

SOLUTIONS TO PSEUDODIFFERENTIAL EQUATIONS USING SPHERICAL RADIAL BASIS FUNCTIONS

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Abstract

Spherical radial basis functions are used to define approximate solutions to pseudodifferential equations of negative order on the unit sphere. These equations arise from geodesy. The approximate solutions are found by the collocation method. A salient feature of our approach in this paper is a simple error analysis for the collocation method using the same argument as that for the Galerkin method.

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1. Introduction

Pseudodifferential equations on the sphere have important applications in geodesy, oceanography, and meteorology [3, 10]. Demands for efficient solutions to these equations are increasing as more and more global data are collected by satellites. In this paper, we study the use of spherical radial basis functions to find approximate solutions to pseudodifferential equations of negative order.

The use of spherical radial basis functions results in meshless methods which, in recent years, have grown in popularity [12, 13]. These methods are an alternative to finite-element methods. The advantage of spherical radial basis functions is in the construction of the finite-dimensional subspace; it is independent of the dimension of the geometry. In particular, when scattered data are given, they can be used as centres to define the spherical radial basis functions without resorting to interpolation.

Collocation solutions to pseudodifferential equations on the sphere by using spherical radial basis functions have been studied by Morton and Neamtu [5]. Error bounds were later improved by Morton [4]. These papers emphasize operators of *positive* orders, and the crux of the analysis therein is the transformation of the collocation problem to a Lagrange or Hermite interpolation problem.

In this paper, we shall consider the collocation method for solving pseudodifferential equations of *negative* order. These equations arise, for example, when one solves,

in the exterior of the sphere, the Dirichlet problem with the Laplacian; see [3, 10]. The novelty of our work compared to [4, 5] lies not only in the negativity of the orders of the operators, but also in the analysis of the error of the approximation. It is well known that in general the collocation method is easier to implement but elicits a more complicated analysis than the Galerkin method. Since the spherical radial basis functions to be used in the approximation are defined from a reproducing kernel, we can view the collocation method as the Galerkin method applied to another pseudodifferential operator. Therefore, we can use the same error analysis for the Galerkin method to study errors in the collocation approximation.

In Section 2 we introduce the definition of pseudodifferential operators in terms of spherical harmonics and describe in detail the problem to be solved. Section 3 is devoted to the introduction of spherical radial basis functions and their approximation properties. The main result of the paper is presented in Section 4.

Throughout this paper, C denotes a generic positive constant which may vary from instance to instance.

2. The problem

In this paper we denote by \mathbb{S} the unit sphere in \mathbb{R}^3 , that is, $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\}$. For $s \in \mathbb{R}$, the Sobolev space H^s on \mathbb{S} is defined as usual (see, for example, [7]),

$$H^s := \left\{ v \in \mathcal{D}'(\mathbb{S}) \mid \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} |\widehat{v}_{\ell,m}|^2 < \infty \right\},$$

where $\mathcal{D}'(\mathbb{S})$ is the space of distributions on \mathbb{S} . The space H^s is equipped with the following norm and inner product:

$$\begin{aligned} \|v\|_s &:= \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} |\widehat{v}_{\ell,m}|^2 \right)^{1/2}, \\ \langle v, w \rangle_s &:= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} \widehat{v}_{\ell,m} \overline{\widehat{w}_{\ell,m}}. \end{aligned} \tag{2.1}$$

When $s = 0$ we write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_0$, which is in fact the L_2 -inner product on \mathbb{S} . We note that

$$|\langle v, w \rangle_s| \leq \|v\|_s \|w\|_s \quad \forall v, w \in H^s, \forall s \in \mathbb{R}, \tag{2.2}$$

and

$$\|v\|_s = \sup_{\substack{w \in H^t \\ w \neq 0}} \frac{\langle v, w \rangle_{(s+t)/2}}{\|w\|_t} \quad \forall v \in H^s, \forall s, t \in \mathbb{R}. \tag{2.3}$$

A pseudodifferential operator L on the sphere is defined by

$$Lv = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{L}(\ell) \widehat{v}_{\ell,m} Y_{\ell,m} \tag{2.4}$$

for all $v \in \mathcal{D}'(\mathbb{S})$ where, for $\ell = 0, 1, 2, \dots$ and $m = -\ell, \dots, \ell$, the functions $Y_{\ell,m}$ are spherical harmonics of degree ℓ (see, for example, [6]), and

$$\widehat{v}_{\ell,m} = \int_{\mathbb{S}} v(\mathbf{x}) Y_{\ell,m}(\mathbf{x}) ds$$

are the Fourier coefficients of v with respect to spherical harmonics. The sequence $(\widehat{L}(\ell))_{\ell \geq 0}$ is referred to as the *spherical symbol* of L .

