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Coexistence of nontrivial solutions of the one-dimensional Ginzburg-Landau equation: A computer-assisted proof

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In this paper, Chebyshev series and rigorous numerics are combined to compute solutions of the Euler-Lagrange equations for the one-dimensional Ginzburg-Landau model of superconductivity. The idea is to recast solutions as fixed points of a Newton-like operator defined on a Banach space of rapidly decaying Chebyshev coefficients. Analytic estimates, the radii polynomials and the contraction mapping theorem are combined to show existence of solutions near numerical approximations. Coexistence of as many as seven nontrivial solutions is proved.

Key words: Ginzburg-Landau equation, boundary value problems, coexistence of nontrivial solutions, rigorous numerics, Chebyshev series, contraction mapping theorem

1 Introduction

According to the Ginzburg-Landau theory of superconductivity [1], the electromagnetic properties of a superconducting material of width $2d$ subjected to a tangential external magnetic field are described by a pair (ϕ, ψ) which minimises the free energy functional

$$G = G(\phi, \psi) = \frac{1}{2d} \int_{-d}^{d} \left(\phi^2(\phi^2 - 2) + \frac{2(\phi')^2}{\kappa^2} + 2\phi^2\psi^2 + 2(\psi' - h_e)^2 \right) d\xi. \tag{1.1}$$

In this context, the functional G is known as the *Ginzburg-Landau energy*, and provides a measure of the difference between normal and superconducting states of the material. The function ϕ measures the density of superconducting electrons and the function ψ is the magnetic field potential. The parameter d is the size of the superconducting material, h_e is the external magnetic field and κ is the Ginzburg-Landau parameter, which is a dimensionless constant distinguishing different superconductors. More precisely, $0 < \kappa < 1/\sqrt{2}$ characterises type I superconductors while $\kappa > 1/\sqrt{2}$ characterises type II superconductors [2] (e.g. see Figure 1(a)).

A standard variational argument shows that the Ginzburg-Landau energy (1.1) has a minimiser, the minimiser is a solution of the Euler-Lagrange equations, and in particular is given by the boundary value problem (BVP)

$$\begin{cases} \phi'' = \kappa^2 \phi(\phi^2 + \psi^2 - 1) \\ \psi'' = \phi^2 \psi \\ \phi'(\pm d) = 0, \psi'(\pm d) = h_e. \end{cases} \tag{1.2}$$

The Ginzburg-Landau BVP (1.2) has been studied by many authors (e.g. see [3–15] and the references therein). This list of references is by no means meant to provide a complete review of the literature of the work done on (1.2). Several people have also studied the Ginzburg-Landau model in higher dimensions [16–19].

A solution (ϕ, ψ) of (1.2) is called *symmetric* if

$$\phi'(0) = 0 \quad \text{and} \quad \psi(0) = 0, \tag{1.3}$$

and called *asymmetric* otherwise. If (1.3) holds, then ϕ is even and ψ is odd. There is a family of solutions of (1.2) of the form

$$(\phi(\xi), \psi(\xi) = (0, h_e \xi + q), \quad q \in \mathbb{R}, \tag{1.4}$$

which are symmetric when $q = 0$ and asymmetric otherwise. We refer to solutions (1.4) as *trivial* and refer to solutions that are not of the form (1.4) as *nontrivial.*

An interesting review of results and open problems about existence, uniqueness and coexistence of nontrivial symmetric and asymmetric solutions of (1.2) can be found in [13]. Moreover, in [13], Aftalion *et al.* present a detailed numerical study of the bifurcations arising in (1.2), where they obtain a complete description of the solutions over the range of physically important parameters (d, κ, h_e). They consider $(d, \kappa) \in D \stackrel{\text{def}}{=} [0, 5] \times [0, 1.4]$, leave h_e as a parameter, and investigate bifurcations of symmetric and asymmetric solutions as h_e varies. They numerically obtain two partitions for D. The first one is $D = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3$ and it characterises the symmetric solutions as follows: given $(d, \kappa) \in \mathscr{S}_i$, there exists h_e such that (1.2) has i symmetric solutions. The second partition is $D = \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$ and it characterises the asymmetric solutions as follows: given $(d, \kappa) \in \mathscr{A}_j$, there exists h_e such that (1.2) has $2j$ asymmetric solutions. Note that asymmetric solutions come in pairs. Indeed, one can easily verify from (1.2) that if $(\phi(\xi), \psi(\xi))$ is an asymmetric solution, then $(\phi(-\xi), -\psi(-\xi))$ is another asymmetric solution. A geometric representation of the two partitions of D can be found in Figure 1(a). The following conjecture follows from the analysis and the numerical investigation of [13].

Conjecture 1.1 *For $i \in \{1, 2, 3\}$, $j \in \{0, 1, 2\}$ and $(d, \kappa) \in \mathscr{S}_i \cap \mathscr{A}_j$, there exists h_e such that there exist i nontrivial symmetric solutions and $2j$ nontrivial asymmetric solutions of* (1.2).

Partial progress has been made toward a proof of Conjecture 1.1, but many cases remain open. Perhaps the most interesting open question arising from Conjecture 1.1 concerns the region $\mathscr{S}_3 \cap \mathscr{A}_2$, where as many as seven solutions may coexist. Seydel is the first in 1983 to give numerical evidence of existence of parameters for which four asymmetric solutions and three symmetric solutions may coexist [5]. In 1996, Hastings *et al.* comment in [8] that *"this [analysis] goes only part way towards verifying the numerical results of Seydel,*

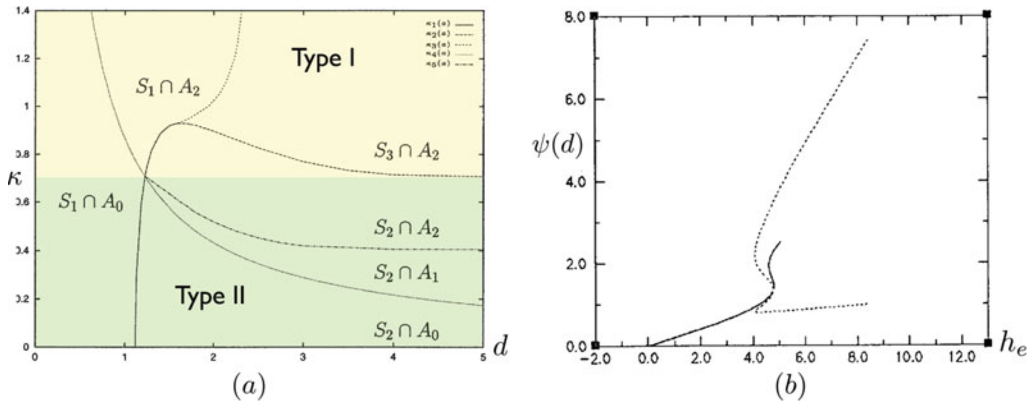

FIGURE 1. (a) Figure taken from [13] with permission from the authors. The regions $\mathscr{S}_i \cap \mathscr{A}_j$ ($i \in \{1,2,3\}$ and $j \in \{0,1,2\}$) are delimited by black lines. The regions corresponding to the two different types of superconductors are pictured in yellow (type I) and green (type II), where $0 < \kappa < 1/\sqrt{2}$ characterises type I while $\kappa > 1/\sqrt{2}$ characterises type II. (b) A global bifurcation diagram when $(d,\kappa) = (2.5,1) \in \mathscr{S}_3 \cap \mathscr{A}_2$ taken from [12] with permission from the authors.

where as many as seven solutions are found in a limited parameter range. This remains an interesting problem for future research." In 2000, Dancer *et al.* in [12] write that *"the initial motivation for our paper was Seydel's bifurcation diagram, and our goal was to prove that in some parameter range the problem could have as many as seven solutions, but unfortunately we have not achieved this goal.* Seydel's bifurcation diagram can be found in Figure 1(b). Besides the region $\mathscr{S}_3 \cap \mathscr{A}_2$, other cases are interesting. For instance, as mentioned in [13], *"it is an interesting open problem to prove that both symmetric and asymmetric solutions coexist in $\mathscr{S}_1 \cap \mathscr{A}_2$."* The goal of the present paper is to prove these open questions for specific parameter values using the rigorous computational methods of [20–24] and more specifically with the approach as introduced in [25].

Our proposed approach to the problem has a different flavor than the standard tools of nonlinear analysis (e.g. bifurcation and perturbation theory, degree theory, global bifurcation theorems). It combines the strength of numerical analysis, approximation theory, spectral methods, fixed point theory, functional analysis and interval arithmetic (e.g. see [26]) to demonstrate that near numerical approximations, there are solutions of (1.2). This approach uses the field of *rigorous numerics* (described in Section 2), and as opposed to classical methods in nonlinear analysis it does not require knowing the existence of a *trivial* solution from which one can perturb. In fact, our method is a perturbative result from a numerical approximation,and this implies that it is extremely suitable to prove conjectures about coexistence of solutions. However, as opposed to global methods, our proposed method works well for specific parameter values, rather than globally (i.e. for all parameters). Let us now present our four main results.

