a_k \in \text{kern } \sigma'$.

If no such $\sigma' \in X$ exists, and if $f = \langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_k\rangle$, then $\sigma' f = 0$ for all $\sigma' \in X$, so $f \equiv \langle 1, -1\rangle \otimes \ldots \otimes \langle 1, -1\rangle$, and hence $D_f = G$. Thus, it is enough to prove that $D_f \subseteq \text{kern } \sigma$.

This is proved by induction on k, the result being true by hypothesis for $k = 1$. For $k > 1$, let $g = \langle 1, a_2\rangle \otimes \ldots \otimes \langle 1, a_k\rangle$, and suppose x is represented by $f \equiv g \oplus a_1 g$. Thus, by 0_4, there exist $y, z \in D_g$ such that x is represented by $\langle y, a_1 z\rangle$. Thus xy is represented as $\langle 1, a_1 yz\rangle$. Now $a_1 \in \text{kern } \sigma$, and $y, z \in D_g \subseteq \text{kern } \sigma$ by induction. Thus $a_1 yz \in \text{kern } \sigma$. Thus, by hypothesis, $xy \in \text{kern } \sigma$. Finally $y \in \text{kern } \sigma$, so $x = (xy)y \in \text{kern } \sigma$.

We use this theorem to prove the following:

LEMMA 4.2. *Suppose $\sigma_1, \ldots, \sigma_n \in X$ is a basis of X, that $\alpha \in \chi(G)$, and that $\alpha\sigma_1, \ldots, \alpha\sigma_n \in X$. Then $\alpha X = X$,*

Proof. Since $\sigma_1, \sigma_1\alpha \in X$ we have $\sigma_1(-1) = (\sigma_1\alpha)(-1) = -1$ by 0_2. Thus $\alpha(-1) = 1$. We wish to show that if $\sigma \in X$, then $\sigma\alpha \in X$. Suppose $x \in G$ is such that $(\sigma\alpha)(x) = 1$, and suppose $\langle 1, x\rangle$ represents y. By the above theorem we are done if we show $(\sigma\alpha)(y) = 1$. There are three cases to consider.

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Case 1. $\alpha(x) = -1$. Then for each i, $\sigma_i(x)$ and $(\sigma_i\alpha)(x)$ have opposite sign. Thus, we can find $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ such that $\sigma_1\alpha^{\epsilon_1}, \ldots, \sigma_n\alpha^{\epsilon_n}$ are all 1 at x (and hence at y). The space generated by these is $n - 1$ dimensional at least. Thus $\bigcap_1^n$ kern $(\sigma_i\alpha^{\epsilon_i})$ is at most one dimensional. Since $D\langle 1, x\rangle$ is in this intersection, it follows that $D\langle 1, x\rangle = \{1, x\}$. Thus $y = 1$ or x. In either case we have $(\sigma\alpha)(y) = 1$.

Case 2. $\alpha(y) = -1$. Then $\alpha(-y) = -1$ (since $\alpha(-1) = 1$), so by the above argument $D\langle 1, -y\rangle$ is at most one dimensional. Since $y \in D\langle 1, x\rangle$ it follows that $-x \in D\langle 1, -y\rangle$. Thus, $-x = 1$ or $-y$. Now $x \in$ kern $\sigma\alpha$, so $x \neq -1$. Thus $x = y$, so $(\sigma\alpha)(y) = 1$.

Case 3. $\alpha(x) = 1, \alpha(y) = 1$. Thus $\sigma(x) = 1$, and $\langle 1, x\rangle$ represents y, $\sigma \in X$, so $\sigma(y) = 1$. Thus $(\sigma\alpha)(y) = 1$.