Let $\mathcal{K}(L) := \{\ell \mid \widehat{L}(\ell) = 0\}$. Then

$$\ker L = \text{span}\{Y_{\ell,m} \mid \ell \in \mathcal{K}(L), m = -\ell, \dots, \ell\}.$$

Denoting $M := \dim \ker L$, we assume that $0 < M < \infty$.

The operator L is said to be of *order* 2α for some $\alpha \in \mathbb{R}$ if there exist positive constants C_1 and C_2 such that

$$C_1(\ell + 1)^{2\alpha} \leq \widehat{L}(\ell) \leq C_2(\ell + 1)^{2\alpha} \quad \forall \ell \notin \mathcal{K}(L). \tag{2.5}$$

Under this assumption one can prove that $L : H^{s+\alpha} \rightarrow H^{s-\alpha}$ is a bounded operator for all $s \in \mathbb{R}$.

The following pseudodifferential operators are commonly seen; see [7].

- (i) The Laplace–Beltrami operator is an operator of order 2 and has as symbol $\widehat{L}(\ell) = \ell(\ell + 1)$. This operator is the restriction of the Laplacian on the sphere.
- (ii) The hypersingular integral operator (without the minus sign) is an operator of order 1 and has as symbol $\widehat{L}(\ell) = \ell(\ell + 1)/(2\ell + 1)$. This operator arises from the boundary-integral reformulation of the Neumann problem with the Laplacian in the interior or exterior of the sphere.
- (iii) The weakly-singular integral operator is an operator of order -1 and has as symbol $\widehat{L}(\ell) = 1/(2\ell + 1)$. This operator arises from the boundary-integral reformulation of the Dirichlet problem with the Laplacian in the interior or exterior of the sphere.

The problem we are solving in this paper is posed as follows.

PROBLEM. Let L be a pseudodifferential operator of order 2α with $\alpha \leq 0$. Given, for some $\sigma \in \mathbb{R}$, $g \in H^{\sigma-\alpha}$ satisfying $\widehat{g}_{\ell,m} = 0$ for all $\ell \in \mathcal{K}(L)$ and $m = -\ell, \dots, \ell$, find $u \in H^{\sigma+\alpha}$ satisfying

$$\begin{aligned} Lu &= g, \\ \mu_i(u) &= a_i, \quad i = 1, \dots, M, \end{aligned} \tag{2.6}$$

where, for $i = 1, \dots, M$, $a_i \in \mathbb{R}$ are given numbers and $\mu_i \in (H^{\sigma+\alpha})^*$, that is, μ_i are continuous linear functionals on $H^{\sigma+\alpha}$. These functionals are assumed to be unisolvent with respect to $\ker L$, namely, for every $v \in \ker L$, if $\mu_i(v) = 0$ for $i = 1, \dots, M$, then $v = 0$.

The following lemma which is proved in [5] yields the existence and uniqueness of the solution to (2.6).

LEMMA 2.1. *There exists a unique solution $u \in H^{\sigma+\alpha}$ to (2.6), which can be written as*

$$u = u_0 + u_1 \quad \text{where } u_0 \in \ker L \text{ and } u_1 \in (\ker L)^\perp_{H^{\sigma+\alpha}}. \tag{2.7}$$

Here, $(\ker L)^\perp_{H^{\sigma+\alpha}}$ is the orthogonal complement of $\ker L$ as a subspace of the Hilbert space $H^{\sigma+\alpha}$. Moreover, u_1 satisfies $Lu_1 = g$ in $H^{\sigma-\alpha}$, and in particular,

$$\langle Lu_1, v \rangle = \langle g, v \rangle \quad \forall v \in H^{\alpha-\sigma}. \tag{2.8}$$

In the following we shall find an approximate solution to (2.6) in terms of spherical radial basis functions using the collocation method.

3. Spherical radial basis functions

Spherical radial basis functions are defined from positive-definite kernels.