Theorem 1.1 *Define* $(d,\kappa) = (3.5, 0.9) \in \mathscr{S}_3 \cap \mathscr{A}_2$. *Then at* $h_e = 0.85$, *there exist three symmetric solutions* $x_s^{(i)} = (\phi_s^{(i)}, \psi_s^{(i)})$ *(*$i = 1,2,3$*) and four asymmetric solutions* $x_a^{(i)} = (\phi_a^{(i)}, \psi_a^{(i)})$ *(*$i = 1,2,3,4$*) of* (1.2). *Each solution is nontrivial and all solutions are distinct. Hence, there are seven coexisting nontrivial solutions.*

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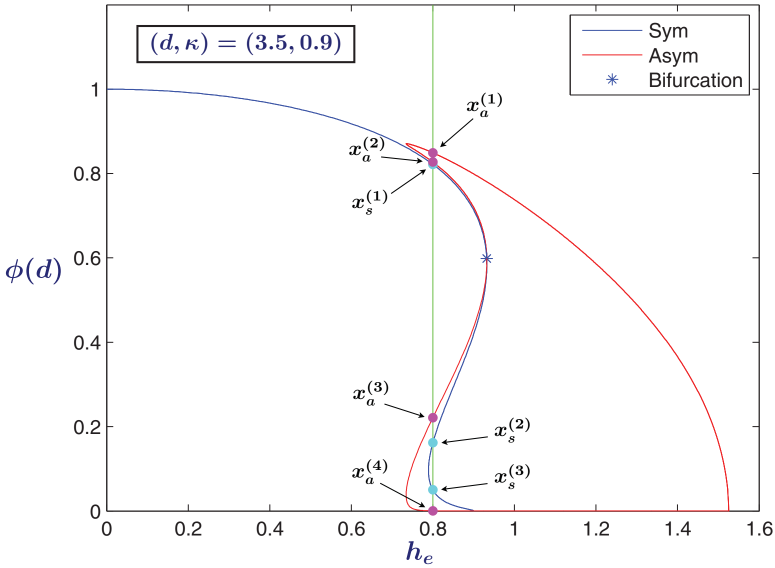

FIGURE 2. The global bifurcation diagram when $(d,\kappa) = (3.5, 0.9) \in \mathscr{S}_3 \cap \mathscr{A}_2$. The data from this diagram was used to obtain the proof of Theorem 1.1. At $h_e = 0.85$, there exist three symmetric solutions $x_s^{(1)},\dots, x_s^{(3)}$ and four asymmetric solutions $x_a^{(1)},\dots, x_a^{(4)}$ of (1.2).

The proof of Theorem 1.1 can be found in Section 4. A geometrical interpretation of the global bifurcation diagram with fixed $(d,\kappa) = (3.5, 0.9)$ and h_e left as a free parameter is depicted in Figure 2. The profile of each of the seven nontrivial coexisting solutions of Theorem 1.1 can be found in Figure 3.

Theorem 1.2 *Let* $(d,\kappa) = (1.6, 1.2) \in \mathscr{S}_1 \cap \mathscr{A}_2$. *Then at* $h_e = 1.1$, *there exist one symmetric solution* $x_s^{(1)} = (\phi_s^{(1)}, \psi_s^{(1)})$ *and four asymmetric solutions* $x_a^{(i)} = (\phi_a^{(i)}, \psi_a^{(i)})$ *(*$i = 1,2,3,4$*) of* (1.2). *Each solution is nontrivial and all solutions are distinct.*

The proof of Theorem 1.2 can be found in Section 4. A geometrical interpretation of the global bifurcation diagram with fixed $(d,\kappa) = (1.6, 1.2)$ and h_e left as a free parameter is depicted in Figure 4. The profile of each of the five nontrivial coexisting solutions of Theorem 1.2 can be found in Figure 5.

Theorem 1.3 *Let* $(d,\kappa) = (4, 0.3) \in \mathscr{S}_2 \cap \mathscr{A}_1$ *and* $h_e = 0.8$. *There exist two symmetric solutions* $x_s^{(i)} = (\phi_s^{(i)}, \psi_s^{(i)})$ *(*$i = 1,2$*) and two asymmetric solutions* $x_a^{(i)} = (\phi_a^{(i)}, \psi_a^{(i)})$ *(*$i = 1,2$*) of* (1.2). *All solutions are nontrivial and distinct.*

The proof of Theorem 1.3 can be found in Section 4. A geometrical interpretation of the global bifurcation diagram with fixed $(d,\kappa) = (4, 0.3)$ and h_e left as a free parameter is depicted in Figure 6.

Theorem 1.4 *Define* $(d,\kappa) = (3, 0.6) \in \mathscr{S}_2 \cap \mathscr{A}_2$. *Then at* $h_e = 0.9$, *there exist two symmetric solutions* $x_s^{(i)} = (\phi_s^{(i)}, \psi_s^{(i)})$ *(*$i = 1,2$*) and four asymmetric solutions* $x_a^{(i)} = (\phi_a^{(i)}, \psi_a^{(i)})$ *(*$i = 1,2,3,4$*) of* (1.2). *All solutions are nontrivial and distinct.*

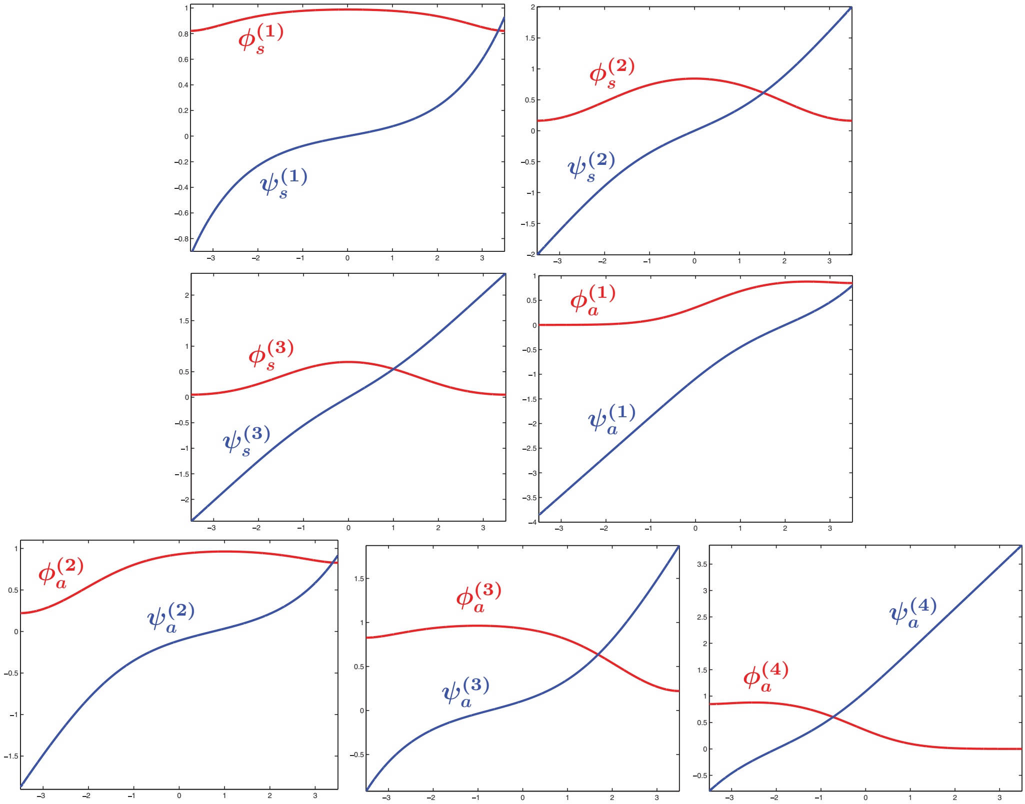

FIGURE 3. For $(d, \kappa) = (3.5, 0.9) \in \mathscr{S}_3 \cap \mathscr{A}_2$ and $h_e = 0.85$, the solution profiles of the seven nontrivial coexisting solutions of Theorem 1.1: three symmetric solutions $x_s^{(i)} = (\phi_s^{(i)}, \psi_s^{(i)})$, $i = 1, 2, 3$, and four asymmetric solutions $x_a^{(i)} = (\phi_a^{(i)}, \psi_a^{(i)})$, $i = 1, 2, 3, 4$. Each solution is defined on $[-d, d] = [-3.5, 3.5]$. Note that $\phi_s^{(1)}(d) \approx 0.821$, $\phi_s^{(2)}(d) \approx 0.161$, $\phi_s^{(3)}(d) \approx 0.050$, $\phi_a^{(1)}(d) \approx 0.849$, $\phi_a^{(2)}(d) \approx 0.827$, $\phi_a^{(3)}(d) \approx 0.221$ and $\phi_a^{(4)}(d) \approx 3.37 \times 10^{-4}$.

The proof of Theorem 1.4 can be found in Section 4. A geometrical interpretation of the global bifurcation diagram with fixed $(d, \kappa) = (3, 0.6)$ and h_e left as a free parameter is depicted in Figure 7.

As mentioned above, the proofs of Theorems 1.1–1.4 are done using *rigorous numerics*, which is a field that aims at constructing algorithms that provide an approximate solution to a problem together with precise bounds within which the solution is guaranteed to exist in the mathematically rigorous sense. In our context, Chebyshev series are combined with rigorous numerics to compute solutions of (1.2). The idea is to recast solutions as fixed points of a Newton-like operator defined on a Banach space of rapidly decaying Chebyshev coefficients and to use the contraction mapping theorem to show existence of solutions near numerical approximations. Note that a similar approach can be used to prove existence of connecting orbits (e.g. see [25]). The radii polynomials (first introduced in [20] to compute equilibria of PDEs) are used to construct sets on which the contraction mapping theorem is applicable, and their construction is a combination of analytic estimates and interval arithmetic computations. The last steps of the proofs of Theorems 1.1–1.4 are are done by running the MATLAB codes which are available at [33].