We are interested in subspaces Y of X which satisfy $\alpha Y = Y$ for some $\alpha \in \chi(G)$, $\alpha \neq 1$. For $\alpha \in \chi(G)$ let X_α denote the set $\{\sigma \in X | \sigma\alpha \in X\} = \alpha X \cap X$. Then clearly X_α is the maximal subset of X satisfying $\alpha X_\alpha = X_\alpha$. X_α is in fact a subspace of X. This is the content of the following lemma:

LEMMA 4.3. *Let α be a character of G, and let X_α be defined as above. Then X_α is a (possibly trivial) subspace of X.*

Proof. Let $\Delta_\alpha = X_\alpha^\perp$. We must show that $X_\alpha = \Delta_\alpha^\perp \cap X$. Now X_α generates $\Delta_\alpha^\perp \cap X$, and $\alpha X_\alpha = X_\alpha$. Thus, applying the previous lemma to the space $(\Delta_\alpha^\perp \cap X, G/\Delta_\alpha)$, we see that $\alpha(\Delta_\alpha^\perp \cap X) = \Delta_\alpha^\perp \cap X$. It follows, by definition of X_α, that $\Delta_\alpha^\perp \cap X = X_\alpha$.

In the following few lemmas we examine the lattice of subspaces X_α, $\alpha \in \chi(G)$. The major goal is the proof of Theorem 4.7.

LEMMA 4.4. *Suppose $\sigma_1, \sigma_1\alpha, \sigma_1\beta, \sigma_1\alpha\beta \in X$. Then either $X_\alpha \subseteq X_\beta$ or $X_\beta \subseteq X_\alpha$.*

Proof. Suppose $X_\alpha \nsubseteq X_\beta$, $X_\beta \nsubseteq X_\alpha$. Choose $\sigma_2 \in X_\alpha$, $\sigma_2 \notin X_\beta$, $\sigma_3 \in X_\beta$, $\sigma_3 \notin X_\alpha$. Consider the elements:

$$\begin{array}{llll}
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_1\sigma_2\sigma_3 \\
\sigma_1\alpha & \sigma_2\alpha & \sigma_3\alpha & \sigma_1\sigma_2\sigma_3\alpha \\
\sigma_1\beta & \sigma_2\beta & \sigma_3\beta & \sigma_1\sigma_2\sigma_3\beta \\
\sigma_1\alpha\beta & \sigma_2\alpha\beta & \sigma_3\alpha\beta & \sigma_1\sigma_2\sigma_3\alpha\beta
\end{array}$$

in $\chi(G)$. The elements in the first column are all in X by assumption. Also $\sigma_2, \sigma_2\alpha \in X$, $\sigma_2\beta \notin X$ by assumptions on σ_2. What about $\sigma_2\alpha\beta$? If $\sigma_2\alpha\beta \in X$ then consider just the first two columns and the element $\gamma = \sigma_1\sigma_2\alpha \in \chi(G)$. We have $\sigma_1\gamma = \sigma_2\alpha$, $(\sigma_1\alpha)(\gamma) = \sigma_2$, $(\sigma_1\beta)\gamma = \sigma_2\alpha\beta$. Thus by Lemma 4.2, the space spanned by $\sigma_1, \sigma_2, \sigma_1\alpha, \sigma_1\beta$ is invariant under γ. In particular, $(\sigma_1\alpha\beta)\gamma = \sigma_2\beta \in X$. This is a contradiction. Thus $\sigma_2\alpha\beta \notin X$. Similarly in the third column we have $\sigma_3, \sigma_3\beta \in X$, $\sigma_3\alpha, \sigma_3\alpha\beta \notin X$. If any of the 4th column are in X, we may

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assume (by replacing σ_2 by $\sigma_2\alpha$ or σ_3 by $\sigma_3\beta$ if necessary) that $\sigma_1\sigma_2\sigma_3 \in X$. We have $\sigma_1, \sigma_2 \in X_\alpha$. If $\sigma_1\sigma_2\sigma_3 \in X_\alpha$ then $\sigma_3 = \sigma_1\sigma_2(\sigma_1\sigma_2\sigma_3) \in X_\alpha$ by Lemma 4.2. But $\sigma_3 \notin X_\alpha$. Thus $\sigma_1\sigma_2\sigma_3 \notin X_\alpha$, i.e. $\sigma_1\sigma_2\sigma_3\alpha \notin X$. Similarly one shows that $\sigma_1\sigma_2\sigma_3\beta \notin X$. Thus of the list of elements, only

$$\begin{matrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1\alpha & \sigma_2\alpha & \\ \sigma_1\beta & & \sigma_3\beta \\ \sigma_1\alpha\beta & & \end{matrix} \quad \text{and possibly one or more of} \left\{ \begin{matrix} \sigma_1\sigma_2\sigma_3 \\ \\ \\ \sigma_1\sigma_2\sigma_3\alpha\beta \end{matrix} \right.$$

are in X. This list can be viewed as the (5 dimensional) subspace of X generated by

$$\sigma_1\alpha,\ \sigma_1\beta,\ \sigma_2,\ \sigma_2\alpha, \text{ and } \sigma_3.$$