3.1. Positive-definite kernels A continuous function $\Phi : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ is called a *positive-definite kernel* on \mathbb{S} if it satisfies:

- (i) $\Phi(\mathbf{x}, \mathbf{y}) = \overline{\Phi(\mathbf{y}, \mathbf{x})}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}$;
- (ii) for every set of distinct points $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ on \mathbb{S} , the $N \times N$ matrix A with entries $A_{i,j} = \Phi(\mathbf{x}_i, \mathbf{x}_j)$ is positive-semidefinite.

If the matrix A is positive-definite then Φ is called a *strictly positive-definite kernel*; see [9, 14].

We define the kernel Φ in terms of a univariate function $\phi : [-1, 1] \rightarrow \mathbb{R}$:

$$\Phi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x} \cdot \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{S}.$$

Assume that ϕ has a series expansion in terms of Legendre polynomials P_ℓ ,

$$\phi(t) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) \widehat{\phi}(\ell) P_\ell(t), \tag{3.1}$$

where

$$\widehat{\phi}(\ell) = 2\pi \int_{-1}^1 \phi(t) P_\ell(t) dt.$$

By using the well-known addition formula for spherical harmonics [6, 7],

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x}) \overline{Y_{\ell,m}(\mathbf{y})} = \frac{2\ell + 1}{4\pi} P_\ell(\mathbf{x} \cdot \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{S}, \tag{3.2}$$

we can write

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \widehat{\phi}(\ell) \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x}) \overline{Y_{\ell,m}(\mathbf{y})}. \tag{3.3}$$

REMARK 3.1. The kernel Φ is called a *zonal* kernel. In [2], a complete characterization of strictly positive-definite kernels is established: the kernel Φ is strictly positive-definite if and only if $\widehat{\phi}(\ell) \geq 0$ for all $\ell \geq 0$, and $\widehat{\phi}(\ell) > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ ; see also [9, 14].

In the following we shall assume that $\widehat{\phi}(\ell) > 0$ for all $\ell \geq 0$, so that the kernel Φ is strictly positive-definite. The native space associated with ϕ is defined by

$$\mathcal{N}_\phi := \left\{ v \in \mathcal{D}'(\mathbb{S}) \mid \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \frac{|\widehat{v}_{\ell,m}|^2}{\widehat{\phi}(\ell)} < \infty \right\}.$$

This space is equipped with an inner product and a norm defined by

$$\langle v, w \rangle_\phi = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \frac{\widehat{v}_{\ell,m} \overline{\widehat{w}_{\ell,m}}}{\widehat{\phi}(\ell)} \quad \text{and} \quad \|v\|_\phi = \left(\sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \frac{|\widehat{v}_{\ell,m}|^2}{\widehat{\phi}(\ell)} \right)^{1/2}. \quad (3.4)$$

If the coefficients $\widehat{\phi}(\ell)$ for $\ell = 0, 1, \dots$ satisfy

$$c_1(\ell + 1)^{-2\tau} \leq \widehat{\phi}(\ell) \leq c_2(\ell + 1)^{-2\tau} \quad (3.5)$$

for some positive constants c_1 and c_2 , and some $\tau \in \mathbb{R}$, then the native space \mathcal{N}_ϕ can be identified with the Sobolev space H^τ , and the corresponding norms are equivalent. In particular, if $\tau > 1$ then the series (3.1) converges pointwise and $\mathcal{N}_\phi \subset C(\mathbb{S})$, which is essentially the Sobolev embedding theorem.

It is noted that, for any $v \in C(\mathbb{S})$,

$$v(\mathbf{x}) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \frac{\widehat{v}_{\ell,m} \widehat{\phi}(\ell) Y_{\ell,m}(\mathbf{x})}{\widehat{\phi}(\ell)} = \langle v, \Phi(\cdot, \mathbf{x}) \rangle_\phi \quad \forall \mathbf{x} \in \mathbb{S}. \quad (3.6)$$

Therefore, Φ is a reproducing kernel of the Hilbert space \mathcal{N}_ϕ .

In the next section we shall use the reproducing kernel Φ to define the spherical radial basis functions, which in turn define our finite-dimensional subspace for the approximation.