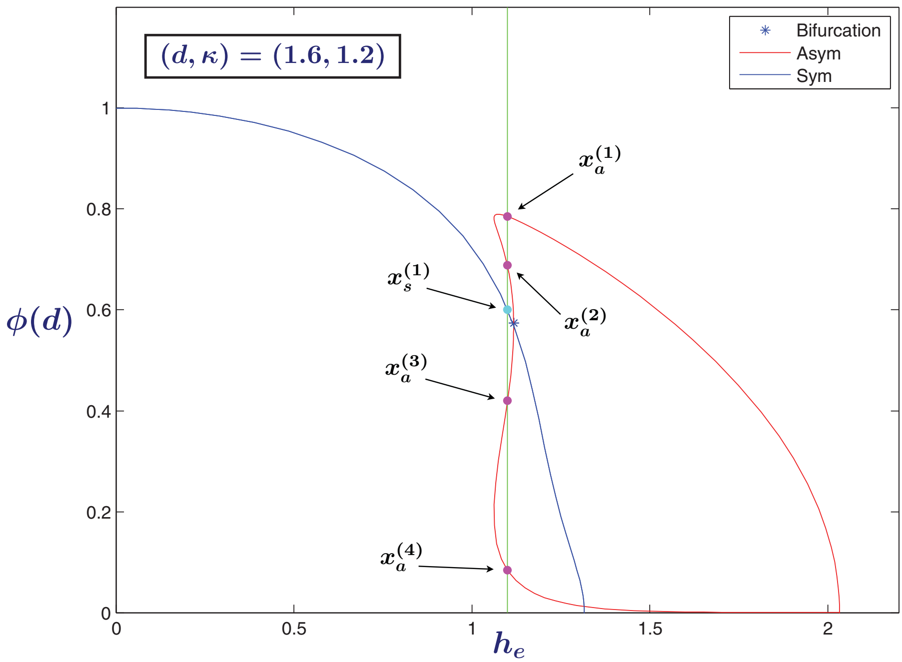

FIGURE 4. The global bifurcation diagram when $(d, \kappa) = (1.6, 1.2) \in \mathscr{S}_1 \cap \mathscr{A}_2$. The data from this diagram was used to obtain the proof of Theorem 1.2. At $h_e = 1.1$, there exist one symmetric solution $x_s^{(1)}$ and four asymmetric solutions $x_a^{(1)}$, $x_a^{(2)}$, $x_a^{(3)}$ and $x_a^{(4)}$ of (1.2).

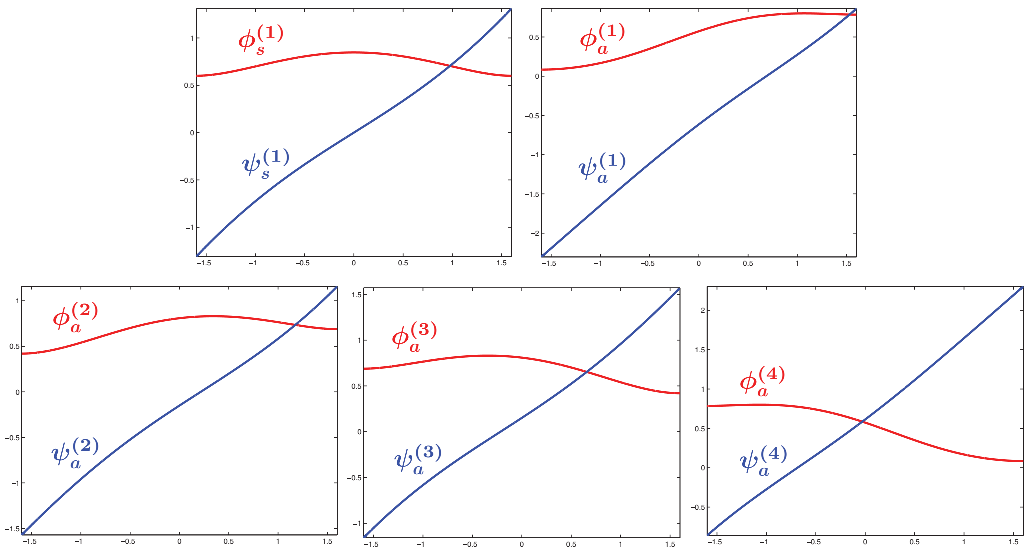

FIGURE 5. For $(d, \kappa) = (1.6, 1.2) \in \mathscr{S}_1 \cap \mathscr{A}_2$ and $h_e = 1.1$, the profiles of the solutions of Theorem 1.2: 1 symmetric solution $x_s^{(1)} = (\phi_s^{(1)}, \psi_s^{(1)})$ and 4 asymmetric solutions $x_a^{(i)} = (\phi_a^{(i)}, \psi_a^{(i)})$, $i = 1, 2, 3, 4$. Each solution is defined on $[-d, d] = [-1.6, 1.6]$. $\phi_s^{(1)}(d) \approx 0.600$, $\phi_a^{(1)}(d) \approx 0.785$, $\phi_a^{(2)}(d) \approx 0.688$, $\phi_a^{(3)}(d) \approx 0.419$ and $\phi_a^{(4)}(d) \approx 0.0842$.

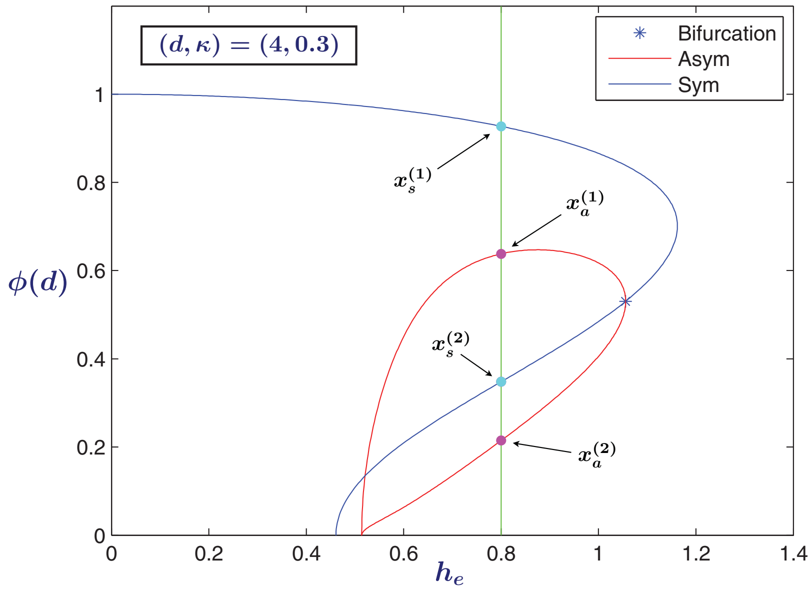

FIGURE 6. The global bifurcation diagram when $(d, \kappa) = (4, 0.3) \in \mathscr{S}_2 \cap \mathscr{A}_1$. The data from this diagram was used to obtain the proof of Theorem 1.3. At $h_e = 0.8$, there exist two symmetric solutions $x_s^{(1)}$, $x_s^{(2)}$ and two asymmetric solutions $x_a^{(1)}$, $x_a^{(2)}$ of (1.2).

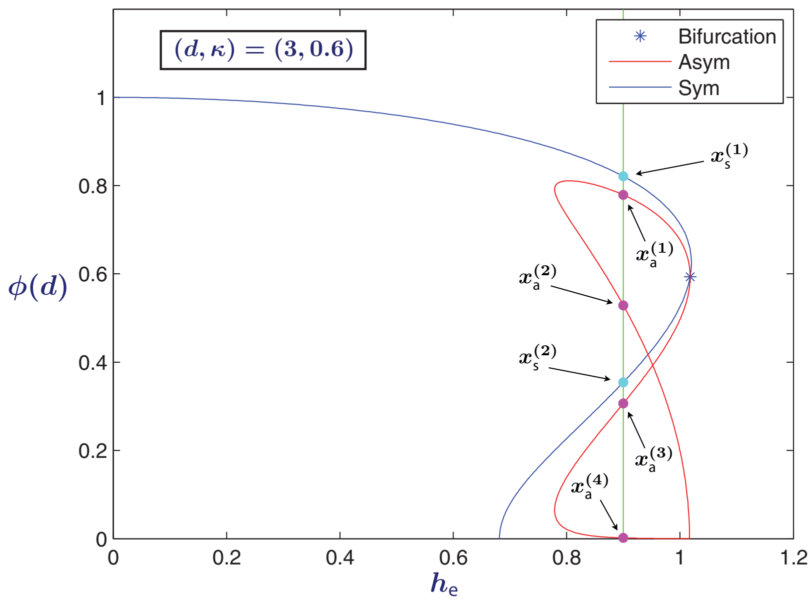

FIGURE 7. The global bifurcation diagram when $(d, \kappa) = (3, 0.6) \in \mathscr{S}_2 \cap \mathscr{A}_2$. The data from this diagram was used to obtain the proof of Theorem 1.4. At $h_e = 0.9$, there exist two symmetric solutions $x_s^{(1)}$, $x_s^{(2)}$ and four asymmetric solutions $x_a^{(1)}$, $x_a^{(2)}$, $x_a^{(3)}$, $x_a^{(4)}$ of (1.2).

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Remark 1.1 *We also obtained rigorous results concerning existence of solutions in $\mathscr{S}_1 \cap \mathscr{A}_0$ and $\mathscr{S}_2 \cap \mathscr{A}_0$, but we do not present them here, as these two regions are better understood theoretically. The codes for the proofs can be found at [33].*

Remark 1.2 *Note that our proposed approach is certainly not the only rigorous computational method that could have been used to prove the above results. For instance, a rigorous numerical integration of the equations combined with studying an appropriate Poincaré section could have been used. The choice of the approach is a matter of taste. However, since the Euler-Lagrange equation* (1.2) *is naturally a boundary value problem, we believe that our collocation-type approach based on Chebyshev series is a natural choice.*

The paper is organised as follows: in Section 2, we introduce the rigorous computational method and the theoretical definition of the radii polynomials; in Section 3, analytic estimates are used to obtain the explicit formulas for the radii polynomials; in Section 4, the proofs of Theorems 1.1,–1.4 are presented.