In terms of this basis we have:

$$\begin{aligned} \sigma_1 &= (\sigma_2\alpha)(\sigma_1\alpha)(\sigma_2) \\ \sigma_3\beta &= (\sigma_1\alpha)(\sigma_1\beta)(\sigma_2)(\sigma_2\alpha)(\sigma_3) \\ \sigma_1\sigma_2\sigma_3 &= (\sigma_1\alpha)(\sigma_2\alpha)(\sigma_3) \\ \sigma_1\sigma_2\sigma_3\alpha\beta &= (\sigma_1\beta)(\sigma_2\alpha)(\sigma_3) \\ \sigma_1\alpha\beta &= (\sigma_1\beta)(\sigma_2\alpha)(\sigma_2) \end{aligned}$$

But, according to Lemma 3.1 this is not a space of orderings, a contradiction.

LEMMA 4.5. *Suppose X_α, X_β have rank* $\geqq 3$. *Then either* $X_\alpha \cap X_\beta = \emptyset$, *or* $|X_\alpha \cap X_\beta| \geqq 2$.

Proof. Suppose the result is false, i.e. $X_\alpha \cap X_\beta = \{\sigma_1\}$. Then $X_\alpha \not\subseteq X_\beta$, $X_\beta \not\subseteq X_\alpha$. Thus there exists $\sigma_2 \in X_\alpha$, $\sigma_2 \notin X_\beta$. We may further assume $\sigma_2 \neq \sigma_1\alpha$; for if $\sigma_1\alpha$ were the only element of X_α not in X_β then (since $|X_\alpha| >$ rank X_α) X_α would have a basis in X_β so by Lemma 4.2 $X_\alpha \subseteq X_\beta$. Similarly there exists $\sigma_3 \in X_\beta$, $\sigma_3 \notin X_\alpha$, $\sigma_3 \neq \sigma_1\beta$.

Now consider the same list of elements as in Lemma 4.4. We have $\sigma_1, \sigma_1\alpha$, $\sigma_1\beta \in X$, but $\sigma_1\alpha\beta \notin X$ by Lemma 4.4, since $X_\alpha \not\subseteq X_\beta$, $X_\beta \not\subseteq X_\alpha$. Also σ_2, $\sigma_2\alpha \in X$, $\sigma_2\beta \notin X$. If $\sigma_2\alpha\beta$ were in X, then $\sigma_2\alpha \in X_\alpha \cap X_\beta$, $\sigma_2\alpha \neq \sigma_1$. This is a contradiction. Thus $\sigma_2\alpha\beta \notin X$. Similarly $\sigma_3, \sigma_3\beta \in X$, $\sigma_3\alpha, \sigma_3\alpha\beta \notin X$. The same argument used in Lemma 4.4 shows that the elements in the last column in X, if any, have the form $\sigma_1\sigma_2\sigma_3$, $\sigma_1\sigma_2\sigma_3\alpha\beta$ by suitable change in notation. From this point on, the proof parallels exactly the latter part of the proof of Lemma 4.4.

LEMMA 4.6. *Suppose* $\alpha \neq 1$, $\beta \neq 1$, *and that* $X_\alpha \cap X_\beta \neq \emptyset$, rank $X_\alpha, X_\beta \geqq 3$. *Then there exists* $\gamma \in \chi(G)$, $\gamma \neq 1$ *such that* $X_\alpha, X_\beta \subseteq X_\gamma$.