3.2. Spherical radial basis functions and approximation properties Let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ be a set of data points on \mathbb{S} . An important parameter characterizing the set X is the *mesh norm* h_X defined by

$$h_X := \sup_{\mathbf{y} \in \mathbb{S}} \min_{1 \leq j \leq N} \cos^{-1}(\mathbf{x}_j \cdot \mathbf{y}).$$

The *spherical radial basis functions* Φ_j , $j = 1, \dots, N$, associated with X and the kernel Φ are defined by

$$\Phi_j(\mathbf{x}) := \Phi(\mathbf{x}, \mathbf{x}_j) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \widehat{\phi}(\ell) \overline{Y_{\ell,m}(\mathbf{x}_j)} Y_{\ell,m}(\mathbf{x}), \quad (3.7)$$

so that

$$(\widehat{\Phi}_j)_{\ell,m} = \widehat{\phi}(\ell) \overline{Y_{\ell,m}(\mathbf{x}_j)}, \quad \ell = 0, 1, 2, \dots, m = -\ell, \dots, \ell.$$

We note that if (3.5) holds then

$$\Phi_j \in H^s \quad \forall s < 2\tau - 1. \tag{3.8}$$

Moreover, for any continuous function v defined on \mathbb{S} , it follows from (3.6) that

$$v(\mathbf{x}_j) = \langle v, \Phi_j \rangle_\phi, \quad j = 1, \dots, N. \tag{3.9}$$

The finite-dimensional subspace to be used in our approximation is now defined by

$$V_X^\phi := \text{span}\{\Phi_1, \dots, \Phi_N\}.$$

We note that if $\tau > 1$, then $V_X^\phi \subset \mathcal{N}_\phi = H^\tau \subset C(\mathbb{S})$. We recall here the approximation property of V_X^ϕ as a subset of Sobolev spaces.

PROPOSITION 3.2. [11, Theorem 3.7] *Assume that (3.5) holds for some $\tau > 1$. For any $s, t \in \mathbb{R}$ satisfying $t \leq s \leq 2\tau$ and $t \leq \tau$, if $v \in H^s$ then there exists $\eta \in V_X^\phi$ such that*

$$\|v - \eta\|_t \leq Ch_X^v \|v\|_s, \tag{3.10}$$

where $v = \min\{s - t, 2(\tau - t), 2\tau + |s|\}$, and where the constant C is independent of v and h_X .

An approximation to (2.6) will be sought in the space V_X^ϕ , as described in the next section.

4. Collocation method

4.1. Approximate solutions Recalling (3.8), we assume that

$$\tau > \frac{\sigma + \alpha + 1}{2}, \tag{4.1}$$

so that $V_X^\phi \subset H^{\sigma+\alpha}$. We shall seek (see Lemma 2.1) an approximate solution $\tilde{u} \in H^{\sigma+\alpha}$ in the form

$$\tilde{u} = \tilde{u}_0 + \tilde{u}_1 \quad \text{where } \tilde{u}_0 \in \ker L \text{ and } \tilde{u}_1 \in V_X^\phi. \tag{4.2}$$

We note that $L\tilde{u}_1 \in H^{\sigma-\alpha}$ and recall that $g \in H^{\sigma-\alpha}$. If $\sigma > \alpha + 1$, then $L\tilde{u}_1$ and g are continuous functions. Therefore, we can find the component $\tilde{u}_1 \in V_X^\phi$ by solving the equations

$$L\tilde{u}_1(\mathbf{x}_j) = g(\mathbf{x}_j), \quad j = 1, \dots, N. \tag{4.3}$$

By writing $\tilde{u}_1 = \sum_{i=1}^N \alpha_i \Phi_i$ we obtain from (4.3) a matrix equation with an $N \times N$ matrix A having entries given by

$$\begin{aligned} A_{i,j} &= L\Phi_i(\mathbf{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{L}(\ell) \widehat{\phi}(\ell) \overline{Y_{\ell,m}(\mathbf{x}_i)} Y_{\ell,m}(\mathbf{x}_j) \\ &= \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) \widehat{L}(\ell) \widehat{\phi}(\ell) P_{\ell}(\mathbf{x}_i \cdot \mathbf{x}_j), \end{aligned}$$

where in the final step we used (3.2). The matrix A is positive-definite due to the strict positive-definiteness of the kernel Ψ defined by

$$\Psi(\mathbf{x}, \mathbf{y}) := \sum_{\ell=0}^{\infty} \widehat{L}(\ell) \widehat{\phi}(\ell) \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x}) \overline{Y_{\ell,m}(\mathbf{y})};$$

see Remark 3.1. Therefore, there exists a unique solution \tilde{u}_1 to (4.3).

Having found \tilde{u}_1 , we find $\tilde{u}_0 \in \ker L$ by solving the equations (see (2.6))

$$\mu_i(\tilde{u}_0) = a_i - \mu_i(\tilde{u}_1), \quad i = 1, \dots, M, \tag{4.4}$$

so that $\mu_i(\tilde{u}) = \mu_i(u)$ for $i = 1, \dots, M$. The unisolvency assumption assures the unique existence of \tilde{u}_0 .