2 The rigorous computational method

The rigorous computational method used here is based on the general method first introduced in [25]. More precisely, the idea is to expand solutions of (1.2) using their Chebyshev series, plug the expansion in the equation, obtain an equivalent infinite dimensional problem of the form $f(x) = 0$ to solve in a Banach space of rapidly decaying Chebyshev coefficients, and finally to get the existence, via a fixed point argument, of a solution of $f(x) = 0$ near a numerical approximation of a finite dimensional projection of f. The fixed point argument is solved by using the radii polynomials, which provide an efficient way of constructing a set on which the contraction mapping theorem is applicable. We begin by setting up the problem $f(x) = 0$.

2.1 Setting up $f(x) = 0$

Setting $u = (u_1, u_2, u_3, u_4) = (\phi, \phi', \psi, \psi')$ and introducing the new independent variable $t = \xi/d$, (1.2) becomes

$$\frac{du}{dt} = \Psi(u) \stackrel{\text{def}}{=} d \begin{pmatrix} u_2 \\ \kappa^2 u_1 \left(u_1^2 + u_3^2 - 1\right) \\ u_4 \\ u_1^2 u_3 \end{pmatrix}, \quad u_2(\pm 1) = 0, \quad u_4(\pm 1) = h_e, \tag{2.1}$$

where $u = u(t)$ is defined for $t \in [-1, 1]$. Let $P(\theta) \stackrel{\text{def}}{=} u(-1) = (\theta_1, 0, \theta_2, h_e)$, where $\theta = (\theta_1, \theta_2) = (u_1(-1), u_3(-1)) \in \mathbb{R}^2$ is a variable that is used to compensate for the fact

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that the values of $u_1(-1)$ and $u_3(-1)$ are not fixed. Letting

$$F(\theta,u)(t) \stackrel{\text{def}}{=} \begin{pmatrix} u_2(1) \\ u_4(1)-h_e \\ P(\theta)+\displaystyle\int_{-1}^{t} \Psi(u(s))ds-u(t) \end{pmatrix}, \quad t\in[-1,1], \tag{2.2}$$

a solution (θ,u) of $F(\theta,u)=0$ is a solution of (2.1) and therefore solves the Euler-Lagrange BVP (1.2), provided that u is sufficiently smooth. Note that the extra variable $\theta\in\mathbb{R}^2$ ensures that the operator (2.2) is not overdetermined. Expand u with Chebyshev series

$$u(t)=a_0+2\sum_{k\geqslant 1} a_k T_k(t)=\sum_{k\in\mathbb{Z}} a_k T_k(t), \tag{2.3}$$

where $T_k:[-1,1]\to[-1,1]$ $(k\geqslant 0)$ are the Chebyshev polynomials defined by $T_0(t)=1$, $T_1(t)=t$ and $T_{k+1}(t)=2tT_k(t)-T_{k-1}(t)$, for $k\geqslant 1$, and where $a_{-k}\stackrel{\text{def}}{=}a_k$, $T_{-k}\stackrel{\text{def}}{=}T_k$, $a_k=(a_k^{(1)},a_k^{(2)},a_k^{(3)},a_k^{(4)})^T\in\mathbb{R}^4$. Define the infinite dimensional vector of Chebyshev coefficients $a=(a_k)_{k\geqslant 0}$. Using that $T_k(1)=1$ for every $k\geqslant 0$, define

$$\eta(\theta,a)\stackrel{\text{def}}{=}(u_2(1),u_4(1)-h_e)=\left(a_0^{(2)}+2\sum_{j\geqslant 1} a_j^{(2)}, a_0^{(4)}+2\sum_{j\geqslant 1} a_j^{(4)}-h_e\right). \tag{2.4}$$

Since Chebyshev polynomials are in fact Fourier series *in disguise* [27], as $T_k(\cos\xi)=\cos(k\xi)$ with $\xi=\arccos t$, the Chebyshev coefficients of the product of two functions is given by the discrete convolution of the Chebyshev coefficients of each function (e.g. see [25]). In the context of the vector field defined in (2.1),

$$\Psi(u(t))=c_0+2\sum_{k\geqslant 1} c_k T_k(t)=\sum_{k\in\mathbb{Z}} c_k T_k(t), \quad (c_{-k}=c_k), \tag{2.5}$$

where

$$c_k=\begin{pmatrix} c_k^{(1)} \\ c_k^{(2)} \\ c_k^{(3)} \\ c_k^{(4)} \end{pmatrix}=d\begin{pmatrix} a_k^{(2)} \\ \kappa^2\left([a^{(1)}a^{(1)}a^{(1)}]_k+[a^{(1)}a^{(3)}a^{(3)}]_k-a_k^{(1)}\right) \\ a_k^{(4)} \\ [a^{(1)}a^{(1)}a^{(3)}]_k \end{pmatrix}, \tag{2.6}$$

and for $i,j\in\{1,3\}$,

$$[a^{(1)}a^{(i)}a^{(j)}]_k=\sum_{\substack{k_1+k_2+k_3=k\\ k_1,k_2,k_3\in\mathbb{Z}}} a_{k_1}^{(1)}a_{k_2}^{(i)}a_{k_3}^{(j)}, \ (a_{-k}^{(\ell)}=a_k^{(\ell)}).$$

Plugging the expansions (2.3) and (2.5) in (2.2), and using the properties $\int T_0(s)ds=T_1(s)$, $\int T_1(s)ds=(T_2(s)+T_0(s))/4$ and $\int T_k(s)ds=\frac{1}{2}\left(\frac{T_{k+1}(s)}{k+1}-\frac{T_{k-1}(s)}{k-1}\right)$ for $k\geqslant 2$,

$$P(\theta)+\int_{-1}^{t}\Psi(u(s))ds-u(t)=\tilde{f}_0+2\sum_{k\geqslant 1}\tilde{f}_k T_k(t),$$

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where $\tilde{f}_0 \stackrel{\text{def}}{=} P(\theta) - a_0 + c_0 - \frac{c_1}{2} - 2\sum_{j\geqslant 2}\frac{(-1)^j}{j^2-1}c_j$ and $\tilde{f}_k \stackrel{\text{def}}{=} \frac{c_{k-1}-c_{k+1}}{2k} - a_k$, for $k \geqslant 1$. Denote $x = (\theta, a)$ so that $x_{-1} = \theta \in \mathbb{R}^2$ and $x_k = a_k \in \mathbb{R}^4$ for $k \geqslant 0$. Finally, define $f(x) = (f_k(x))_{k\geqslant -1}$ component-wise by

$$f_k(x) = \begin{cases} (a_0^{(2)} + 2\displaystyle\sum_{j\geqslant 1} a_j^{(2)}, a_0^{(4)} + 2\sum_{j\geqslant 1} a_j^{(4)} - h_e), & k = -1, \\ P(\theta) - a_0 + c_0 - \dfrac{c_1}{2} - 2\displaystyle\sum_{j\geqslant 2}\frac{(-1)^j}{j^2-1}c_j, & k = 0, \\ -2ka_k + c_{k-1} - c_{k+1}, & k \geqslant 1, \end{cases} \tag{2.7}$$

where $f_0 = \tilde{f}_0$ and $f_k \stackrel{\text{def}}{=} 2k\tilde{f}_k$ for $k \geqslant 1$. Note that an equivalent simpler formulation for $f_0(x)$ in (2.7) is given by $f_0(x) = P(\theta) - a_0 - 2\sum_{j=1}^{\infty}(-1)^j a_j$ (see [25]). However, we chose to work with f_0 as in (2.7) because dividing by $j^2 - 1$ gives slightly better estimates.

The importance of introducing the operator (2.7) is that solutions of $f(x) = 0$ correspond to solutions of the BVP (1.2) (see Lemma 2.1). Hence, coexistence of solutions of the Euler-Lagrange equations reduces to demonstrating that the operator f defined component-wise by (2.7) has several coexisting nontrivial roots.

The next step is to introduce the Banach space X^s of fast decaying Chebyshev coefficients with algebraic decay k^s on which the operator f is defined, and to introduce the equivalent fixed point problem $T(x) = x$. Note that the fixed point operator T is defined as a Newton-like operator defined near a numerical approximation $\bar{x} \in X^s$. The idea is that locally, the operator T should be a contraction on a small ball containing $\bar{x}$. The way to find the ball is done using the radii polynomials. This is described in Section 2.3.

2.2 The Banach space X^s and the fixed point problem $T(x) = x$

Let $\|\theta\|_\infty = \max\{|\theta_1|, |\theta_2|\}$, $\|a_k\|_\infty = \max\limits_{i=1,\dots,4}\{|a_k^{(i)}|\}$ for $k \geqslant 0$ and define the weights

$$\omega_k^s \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } k = 0 \\ |k|^s, & \text{if } k \neq 0. \end{cases} \tag{2.8}$$

The Banach space on which we solve the problem $f(x) = 0$ is defined by

$$X^s = \left\{ x = (x_k)_{k\geqslant -1} \,:\, \|x\|_s \stackrel{\text{def}}{=} \sup_{k\geqslant -1}\{\|x_k\|_\infty \omega_k^s\} < \infty \right\}, \tag{2.9}$$

which is a space of algebraically decaying sequences with decay rate $s > 1$. Next, we show that $f : X^s \to X^{s-1}$ and that if $x \in X^s$ solves $f(x) = 0$ for some $s > 1$, then $x \in X^{s_0}$ for any $s_0 > 1$. Hence, if $x = (\theta, a)$ is a solution of $f(x) = 0$ then the Chebyshev coefficients a of (2.3) decay faster than any algebraic decay. This comes as no surprise as a solution $u = (u_1, u_2, u_3, u_4) = (\phi, \phi', \psi, \psi')$ of the analytic differential equation (1.2) is analytic. This implies that the Chebyshev expansion x of any solution of (1.2) is in the space X^s.