Proof. If either $X_\alpha \subseteq X_\beta$ or $X_\beta \subseteq X_\alpha$ we are done. Thus we may assume $X_\alpha \not\subseteq X_\beta$, $X_\beta \not\subseteq X_\alpha$. By Lemma 4.5 there exist $\sigma_1, \sigma_2 \in X_\alpha \cap X_\beta$, $\sigma_1 \neq \sigma_2$. Take $\gamma = \sigma_1\sigma_2$. Then σ_1, $\sigma_1\alpha$, $\sigma_1\gamma = \sigma_2$, $\sigma_1\alpha\gamma = \sigma_2\alpha$ are all in X, so by Lemma 4.4 either $X_\alpha \subseteq X_\gamma$ or $X_\gamma \subseteq X_\alpha$. Now $\sigma_1\beta\gamma = \sigma_2\beta \in X$ so $\sigma_1\beta \in X_\gamma$. On the other hand, $\sigma_1\beta\alpha \notin X$ by Lemma 4.4, so $\sigma_1\beta \notin X_\alpha$. Thus $X_\alpha \subseteq X_\gamma$. A similar argument shows $X_\beta \subseteq X_\gamma$.

THEOREM 4.7. *Let X be a connected space*, rank $X \neq 1$. *Then there exist $\alpha \in \chi(G), \alpha \neq 1$ such that $\alpha X = X$.*

Proof. Since rank $X \neq 1$ and X is connected there exists $\sigma_1, \sigma_2 \in X$, $\sigma_1 \neq \sigma_2$, $\sigma_1 \backsim_S \sigma_2$. Take $\alpha = \sigma_1\sigma_2$. Then X_α has rank $\geqq 3$. Of all $\alpha \in \chi(G)$ satisfying $\alpha \neq 1$, rank $X_\alpha \geqq 3$, pick one such that X_α is maximal. If $X_\alpha \neq X$, then there exist (since X is connected) elements $\sigma_1, \sigma_2 \in X$, $\sigma_1 \in X_\alpha$, $\sigma_2 \notin X_\alpha$, $\sigma_1 \backsim_S \sigma_2$. Let $\beta = \sigma_1\sigma_2$. Then $\sigma_1 \in X_\alpha \cap X_\beta$ so by Lemma 4.6, there exists $\gamma \neq 1$, $\gamma \in \chi(G)$ such that $X_\alpha \subseteq X_\gamma$, $X_\beta \subseteq X_\gamma$. Since $\sigma_2 \in X_\beta \subseteq X_\gamma$ it follows that X_γ contains X_α properly. This is a contradiction and $X_\alpha = X$.

Now let T denote the set of all $\alpha \in \chi(G)$ such that $\alpha X = X$. T is clearly a subgroup of $\chi(G)$, and will be referred to as the *translation group of* X. Since we are assuming X is connected, we have, by the above theorem, that $T \neq 1$, if rank $X \neq 1$. Let $G' = T^\perp$ and let X' denote the set of all restrictions $\sigma_{|G'}$, $\sigma \in X$.

THEOREM 4.8. *Let X be a connected space, and define X', G' as above. Then (X', G') is a space of orderings.*

Proof. It is clear that $-1 \in G'$ and that X' is a subset of $\chi(G')$ satisfying 0_1, 0_2, and 0_3. 0_4 follows from the following lemma.

LEMMA 4.9. *Let f be a form over G' which is not isotropic* (mod X'). *Then as a form over G, f only represents elements of G'.*

Assuming this lemma, let f, g be forms over G' such that $f \oplus g$ represents $x \in G'$ (mod X'). We may assume neither f nor g is isotropic (mod X'). By 0_4 for X, there exist $y, z \in G$ represented by f, g respectively (mod X) such that $\langle y, z\rangle$ represents x (mod X). By the lemma it follows that $y, z \in G'$ and that f, g, in fact, represent y, z respectively mod X'. Also it is clear that $\langle y, z\rangle$ represents x (mod X').