In the remainder, we shall convert the collocation Equation (4.3) into a Galerkin equation, and use the known error analysis for Galerkin methods to estimate the error $u - \tilde{u}$.

4.2. Error analysis Recalling (3.9), we can rewrite (4.3) as

$$\langle L\tilde{u}_1, \Phi_j \rangle_{\phi} = \langle g, \Phi_j \rangle_{\phi}, \quad j = 1, \dots, N. \tag{4.5}$$

In order to see that the above equation is a Galerkin equation, we introduce a new pseudodifferential operator L_{ϕ} and a new function $g_{\phi} \in H^{\sigma-\alpha-2\tau}$:

$$L_{\phi} v := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{L}(\ell)}{\widehat{\phi}(\ell)} \widehat{v}_{\ell,m} Y_{\ell,m} \quad \text{and} \quad g_{\phi} := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{g}_{\ell,m}}{\widehat{\phi}(\ell)} Y_{\ell,m}.$$

The operator L_{ϕ} has as symbol $\widehat{L}(\ell)/\widehat{\phi}(\ell)$, and is of order $2(\tau + \alpha)$ due to (2.5) and (3.5). Therefore, $L_{\phi} : H^{\tau+\alpha} \rightarrow H^{-\tau-\alpha}$ is bounded.

Using L_{ϕ} and g_{ϕ} , we can rewrite Equation (4.5) as

$$\langle L_{\phi}\tilde{u}_1, \Phi_j \rangle = \langle g_{\phi}, \Phi_j \rangle, \quad j = 1, \dots, N. \tag{4.6}$$

The exact solution u_1 defined in Lemma 2.1 satisfies a similar equation as shown in the following lemma.

LEMMA 4.1. Assume that $g \in H^{\sigma-\alpha}$ with $\sigma \geq \tau$. Let u_1 be given as in Lemma 2.1. Then

$$\langle L_\phi u_1, v \rangle = \langle g_\phi, v \rangle \quad \forall v \in H^{\tau+\alpha}. \tag{4.7}$$

PROOF. First we note that $g_\phi \in H^{-\tau-\alpha}$ under the assumption $\sigma \geq \tau$. On the other hand, L_ϕ maps $H^{\tau+\alpha}$ to $H^{-\tau-\alpha}$. Therefore, equation (4.7) is well defined.

Since $\sigma \geq \tau$, then $H^{\alpha-\tau} \subset H^{\alpha-\sigma}$, which together with (2.8) implies that

$$\langle Lu_1, w \rangle = \langle g, w \rangle \quad \forall w \in H^{\alpha-\tau}. \tag{4.8}$$

For any $v \in H^{\tau+\alpha}$, let

$$V := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{v}_{\ell,m}}{\widehat{\phi}(\ell)} Y_{\ell,m}.$$

Then $V \in H^{\alpha-\tau}$. By using (4.8), we obtain

$$\begin{aligned} \langle L_\phi u_1, v \rangle &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{L}(\ell)}{\widehat{\phi}(\ell)} (\widehat{u_1})_{\ell,m} \overline{\widehat{v}_{\ell,m}} \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{L}(\ell) (\widehat{u_1})_{\ell,m} \overline{\widehat{V}_{\ell,m}} \\ &= \langle Lu_1, V \rangle = \langle g, V \rangle \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{g}_{\ell,m} \overline{\widehat{v}_{\ell,m}}}{\widehat{\phi}(\ell)} = \langle g_\phi, v \rangle, \end{aligned}$$

finishing the proof of the lemma. □

Assumptions (4.1) and $\sigma \geq \tau$ imply that $V_X^\phi \subset H^{\sigma+\alpha} \subset H^{\tau+\alpha}$. Due to (4.6) and (4.7), \widetilde{u}_1 is a Galerkin solution to the equation $L_\phi u_1 = g_\phi$, with the energy space being $H^{\tau+\alpha}$, namely,

$$\langle L_\phi(u_1 - \widetilde{u}_1), v \rangle = 0 \quad \forall v \in V_X^\phi. \tag{4.9}$$

It is noted that in general $V_X^\phi \not\subset (\ker L)^\perp_{H^{\tau+\alpha}}$. However, \widetilde{u} can be rewritten in a form similar to (2.7) as follows. Let

$$u_0^* := \widetilde{u}_0 + \sum_{\ell \in \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} (\widehat{\widetilde{u}_1})_{\ell,m} Y_{\ell,m} \quad \text{and} \quad u_1^* := \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} (\widehat{\widetilde{u}_1})_{\ell,m} Y_{\ell,m}. \tag{4.10}$$

Then

$$\widetilde{u} = u_0^* + u_1^* \quad \text{with} \quad u_0^* \in \ker L \quad \text{and} \quad u_1^* \in (\ker L)^\perp_{H^{\tau+\alpha}}. \tag{4.11}$$

It should be noted that u_1^* may not belong to V_X^ϕ .