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Lemma 2.1 *Let $f(x) = (f_k(x))_{k\geqslant -1}$ as in (2.7). Then the following statements hold.*

(a) $f : X^s \to X^{s-1}$, $s > 1$.

(b) If $x \in X^s$ is a solution of $f(x) = 0$, then $x \in X^{s_0}$ for any $s_0 > 1$.

(c) The Chebyshev series $u(t) = a_0 + 2\sum_{k\geqslant 1} a_k T_k(t)$ and $\theta = (u_1(-1), u_3(-1))$ solve $F(\theta, u) = 0$ where F is the integral operator (2.2) if and only if $x = (\theta, a) \in X^s$ solves $f(x) = 0$.

(d) Any solution $x = (\theta, u)$ of $f(x) = 0$ yields a unique solution (ϕ, ψ) of the Euler-Lagrange boundary value problem (1.2).

Proof First of all, for any $s > 1$, the space of scalar algebraically decaying sequences

$$\Omega^s \stackrel{\text{def}}{=} \{a = (a_k)_{k\in\mathbb{N}} : a_k \in \mathbb{R},\ \sup_{k\geqslant 0}\{|a_k|\omega_k^s\} < \infty\} \tag{2.10}$$

is an algebra under discrete convolution. Indeed, for any $a, b \in \Omega^s$, there exists a constant $\alpha = \alpha(a, b) < \infty$ such that $|(a * b)_k| = |\sum_{\substack{k_1+k_2=k \\ k_i\in\mathbb{Z}}} a_{|k_1|}b_{|k_2|}| \leqslant \frac{\alpha}{\omega_k^s}$ (e.g. see [28] if $s \geqslant 2$ and [29] if $s \in (1, 2)$). This implies that $a * b \in \Omega^s$, and hence that $(\Omega^s, *)$ is an algebra.

(a) Consider $x = (\theta, a) \in X^s$. For each $i = 1, 2, 3, 4$, $|a_k^{(i)}| \leqslant \|a\|_s/\omega_k^s$. Recalling (2.4), $\|f_{-1}(x)\|_\infty = \|\eta(\theta, a)\|_\infty < \infty$ since $s > 1$. Consider the Chebyshev coefficients $(c_k)_{k\geqslant 0}$ of $\Psi(u)$ defined in (2.6). Since Ω^s is an algebra, then $\|c\|_s < \infty$. Hence, $\|c_k\|_\infty \leqslant \frac{\|c\|_s}{\omega_k^s}$ and therefore $\|\sum_{j\geqslant 2} \frac{c_j}{j^2-1}\|_\infty \leqslant \|c\|_s \sum_{j\geqslant 2} \frac{1}{j^s(j^2-1)} < \infty$. This implies that $\|f_0(x)\|_\infty < \infty$. Moreover, there exists a constant $\alpha_1 < \infty$ such that $\|f_k(x)\|_\infty = \| - 2ka_k + c_{k-1} - c_{k+1}\|_\infty \leqslant \frac{2\|a\|_s}{\omega_k^{s-1}} + \frac{\|c\|_s}{\omega_{k+1}^s} + \frac{\|c\|_s}{\omega_{k-1}^s} \leqslant \frac{\alpha_1}{\omega_k^{s-1}}$ for all $k \geqslant 1$. It follows that $\|f(x)\|_{s-1} < \infty$ and therefore that $f(x) \in X^{s-1}$.

(b) If $x \in X^s$ is a solution of $f(x) = 0$, then for any $k \geqslant 1$, $f_k(x) = -2ka_k + c_{k-1} - c_{k+1} = 0$, or in other words $a_k = -\frac{1}{2k}(c_{k+1} - c_{k-1})$. Since $\|c_k\|_\infty \leqslant \frac{\|c\|_s}{\omega_k^s}$, there exists a constant $\alpha_2 < \infty$ such that

$$\sup_{k\geqslant 1}\{\|a_k\|_\infty \omega_k^{s+1}\} \leqslant \sup_{k\geqslant 1}\left\{\frac{1}{2k}(\|c_{k+1}\|_\infty + \|c_{k-1}\|_\infty)\omega_k^{s+1}\right\} \leqslant \alpha_2.$$

That shows that $x = (\theta, a) \in X^{s+1}$. Repeating the same argument inductively and using the fact that $X^{s_1} \subset X^{s_2}$ for any $s_1 \geqslant s_2$, one gets that $x \in X^{s_0}$ for all $s_0 > 1$.

(c) By construction of f in (2.7) and by part (b), one immediately verifies that (θ, u), with $u(t) = a_0 + 2\sum_{k\geqslant 1} a_k T_k(t)$ and $\theta = (u_1(-1), u_3(-1))$, is a solution of $F(\theta, u) = 0$ where F is the integral operator (2.2) if and only if $x = (\theta, a) \in X^s$ solves $f(x) = 0$.

(d) It follows from (c) and by construction that any solution $x = (\theta, u)$ of $f(x) = 0$ yields a unique solution (ϕ, ψ) of the Euler-Lagrange boundary value problem (1.2).

□

The strategy to find solutions of $f(x) = 0$ is now to consider an equivalent fixed point operator $T : X^s \to X^s$ whose fixed points are in one-to-one correspondence with the zeros of f. More precisely, the operator T is a Newton-like operator about an

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approximate solution $\bar{x}$ of f. In order to compute this numerical approximation we introduce a Galerkin projection. Let $m > 1$ and define the finite dimensional projection $\Pi_m : X^s \to X^s_m$ by $\Pi_m x = (x_k)_{k=-1}^{m-1}$. Note that $X^s_m \cong \mathbb{R}^{4m+2}$. The Galerkin projection of f is defined by

$$f^{(m)} : X^s_m \to X^s_m : x_F \mapsto \Pi_m f(x_F, 0_\infty), \tag{2.11}$$

where $0_\infty \stackrel{\text{def}}{=} (0,0,0,\ldots)$. Identifying $(x_F, 0_\infty)$ with $x_F \in X^s_m \cong \mathbb{R}^{4m+2}$, we think of $f^{(m)} : \mathbb{R}^{4m+2} \to \mathbb{R}^{4m+2}$. Using Newton's method, assume that we have computed $\bar{x}_F \in \mathbb{R}^{4m+2}$ such that $f^{(m)}(\bar{x}_F) \approx 0$ and let $\bar{x} = (\bar{\theta}, \bar{a}) = (\bar{\theta}, \bar{a}_F, 0_\infty) = (\bar{x}_F, 0_\infty) \in X^s$. Let $B_{\bar{x}}(r) = \bar{x} + B(r)$, the closed ball in X^s of radius r centred at $\bar{x}$, with

$$B(r) = \left\{ x \in X^s : \ \|x\|_s = \sup_{k \geqslant -1} \{ \|x_k\|_\infty \omega_k^s \} \leqslant r \right\} = \prod_{k \geqslant -1} \left[-\frac{r}{\omega_k^s}, \frac{r}{\omega_k^s} \right]^{\zeta(k)}, \tag{2.12}$$

where $\zeta(-1) = 2$ and $\zeta(k) = 4$ for $k \geqslant 0$. In order to define the fixed point operator T, we introduce a $(4m+2) \times (4m+2)$ matrix $A_m \approx (Df^{(m)}(\bar{x}_F))^{-1}$, which is obtained using the computer. Assume that the finite dimensional matrix A_m is invertible (this hypothesis can be rigorously verified with interval arithmetic). Define the linear invertible operator $A : X^s \to X^{s+1}$ by

$$(Ax)_k = \begin{cases} (A_m(\Pi_m x))_k, & k = -1, \ldots, m-1 \\ \left(\frac{1}{-2k} \right) x_k, & k \geqslant m. \end{cases} \tag{2.13}$$

The choice of multiplying x_k by $\frac{1}{-2k}$ for $k \geqslant m$ in the tail of the operator A comes from the fact that the linear part of $f_k = -2ka_k + c_{k-1} - c_{k+1}$ is given by $-2k$. Hence, in the tail of A, we only consider the inverse of the linear part.

Finally define the Newton-like operator $T : X^s \to X^s$ about the numerical solution $\bar{x}$ by

$$T(x) = x - Af(x). \tag{2.14}$$

2.3 Finding $r > 0$ such that T maps $B_{\bar{x}}(r)$ into itself and that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction

The next step is to determine a positive radius r of the ball $B_{\bar{x}}(r)$ so that T maps $B_{\bar{x}}(r)$ into itself and that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction. If such $r > 0$ exists, an application of the contraction mapping theorem yields the existence of a unique fixed point $\tilde{x}$ of T within the closed ball $B_{\bar{x}}(r)$. By invertibility of the linear operator A, one can conclude that $\tilde{x}$ is the unique solution of $f(x) = 0$ in the ball $B_{\bar{x}}(r)$. By Lemma 2.1, this unique solution represents a solution $u(t)$ of the operator (2.1). The task of finding $r > 0$ is achieved with the notion of the radii polynomials (originally introduced in [20] to compute equilibria of PDEs), which provide an efficient way of constructing a set on which the contraction mapping theorem is applicable. Their construction depends on some bounds that we introduce shortly. Before that, we introduce the notation $\preceq$ to denote component-wise inequality, that is given two vectors v and w, $v \preceq w$ if and only if $v_i \leqslant w_i$ for all i. Similarly, the notation $\prec$ denotes component-wise strict inequality. The radii polynomials are in fact polynomial bound inequalities in the variable radius r which represent sufficient

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conditions to have that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction. These polynomials are defined in terms of two bounds: Y and Z.