Proof of Lemma 4.9. The proof is by induction on the dimension n of f. The result is clear if $n = 1$, so we may assume $f = \langle a, a_2, \ldots, a_n\rangle$, $n \geqq 2$. Let $x \in D_f$. By 0_4 there exist $b, b_3, \ldots, b_n \in G$ such that $\langle a_2, \ldots, a_n\rangle \equiv \langle b, b_3, \ldots, b_n\rangle$ (mod X), and such that $\langle a, b\rangle$ represents x (mod X). By induction $b, b_3, \ldots, b_n \in G'$ so $f \equiv \langle a, b, b_3, \ldots, b_n\rangle$ (mod X'). We are assuming f is not isotropic (mod X'), so $b \neq -a$. Thus $-ab \neq 1$, so there exists an order $\sigma \in X$ such that $\sigma(ab) = 1$. Now suppose $x \notin G'$. Then there exists $\alpha \in T$ such that $\alpha(x) = -1$. Thus $\alpha(ax) = -1$, so $\sigma(ax) = -(\sigma\alpha)(ax)$. Thus, by replacing σ by $\sigma\alpha$ if necessary, we may assume $\sigma(ax) = -1$. But $\langle a, b\rangle$ represents x, so $\langle 1, ab\rangle$ represents ax. Thus, since $\sigma(ab) = 1$, we must have $\sigma(ax) = 1$. This is a contradiction.

Remarks. 1. The space (X, G) is completely determined (up to equivalence) by the space (X', G') and the rank of the group T: For G can be identified with the group $G' \times T$ and, under this identification, X is identified with the set of all extensions of X' to $G' \times T$.

2. Let $r = \text{rank } T$. Thus $r > 0$ if rank $X \neq 1$. Also rank $X = \text{rank } X' + r$. Thus, except in the trivial case rank $X = 1$, X' has strictly lower rank than X so its structure is known inductively. An additional property of X' is known. Namely, the translation group of X' is trivial (for if X' had a non-trivial translation group, then we could enlarge T). Thus, either rank $X' = 1$, or X' is a disconnected space of rank $\geqq 3$.

3. One can show that the Witt ring of (X, G) is just the group ring $W'[T]$, where W' denotes the Witt ring of (X', G').

4. Let us combine the above structure results. Let (X, G) be any finite space of orderings. Then (X, G) decomposes canonically as the sum of its connected components:

$$(X, G) = \bigoplus_{i=1}^{k} (X_i, G_i) \quad k \geqq 1.$$

Let T_i be the translation group of X_i, $G_i' = T_i^{\perp}$, $X_i' = X_{i|G_i'}$, and $r_i = \text{rank } T_i$. Then (X, G) is completely described in terms of the spaces (X_i', G_i'), $i = 1, \ldots, k$ together with the integers $r_1, \ldots, r_k$. Moreover, either (X_i', G_i') has rank $\geqq 3$ and has more than one component and $r_i > 0$, or (X_i', G_i') has rank 1, and $r_i \geqq 0$. Thus, we have an inductive description of all finite spaces of orderings, and can even "count" the number of non-equivalent spaces of any given rank.

THEOREM 4.10. *Let (X, G) be a finite space of orderings. Then there exists a Pythagorian Field F such that $(X, G) \backsim (X_F, G_F)$.*

Proof. This is clear when the above results are combined with the results in [**2**] or [**3**]. For completeness we give the following proof, by induction on the rank n of X. The case $n = 1$ is clear. For $n > 1$, there are two cases:

(i) (X, G) is not connected. Then $(X, G) \backsim (X_1, G_1) \oplus (X_2, G_2)$ where (X_i, G_i) $i = 1, 2$ are spaces of lower rank. By induction there exist Pythagorian fields F_i such that $(X_{F_i}, G_{F_i}) \backsim (X_i, G_i)$, $i = 1, 2$. By [**3**] (also implicit in [**2**]) there exists a Pythagorian Field F such that $(X_F, G_F) \backsim (X_{F_1}, G_{F_1}) \oplus (X_{F_2}, G_{F_2})$. It follows that $(X, G) \backsim (X_F, G_F)$.

(ii) (X, G) is connected. Let T denote the translation group of G, $G' = T^{\perp}$, $X' = \{\sigma_{|G_i} | \sigma \in X\}$. Now $T \neq 1$ by Theorem 4.8, so (X', G') has strictly lower rank than (X, G). Thus, by induction, there exists a Pythagorian field K such that $(X', G') \backsim (X_K, G_K)$. Then it is clear that $(X, G) \backsim (X_F, G_F)$ where F denotes the power series field

$$K((t_1)) \ldots ((t_r)), \quad r = \text{rank } T.$$

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