Comparing (2.7) and (4.11) suggests that $\|u - \widetilde{u}\|_{\tau+\alpha}$ can be estimated by estimating $\|u_0 - u_0^*\|_{\tau+\alpha}$ and $\|u_1 - u_1^*\|_{\tau+\alpha}$. It turns out that an estimate for the latter is sufficient, as given in the following two lemmas.

LEMMA 4.2. *Assume that $\sigma \geq \tau$. If the functionals μ_i , $i = 1, \dots, M$, are bounded in $H^{\tau+\alpha}$, then*

$$\|u_0 - u_0^*\|_{\tau+\alpha} \leq C \|u_1 - u_1^*\|_{\tau+\alpha},$$

where u_0, u_1, u_0^* and u_1^* are defined by (2.7) and (4.10).

PROOF. For $i = 1, \dots, M$, it follows from the linearity of μ_i and $\mu_i(u) = \mu_i(\tilde{u})$, see (4.4), that

$$\mu_i(u_0) + \mu_i(u_1) = \mu_i(u_0^*) + \mu_i(u_1^*),$$

implying that

$$\mu_i(u_0) - \mu_i(u_0^*) = \mu_i(u_1^*) - \mu_i(u_1).$$

Therefore,

$$|\mu_i(u_0 - u_0^*)| = |\mu_i(u_1 - u_1^*)| \leq \|\mu_i\| \|u_1 - u_1^*\|_{\tau+\alpha},$$

where $\|\mu_i\|$ is the operator norm of μ_i with respect to the $H^{\tau+\alpha}$ -norm. This result holds for all $i = 1, \dots, M$, implying that

$$\|u_0 - u_0^*\|_{\mu} \leq \mathcal{M} \|u_1 - u_1^*\|_{\tau+\alpha},$$

where $\mathcal{M} := \max_{i=1, \dots, M} \|\mu_i\|$, and where, for any $v \in \ker L$,

$$\|v\|_{\mu} := \max_{i=1, \dots, M} |\mu_i(v)|.$$

(The unisolvency assumption assures us that the norm $\|\cdot\|_{\mu}$ is well defined. Indeed, if $\|v\|_{\mu} = 0$ for some $v \in \ker L$, then the unisolvency of μ_i confirms that $v = 0$.) The subspace $\ker L$ being finite-dimensional (so that all norms are equivalent), we deduce that

$$\|u_0 - u_0^*\|_{\tau+\alpha} \leq C \|u_1 - u_1^*\|_{\tau+\alpha},$$

proving the lemma. □

LEMMA 4.3. *Under the assumptions of Lemma 4.2,*

$$\|u - \tilde{u}\|_{\tau+\alpha} \leq C \|u_1 - u_1^*\|_{\tau+\alpha}.$$

PROOF. Noting (2.7) and (4.11), the norm $\|u - \tilde{u}\|_{\tau+\alpha}$ can be rewritten as

$$\begin{aligned} \|u - \tilde{u}\|_{\tau+\alpha}^2 &= \sum_{\ell \in \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2(\tau+\alpha)} |\widehat{u}_{\ell,m} - \widehat{(\tilde{u})}_{\ell,m}|^2 \\ &\quad + \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2(\tau+\alpha)} |\widehat{u}_{\ell,m} - \widehat{(\tilde{u})}_{\ell,m}|^2 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\ell \in \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2(\tau+\alpha)} |\widehat{u_0}_{\ell,m} - \widehat{u_0^*}_{\ell,m}|^2 \\
 &\quad + \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2(\tau+\alpha)} |\widehat{u_1}_{\ell,m} - \widehat{u_1^*}_{\ell,m}|^2 \\
 &= \|u_0 - u_0^*\|_{\tau+\alpha}^2 + \|u_1 - u_1^*\|_{\tau+\alpha}^2.
 \end{aligned}$$

The required result now follows from Lemma 4.2. □

In the following, the notation $a \simeq b$ means

$$c_1 a \leq b \leq c_2 a,$$

where c_1 and c_2 are positive constants independent of the variables in concern. The following simple results are often used in the remainder of the paper.