The bound $Y = (Y_k)_{k \geqslant -1}$ satisfies

$$\left| \left[T(\bar{x}) - \bar{x} \right]_k \right| \preceq Y_k, \quad k \geqslant -1, \tag{2.15}$$

where $Y_{-1} \in \mathbb{R}^2_+$ and $Y_k \in \mathbb{R}^4_+$ for $k \geqslant 0$. The bound $Z(r) = (Z_k(r))_{k \geqslant -1}$ satisfies

$$\sup_{\xi_1, \xi_2 \in B(r)} \left| \left[DT(\bar{x} + \xi_1)\xi_2 \right]_k \right| \preceq Z_k(r), \quad k \geqslant -1, \tag{2.16}$$

where $Z_{-1}(r) \in \mathbb{R}^2_+$ and $Z_k(r) \in \mathbb{R}^4_+$ for $k \geqslant 0$. Since the vector field $\Psi(u)$ defined in (2.1) is cubic, we can compute a cubic polynomial expansion in r for $Z_k(r)$. Consider now a computational parameter $M \geqslant 3m - 1$ where m fixes the dimension of the Galerkin projection (2.11). Then the bounds Y and Z satisfying (2.15) and (2.16) can be constructed such that

R1. $Y_k = 0 \in \mathbb{R}^4$ for all $k \geqslant 3m - 1$, since for any $k \geqslant 3m - 2$, $\bar{a}_k = 0$ and $c_k = c_k(\bar{a}) = 0$, and therefore $f_k(\bar{x}) = -2k\bar{a}_k + c_{k-1}(\bar{a}) - c_{k+1}(\bar{a}) = 0$, for all $k \geqslant 3m - 1$.

R2. There exists (using the analytic estimates introduced in [28]) a *uniform* polynomial bound $\bar{Z}_{M+1}(r) \in \mathbb{R}^4_+$ such that for all $k \geqslant M + 1$,

$$Z_k(r) \preceq \frac{\bar{Z}_{M+1}(r)}{\omega_k^s}. \tag{2.17}$$

Defintion 2.1 *Let* $\mathbf{1}_{\zeta(k)} \stackrel{\text{def}}{=} \overbrace{(1, \ldots, 1)}^{\zeta(k)\ \text{times}} \in \mathbb{R}^{\zeta(k)}$. *The finite radii polynomials are*

$$p_k(r) = Y_k + Z_k(r) - \frac{r}{\omega_k^s} \mathbf{1}_{\zeta(k)}, \quad k = -1, \ldots, M, \tag{2.18}$$

and the tail radii polynomial is

$$p_M(r) = \bar{Z}_{M+1}(r) - r\mathbf{1}_4. \tag{2.19}$$

Theorem 2.1 *If there exists* $r > 0$ *such that* $p_k(r) \prec 0$ *for all* $k = -1, \ldots, M + 1$, *then* $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ *is a contraction and therefore there exists a unique* $\tilde{x} \in B_{\bar{x}}(r)$ *such that* $T(\tilde{x}) = \tilde{x}$. *Hence, there exists a unique* $\tilde{x} \in B_{\bar{x}}(r)$ *such that* $f(\tilde{x}) = 0$.

Proof The proof is a consequence of the contraction mapping theorem. We refer to Corollary 3.6 in [23] for a complete proof. □

The strategy to rigorously compute solutions of f defined in (2.7) is to construct the radii polynomials of Definition 2.1, to satisfy the hypothesis of Theorem 2.1, and to use

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the result of Lemma 2.1 to conclude that $u(t) = a_0 + 2\sum_{k\geqslant 1} a_k T_k(t)$ is a solution of $F(\theta, u) = 0$ with $\theta = (u_1(-1), u_3(-1))$, where F is the integral operator (2.2).

While the computation of the bound Y satisfying (2.15) is rather straightforward, the computation of the polynomial bound $Z(r)$ satisfying (2.16) is more involved. In order to simplify its computation, we introduce the linear invertible operator $A^\dagger : X^s \to X^{s-1}$ by

$$(A^\dagger x)_k = \begin{cases} (Df^{(m)}(\bar{x}_F)(\Pi_m x))_k, & k = -1, \dots, m-1 \\ (-2k)\, x_k, & k \geqslant m. \end{cases} \tag{2.20}$$

We then split $T(x) = x - Af(x) = (I - AA^\dagger)x - A(f(x) - A^\dagger x)$. Letting $\xi_1 = wr$, $\xi_2 = vr$ with $w, v \in B(1)$, one has that

$$\begin{aligned} DT(\bar{x} + \xi_1)\xi_2 &= (I - AA^\dagger)\xi_2 - A\left(Df(\bar{x} + \xi_1)\xi_2 - A^\dagger \xi_2\right) \\ &= \left[(I - AA^\dagger)v\right] r - A\left(Df(\bar{x} + wr)vr - A^\dagger vr\right). \end{aligned} \tag{2.21}$$

The first term of (2.21) is of the form ϵr, where $\epsilon \stackrel{\text{def}}{=} (I - AA^\dagger)v \in X^s$ should be small. The coefficient of r in the second term $[Df(\bar{x} + wr)vr - A^\dagger vr]_k$ should be small for large Galerkin projection dimension m. Hence, for m large enough, the coefficient of r in the radii polynomials of Definition 2.1 can expected to be negative, and therefore the hypothesis of Theorem 2.1 can be satisfied. We now derive explicitly the radii polynomials.

3 Explicit construction of the radii polynomials

In this section, the computation of the bounds involved in the radii polynomials are presented in greater detail. Fix a dimension m for the Galerkin projection (2.11) and consider $\bar{x} = (\bar{\theta}, \bar{a}) = (\bar{x}_F, 0_\infty)$ such that $f^{(m)}(\bar{x}_F) \approx 0$, where f is the operator given in (2.7). We fix the decay rate $s = 2$. Recalling **R1**, we obtain

$$\left[T(\bar{x}) - \bar{x}\right]_k = \left[-Af(\bar{x})\right]_k = \begin{cases} \left[-A_m f^{(m)}(\bar{x})\right]_k, & k = -1, 0, \dots, m-1 \\ \dfrac{1}{2k} f_k(\bar{x}), & m \leqslant k < 3m-1 \\ 0, & k \geqslant 3m-1. \end{cases}$$

Then, compute $Y_{-1} \dots, Y_{3m-2}$ using interval arithmetic with the formulas

$$Y_k = \begin{cases} \left|\left[-A_m f^{(m)}(\bar{x})\right]_k\right|, & k = -1, 0, \dots, m-1 \\ \dfrac{1}{2k} |f_k(\bar{x})|, & m \leqslant k < 3m-1 \\ 0, & k \geqslant 3m-1. \end{cases} \tag{3.1}$$

To simplify the computation and the presentation of the coefficients of $Z_k(r)$, we consider vectors $z_k^{(1)}$, $z_k^{(2)}$ and $z_k^{(3)}$ such that

$$[Df(\bar{x} + wr)vr - A^\dagger vr]_k \prec z_k^{(1)} r + z_k^{(2)} r^2 + z_k^{(3)} r^3. \tag{3.2}$$

We use the following notation, $\bar{x} = (\bar{\theta}, \bar{a})$, $w = (\tilde{\theta}_1, \tilde{a}_1)$, $v = (\tilde{\theta}_2, \tilde{a}_2)$. Before defining the

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vectors $z_k^{(1)}$, $z_k^{(2)}$ and $z_k^{(3)}$, let us introduce the explicit computation of $Df_k(\bar{x}+wr)v$ for each $k \geqslant -1$.

Computation of $Df_k(\bar{x}+wr)v$

$\underline{k=-1}$: It follows from definition of η in (2.4) that

$$Df_{-1}(\bar{x}+wr)v = \begin{pmatrix} [\tilde{a}_2^{(2)}]_0 + 2\sum_{k\geqslant 1}[\tilde{a}_2^{(2)}]_k \\ [\tilde{a}_2^{(4)}]_0 + 2\sum_{k\geqslant 1}[\tilde{a}_2^{(4)}]_k \end{pmatrix}.$$

$\underline{k=0}$: For $i,j \in \{1,3\}$, set

$$[s_{1ij}]_k(\tau) \stackrel{\text{def}}{=} \left[\left(\bar{a}^{(1)}+\tilde{a}_1^{(1)}r+\tau\tilde{a}_2^{(1)}\right)\left(\bar{a}^{(i)}+\tilde{a}_1^{(i)}r+\tau\tilde{a}_2^{(i)}\right)\left(\bar{a}^{(j)}+\tilde{a}_1^{(j)}r+\tau\tilde{a}_2^{(j)}\right)\right]_k.$$

Computing the derivative with respect to τ and evaluating at $\tau=0$ yields

$$\begin{aligned}
[s_{111}]_k' &\stackrel{\text{def}}{=} \left.\frac{\partial[s_{111}]_k}{\partial\tau}\right|_{\tau=0} = \left[3(\bar{a}^{(1)})^2\tilde{a}_2^{(1)} + 6\bar{a}^{(1)}\tilde{a}_1^{(1)}\tilde{a}_2^{(1)}r + 3(\tilde{a}_1^{(1)})^2\tilde{a}_2^{(1)}r^2\right]_k \\
[s_{113}]_k' &\stackrel{\text{def}}{=} \left.\frac{\partial[s_{113}]_k}{\partial\tau}\right|_{\tau=0} = \left[(\bar{a}^{(1)})^2\tilde{a}_2^{(3)} + 2\bar{a}^{(1)}\tilde{a}_2^{(1)}\bar{a}^{(3)} + 2\bar{a}^{(1)}\tilde{a}_1^{(1)}\tilde{a}_2^{(3)}r + 2\bar{a}^{(1)}\tilde{a}_2^{(1)}\tilde{a}_1^{(3)}r\right. \\
&\qquad\left. + 2\tilde{a}_1^{(1)}\tilde{a}_2^{(1)}\bar{a}^{(3)}r + (\tilde{a}_1^{(1)})^2\tilde{a}_2^{(3)}r^2 + 2\tilde{a}_1^{(1)}\tilde{a}_2^{(1)}\tilde{a}_1^{(3)}r^2\right]_k \\
[s_{133}]_k' &\stackrel{\text{def}}{=} \left.\frac{\partial[s_{133}]_k}{\partial\tau}\right|_{\tau=0} = \left[\tilde{a}_2^{(1)}(\bar{a}^{(3)})^2 + 2\bar{a}^{(1)}\bar{a}^{(3)}\tilde{a}_2^{(3)} + 2\tilde{a}_2^{(1)}\bar{a}^{(3)}\tilde{a}_1^{(3)}r + 2\tilde{a}_1^{(1)}\bar{a}^{(3)}\tilde{a}_2^{(3)}r\right. \\
&\qquad\left. + 2\bar{a}^{(1)}\tilde{a}_1^{(3)}\tilde{a}_2^{(3)}r + \tilde{a}_2^{(1)}(\tilde{a}_1^{(3)})^2r^2 + 2\tilde{a}_1^{(1)}\tilde{a}_1^{(3)}\tilde{a}_2^{(3)}r^2\right]_k.
\end{aligned}$$