LEMMA 4.4. *If $v \in H^{\tau+\alpha}$ satisfies $\widehat{v}_{\ell,m} = 0$ for all $\ell \in \mathcal{K}(L)$ and $m = -\ell, \dots, \ell$, then*

$$\langle L_\phi v, v \rangle \simeq \|v\|_{\tau+\alpha}^2. \tag{4.12}$$

Moreover, if $v, w \in H^{\tau+\alpha}$, then

$$|\langle L_\phi v, w \rangle| \leq C \|v\|_{\tau+\alpha} \|w\|_{\tau+\alpha}. \tag{4.13}$$

PROOF. Let $v \in H^{\tau+\alpha}$ be such that $\widehat{v}_{\ell,m} = 0$ for all $\ell \in \mathcal{K}(L)$ and for all $m = -\ell, \dots, \ell$. Noting (2.5) and (3.5), then

$$\begin{aligned}
 \langle L_\phi v, v \rangle &= \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} \frac{\widehat{L}(\ell)}{\widehat{\phi}(\ell)} |\widehat{v}_{\ell,m}|^2 \\
 &\simeq \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2(\tau+\alpha)} |\widehat{v}_{\ell,m}|^2 \\
 &= \|v\|_{\tau+\alpha}^2,
 \end{aligned}$$

proving (4.12). Inequality (4.13) can be proved similarly by using (2.5), (3.5) and the Cauchy–Schwarz inequality. □

The following lemma is technically C ea’s lemma (see, for example, [1]).

LEMMA 4.5. *Under the assumptions of Lemma 4.2,*

$$\|u_1 - u_1^*\|_{\tau+\alpha} \leq C \|u_1 - v\|_{\tau+\alpha} \quad \forall v \in V_X^\phi.$$

PROOF. It follows from (4.10) that

$$\langle L_\phi w, u_1^* \rangle = \langle L_\phi w, \widetilde{u}_1 \rangle \quad \forall w \in H^{\tau+\alpha}, \tag{4.14}$$

and from (4.10) and (4.9) that

$$\langle L_\phi(u_1 - u_1^*), v \rangle = 0 \quad \forall v \in V_X^\phi. \tag{4.15}$$

Due to (4.12),

$$\begin{aligned} \|u_1 - u_1^*\|_{\tau+\alpha}^2 &\simeq \langle L_\phi(u_1 - u_1^*), u_1 - u_1^* \rangle \\ &= \langle L_\phi(u_1 - u_1^*), u_1 \rangle - \langle L_\phi(u_1 - u_1^*), u_1^* \rangle, \end{aligned}$$

which together with (4.14) implies that

$$\|u_1 - u_1^*\|_{\tau+\alpha}^2 \simeq \langle L_\phi(u_1 - u_1^*), u_1 \rangle - \langle L_\phi(u_1 - u_1^*), \tilde{u}_1 \rangle.$$

By noting that $\tilde{u}_1 \in V_X^\phi$ and using (4.15) we deduce, for any $v \in V_X^\phi$, that

$$\begin{aligned} \|u_1 - u_1^*\|_{\tau+\alpha}^2 &\simeq \langle L_\phi(u_1 - u_1^*), u_1 \rangle \\ &= \langle L_\phi(u_1 - u_1^*), u_1 \rangle - \langle L_\phi(u_1 - u_1^*), v \rangle \\ &= \langle L_\phi(u_1 - u_1^*), u_1 - v \rangle. \end{aligned}$$

It follows from (4.13) that

$$\|u_1 - u_1^*\|_{\tau+\alpha}^2 \leq C \|u_1 - u_1^*\|_{\tau+\alpha} \|u_1 - v\|_{\tau+\alpha}.$$

The required result now follows by cancelling similar terms. □

By using Lemmas 4.3 and 4.5 and Proposition 3.2, we can now prove the main result of this paper.

THEOREM 4.6. *Let (3.5) and (4.1) be satisfied. Assume that $1 < \tau \leq \sigma$. Assume further that $u \in H^s$ for some $s \in \mathbb{R}$ satisfying $\tau + \alpha \leq s \leq 2\tau$. If $\mu_i \in (H^{\tau+\alpha})^*$ for $i = 1, \dots, M$, then*

$$\|u - \tilde{u}\|_{\tau+\alpha} \leq Ch_X^{v_1} \|u\|_s, \tag{4.16}$$

where $v_1 = \min\{s - \tau - \alpha, -2\alpha, 2\tau + |s|\}$.

Moreover, if $\mu_i \in (H^t)^*$, $i = 1, \dots, M$, for some $t \in \mathbb{R}$ satisfying $2\alpha \leq t \leq \tau + \alpha$, then

$$\|u - \tilde{u}\|_t \leq Ch_X^{v_1+v_2} \|u\|_s, \tag{4.17}$$

where $v_2 = \min\{\tau + \alpha - t, -2\alpha, 2\tau + |2\tau + 2\alpha - t|\}$. The constant C is independent of h_X .

PROOF. We first prove (4.16). Since $\tau + \alpha \leq \tau$ and $\tau + \alpha \leq s \leq 2\tau$, Proposition 3.2 assures us that there exists $v \in V_X^\phi$ satisfying

$$\|u_1 - v\|_{\tau+\alpha} \leq Ch_X^{v_1} \|u_1\|_s,$$

where $v_1 = \min\{s - \tau - \alpha, -2\alpha, 2\tau + |s|\}$. It follows from Lemma 4.5 and the above inequality that

$$\|u_1 - u_1^*\|_{\tau+\alpha} \leq Ch_X^{v_1} \|u_1\|_s. \tag{4.18}$$

This inequality, Lemma 4.3, and the inequality $\|u_1\|_s \leq \|u\|_s$ imply (4.16).

Estimate (4.17) can be obtained by using the well-known duality argument. We include the proof here for completeness. Under the assumption $\mu_i \in (H^t)^*$ for $i = 1, \dots, M$, we can use the same arguments as in the proofs of Lemmas 4.2 and 4.3 to obtain

$$\|u - \tilde{u}\|_t \leq C \|u_1 - u_1^*\|_t.$$

By using (2.3), and noting that the order of L_ϕ is $2(\tau + \alpha)$, we deduce from the above inequality that

$$\begin{aligned} \|u - \tilde{u}\|_t &\leq C \sup_{\substack{v \in H^{2\tau+2\alpha-t} \\ v \neq 0}} \frac{\langle u_1 - u_1^*, v \rangle_{\tau+\alpha}}{\|v\|_{2\tau+2\alpha-t}} \\ &\leq C \sup_{\substack{v \in H^{2\tau+2\alpha-t} \\ v \neq 0}} \frac{\langle L_\phi(u_1 - u_1^*), v \rangle}{\|v\|_{2\tau+2\alpha-t}}. \end{aligned}$$

It follows from (4.15) and (4.13) that, for any $\eta \in V_X^\phi$,

$$\begin{aligned} \|u - \tilde{u}\|_t &\leq C \sup_{\substack{v \in H^{2\tau+2\alpha-t} \\ v \neq 0}} \frac{\langle L_\phi(u_1 - u_1^*), v - \eta \rangle}{\|v\|_{2\tau+2\alpha-t}} \\ &\leq C \|u_1 - u_1^*\|_{\alpha+\tau} \sup_{\substack{v \in H^{2\tau+2\alpha-t} \\ v \neq 0}} \frac{\|v - \eta\|_{\alpha+\tau}}{\|v\|_{2\tau+2\alpha-t}}. \end{aligned}$$

By using Proposition 3.2, noting that $2\tau + 2\alpha - t \leq 2\tau$ due to $2\alpha \leq t$ and $\tau + \alpha \leq 2\tau + 2\alpha - t$ due to $t \leq \tau + \alpha$, we can choose $\eta \in V_X^\phi$ such that

$$\sup_{\substack{v \in H^{2\tau+2\alpha-t} \\ v \neq 0}} \frac{\|v - \eta\|_{\alpha+\tau}}{\|v\|_{2\tau+2\alpha-t}} \leq Ch_X^{v_2}.$$

The desired result now follows from (4.18) and the inequality $\|u_1\|_s \leq \|u\|_s$. □

5. Conclusions

We have presented a collocation solution to pseudodifferential equations of negative order on the sphere, using spherical radial basis functions. The analysis follows that of the Galerkin method thanks to an observation that the collocation equation can be rewritten as a Galerkin equation. Numerical results for the case of an operator of order -1 can be found in a forthcoming paper [8].

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