Therefore,

$$Df_0(\bar{x}+wr)v = -[\tilde{a}_2]_0 +$$
$$d\begin{pmatrix} (\tilde{\theta}_2)_1 + [\tilde{a}_2^{(2)}]_0 - \frac{1}{2}[\tilde{a}_2^{(2)}]_1 - 2\sum_{k=2}^{\infty}[\tilde{a}_2^{(2)}]_k\frac{(-1)^k}{k^2-1} \\ \kappa^2\left([s_{111}]_0' + [s_{133}]_0' - [\tilde{a}_2^{(1)}]_0 - \dfrac{1}{2}\left([s_{111}]_1' + [s_{133}]_1' - [\tilde{a}_2^{(1)}]_1\right) - 2\displaystyle\sum_{k=2}^{\infty}\left([s_{111}]_k' + [s_{133}]_k' - [\tilde{a}_2^{(1)}]_k\right)\frac{(-1)^k}{k^2-1}\right) \\ (\tilde{\theta}_2)_2 + [\tilde{a}_2^{(4)}]_0 - \dfrac{1}{2}[\tilde{a}_2^{(4)}]_1 - 2\displaystyle\sum_{k=2}^{\infty}[\tilde{a}_2^{(4)}]_k\frac{(-1)^k}{k^2-1} \\ [s_{113}]_0' - \dfrac{1}{2}[s_{113}]_1' - 2\displaystyle\sum_{k=2}^{\infty}[s_{113}]_k'\frac{(-1)^k}{k^2-1} \end{pmatrix}.$$

$\underline{k\geqslant 1}$: Using that $f_k(x) = -2ka_k + c_{k-1} - c_{k+1}$ for $k \geqslant 1$, one gets that

$$Df_k(\bar{x}+wr)v = -2k[\tilde{a}_2]_k +$$
$$d\begin{pmatrix} [\tilde{a}_2^{(2)}]_{k-1} - [\tilde{a}_2^{(2)}]_{k+1} \\ \kappa^2\left([s_{111}]_{k-1}' + [s_{133}]_{k-1}' - [\tilde{a}_2^{(1)}]_{k-1} - [s_{111}]_{k+1}' - [s_{133}]_{k+1}' + [\tilde{a}_2^{(1)}]_{k+1}\right) \\ [\tilde{a}_2^{(4)}]_{k-1} - [\tilde{a}_2^{(4)}]_{k+1} \\ [s_{113}]_{k-1}' - [s_{113}]_{k+1}' \end{pmatrix}.$$

For $w \in B(1)$, $|w_{-1}|_\infty, |w_0|_\infty \leqslant 1$, $|w_k|_\infty \leqslant k^{-s}$, $w_k = ([\tilde{a}_1^{(1)}]_k, [\tilde{a}_1^{(2)}]_k, [\tilde{a}_1^{(3)}]_k, [\tilde{a}_1^{(4)}]_k)$, and similarly for v. Hence, for $k \geqslant 0$, $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$, $|(\tilde{a}_i^{(j)})_k| \leqslant \frac{1}{\omega_k^s}$.

We introduce the notation $\omega^{-s} \stackrel{\text{def}}{=} (\omega_k^{-s})_{k\geqslant -1}$, $\omega_F^{-s} \stackrel{\text{def}}{=} (\omega_k^{-s})_{k=-1}^{m-1} \in \mathbb{R}^{m+1}$ and $\omega_I^{-s} \stackrel{\text{def}}{=} \omega^{-s} - \omega_F^{-s}$. Also $\bar{A}^{(i)} \stackrel{\text{def}}{=} \max_{k\in\{0,1,\ldots,m-1\}} \{\omega_k^s |\bar{a}_k^{(i)}|\}$ for $i = 1,2,3,4$, which implies that $|\bar{a}_k^{(i)}| \leqslant \frac{\bar{A}^{(i)}}{\omega_k^s}$. Before obtaining the bounds $z_k^{(1)}$, $z_k^{(2)}$ and $z_k^{(3)}$ in (3.2), we need some analytic estimates, which are explained in detail in Appendix A.

Lemma 3.1 *Consider the decay rate $s = 2$ and $a, b, c \in \Omega^s$, where Ω^s is defined in* (2.10) *with norm $\|a\|_s = \sup_{k\geqslant 0}\{|a_k|\omega_k^s\}$. Consider $\alpha_k^{(3)}$ as defined in* (A 3). *Then, for any $k \geqslant 0$,*

$$|(abc)_k| \leqslant (\|a\|_2\|b\|_2\|c\|_2)\frac{\alpha_M^{(3)}}{\omega_k^2}. \tag{3.3}$$

Proof *The result follows from Lemmas A.4 and A.5.* □

The bound of Lemma 3.1 can be improved by performing some interval arithmetic computations.

Lemma 3.2 *Consider the decay rate $s = 2$ and $a, b, c \in \Omega^s$. Consider a computational parameter M and define $a^{(M)} = (a_0, a_1, \ldots, a_{M-1}) \in \mathbb{R}^M$. Define $b^{(M)}$, $c^{(M)}$ similarly. Consider $\varepsilon_k^{(3)} = \varepsilon_k^{(3)}(2, M)$ as in* (A 4). *Then, for $k \in \{0, \ldots, M-1\}$,*

$$|(abc)_k| \leqslant \left|\left(a^{(M)}b^{(M)}c^{(M)}\right)_k\right| + 3(\|a\|_2\|b\|_2\|c\|_2)\varepsilon_k^{(3)}.$$

Proof *The result follows from Lemma A.6.* □

The convolutions terms $|(a^{(M)}b^{(M)}c^{(M)})_k|$ can be bounded with the Fast Fourier Transform (FFT) algorithm on the computer together with interval arithmetic (e.g. see [30]).

Computation of the bounds $z_k^{(1)}, z_k^{(2)}, z_k^{(3)}$

Case $k = -1$: Since $\left[Df(\bar{x} + rw)v - A^\dagger v\right]_{-1} = \left(2\sum_{k\geqslant m}[\tilde{a}_2^{(2)}]_k, 2\sum_{k\geqslant m}[\tilde{a}_2^{(4)}]_k\right)^T$, then

$$|\left[Df(\bar{x} + rw)v - A^\dagger v\right]_{-1}| \leq z_{-1}^{(1)}\mathbf{1}_2 \stackrel{\text{def}}{=} \left(\sum_{k=m}^{M-1}\frac{2}{k^s} + \frac{2}{(M-1)^{s-1}(s-1)}\right)\mathbf{1}_2.$$

Case $k = 0$: Let $[S_{111}]_k \stackrel{\text{def}}{=} [s_{111}]'_k - ([s_{111}]'_k)_F$, $[S_{113}]_k \stackrel{\text{def}}{=} [s_{113}]'_k - ([s_{113}]'_k)_F$ and $[S_{133}]_k \stackrel{\text{def}}{=}$

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$[s_{133}]'_k - ([s_{133}]'_k)_F$. Then,

$$\left[Df(\bar{x}+rw)v - A^\dagger v\right]_0 =$$

$$d\begin{pmatrix} -2\sum_{k\geqslant m}[\tilde{a}_2^{(2)}]_k \frac{(-1)^k}{k^2-1} \\ \kappa^2\left([S_{111}]_0 + [S_{133}]_0 - \frac{1}{2}\left([S_{111}]_1 + [S_{133}]_1\right) - 2\sum_{k\geqslant m}\left([S_{111}]_k + [S_{133}]_k - [\tilde{a}_2^{(1)}]_k\right)\frac{(-1)^k}{k^2-1}\right) \\ -2\sum_{k\geqslant m}[\tilde{a}_2^{(4)}]_k \frac{(-1)^k}{k^2-1} \\ [S_{113}]_0 - \frac{1}{2}[S_{113}]_1 - 2\sum_{k\geqslant m}[S_{113}]_k \frac{(-1)^k}{k^2-1} \end{pmatrix}.$$

In order to find the bounds $z_0^{(1)}$, $z_0^{(2)}$, and $z_0^{(3)}$ satisfying (3.2) for $k=0$, we need first to bound the terms $|[S_{111}]_k|$, $|[S_{113}]_k|$ and $|[S_{133}]_k|$. Using the analytic estimates of Lemma 3.1 and Lemma 3.2, we can compute upper bounds for $\left[\omega_I^{-s}|a^{(i)}||a^{(j)}|\right]_k$, $\left[\omega^{-s}\omega^{-s}|a^{(i)}|\right]_k$ and $\left[\omega^{-s}\omega^{-s}\omega^{-s}\right]_k$ to obtain

$$|[S_{111}]_k| \leqslant 3\left[\omega_I^{-s}|\bar{a}^{(1)}|^2\right]_k + 6\left[(\omega^{-s})^2|\bar{a}^{(1)}|\right]_k r + 3\left[(\omega^{-s})^3\right]_k r^2$$

$$\leqslant \begin{cases} 3\left[\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant|k_1|<M}} [\omega_I^{-s}]_{k_1}[|\bar{a}^{(1)}|]_{k_2}[|\bar{a}^{(1)}|]_{k_3} + 3(\bar{A}^{(1)})^2\varepsilon_k^{(3)}\right] \\ \quad +6\left[\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant|k_i|<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[|\bar{a}^{(1)}|]_{k_3} + 3\bar{A}^{(1)}\varepsilon_k^{(3)}\right] r + 3\frac{\alpha_M^{(3)}}{\omega_k^s}r^2, & 0\leqslant k<M \\ \frac{\alpha_M^{(3)}}{\omega_k^s}\left(3(\bar{A}^{(1)})^2 + 6\bar{A}^{(1)}r + 3r^2\right), & k\geqslant M, \end{cases}$$

$$|[S_{113}]_k| \leqslant \left[\omega_I^{-s}|\bar{a}^{(1)}|^2 + 2\omega_I^{-s}|\bar{a}^{(3)}||\bar{a}^{(1)}|\right]_k + \left[4(\omega^{-s})^2|\bar{a}^{(1)}| + 2(\omega^{-s})^2|\bar{a}^{(3)}|\right]_k r + 3\left[(\omega^{-s})^3\right]_k r^2$$

$$\leqslant \begin{cases} \sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant|k_1|<M}} [\omega_I^{-s}]_{k_1}[|\bar{a}^{(1)}|]_{k_2}[|\bar{a}^{(1)}|]_{k_3} + 3(\bar{A}^{(1)})^2\varepsilon_k^{(3)} \\ \quad +2\left(\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant|k_1|<M}} [\omega_I^{-s}]_{k_1}[|\bar{a}^{(1)}|]_{k_2}[|\bar{a}^{(3)}|]_{k_3} + 3\bar{A}^{(1)}\bar{A}^{(3)}\varepsilon_k^{(3)}\right) \\ \quad +\left[4\left(\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant|k_i|<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[|\bar{a}^{(1)}|]_{k_3} + 3\bar{A}^{(1)}\varepsilon_k^{(3)}\right)\right. \\ \quad \left. +2\left(\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant|k_i|<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[|\bar{a}^{(3)}|]_{k_3} + 3\bar{A}^{(3)}\varepsilon_k^{(3)}\right)\right] r + 3\frac{\alpha_M^{(3)}}{\omega_k^s}r^2, & 0\leqslant k<M \\ \frac{\alpha_M^{(3)}}{\omega_k^s}\left((\bar{A}^{(1)})^2 + 2\bar{A}^{(1)}\bar{A}^{(3)} + \left(4\bar{A}^{(1)} + 2\bar{A}^{(3)}\right)r + 3r^2\right), & k\geqslant M, \end{cases}$$

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$$|[S_{133}]_k| \leqslant \big[\omega_I^{-s}|\bar a^{(3)}|^2 + 2\omega_I^{-s}|\bar a^{(1)}||\bar a^{(3)}|\big]_k + \big[4(\omega^{-s})^2|\bar a^{(3)}| + 2(\omega^{-s})^2|\bar a^{(1)}|\big]_k r + 3\big[(\omega^{-s})^3\big]_k r^2$$

$$\leqslant \begin{cases} \displaystyle\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant|k_1|<M}} [\omega_I^{-s}]_{k_1}[|\bar a^{(3)}|]_{k_2}[|\bar a^{(3)}|]_{k_3} + 3(\bar A^{(3)})^2\varepsilon_k^{(3)} \\ \displaystyle +2\Bigg(\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant|k_1|<M}} [\omega_I^{-s}]_{k_1}[|\bar a^{(1)}|]_{k_2}[|\bar a^{(3)}|]_{k_3} + 3\bar A^{(1)}\bar A^{(3)}\varepsilon_k^{(3)}\Bigg) \\ \displaystyle +\Bigg[4\Bigg(\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant|k_i|<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[|\bar a^{(3)}|]_{k_3} + 3\bar A^{(3)}\varepsilon_k^{(3)}\Bigg) \\ \displaystyle +2\Bigg(\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant|k_i|<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[|\bar a^{(1)}|]_{k_3} + 3\bar A^{(1)}\varepsilon_k^{(3)}\Bigg)\Bigg] r + 3\frac{\alpha_M^{(3)}}{\omega_k^s}r^2, & 0\leqslant k<M \\ \displaystyle \frac{\alpha_M^{(3)}}{\omega_k^s}\big((\bar A^{(3)})^2 + 2\bar A^{(1)}\bar A^{(3)} + \big(4\bar A^{(3)} + 2\bar A^{(1)}\big)r + 3r^2\big), & k\geqslant M. \end{cases}$$

The finite sums appearing in the upper bounds of $[S_{111}]_k$, $[S_{113}]_k$ and $[S_{133}]_k$ when $k<M$ are computed using the FFT algorithm together with interval arithmetic. Defining $V_{111}, V_{113}, V_{133}$ as in Table 1 and $W_{111}, W_{113}, W_{133}$ as follows

$$\begin{aligned} &W_{111}^{(1)} = 3(\bar A^{(1)})^2, \quad W_{111}^{(2)} = 6\bar A^{(1)}, \quad W_{111}^{(3)} = 3 \\ &W_{113}^{(1)} = (\bar A^{(1)})^2 + 2\bar A^{(1)}\bar A^{(3)}, \quad W_{113}^{(2)} = 4\bar A^{(1)} + 2\bar A^{(3)}, \quad W_{113}^{(3)} = 3 \\ &W_{133}^{(1)} = (\bar A^{(3)})^2 + 2\bar A^{(1)}\bar A^{(3)}, \quad W_{133}^{(2)} = 4\bar A^{(3)} + 2\bar A^{(1)}, \quad W_{133}^{(3)} = 3, \end{aligned}$$

and collecting the coefficients of r, we obtain the upper bounds

$$\begin{aligned} |[S_{1ij}]_k| &\leqslant [V_{1ij}^{(1)}]_k + [V_{1ij}^{(2)}]_k r + [V_{1ij}^{(3)}]_k r^2, && 0\leqslant k<M, \\ |[S_{1ij}]_k| &\leqslant \frac{\alpha_M^{(3)}}{\omega_k^s}\Big(W_{1ij}^{(1)} + W_{1ij}^{(2)}r + W_{1ij}^{(3)}r^2\Big), && k\geqslant M. \end{aligned}$$

Finally, using the estimates

$$\begin{aligned} \left|\sum_{k=m}^{\infty}[S_{1ij}]_k\frac{(-1)^k}{k^2-1}\right| &\leqslant \sum_{k=m}^{M-1}\frac{|[S_{1ij}]_k|}{k^2-1} + \sum_{k\geqslant M}\frac{|[S_{1ij}]_k|}{k^2-1} \\ &\leqslant \sum_{k=m}^{M-1}\frac{(V_{1ij}^{(1)})_k + (V_{1ij}^{(2)})_k r + (V_{1ij}^{(3)})_k r^2}{k^2-1} \\ &\quad + \alpha_M^{(3)}\Big(W_{1ij}^{(1)} + W_{1ij}^{(2)}r + W_{1ij}^{(3)}r^2\Big)\sum_{k\geqslant M}\frac{1}{k^s(k^2-1)} \\ &\leqslant \sum_{k=m}^{M-1}\frac{(V_{1ij}^{(1)})_k + (V_{1ij}^{(2)})_k r + (V_{1ij}^{(3)})_k r^2}{k^2-1} \\ &\quad + \alpha_M^{(3)}\Big(W_{1ij}^{(1)} + W_{1ij}^{(2)}r + W_{1ij}^{(3)}r^2\Big)\frac{1}{(M^2-1)(M-1)^{s-1}(s-1)} \end{aligned}$$

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Table 1. *The formulas for* $V_{1ij}^{(\ell)}$ *for* $i, j \in \{1, 3\}$ *and for* $\ell = 1, 2, 3$

$V_{111}^{(1)}$	$3\left[\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant\lvert k_1\rvert<M}} [\omega_I^{-s}]_{k_1}[\lvert\bar a^{(1)}\rvert]_{k_2}[\lvert\bar a^{(1)}\rvert]_{k_3} + 3(\bar A^{(1)})^2\varepsilon_k^{(3)}\right]$
$V_{111}^{(2)}$	$6\left[\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant\lvert k_i\rvert<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[\lvert\bar a^{(1)}\rvert]_{k_3} + 3\bar A^{(1)}\varepsilon_k^{(3)}\right]$
$V_{111}^{(3)}$	$3\frac{\alpha_M^{(3)}}{\omega_k^s}$
$V_{113}^{(1)}$	$\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant\lvert k_1\rvert<M}} [\omega_I^{-s}]_{k_1}[\lvert\bar a^{(1)}\rvert]_{k_2}[\lvert\bar a^{(1)}\rvert]_{k_3} + 3(\bar A^{(1)})^2\varepsilon_k^{(3)} + 2\left[\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant\lvert k_1\rvert<M}} [\omega_I^{-s}]_{k_1}[\lvert\bar a^{(1)}\rvert]_{k_2}[\lvert\bar a^{(3)}\rvert]_{k_3} + 3\bar A^{(1)}\bar A^{(3)}\varepsilon_k^{(3)}\right]$
$V_{113}^{(2)}$	$4\left[\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant\lvert k_i\rvert<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[\lvert\bar a^{(1)}\rvert]_{k_3} + 3\bar A^{(1)}\varepsilon_k^{(3)}\right] + 2\left[\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant\lvert k_i\rvert<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[\lvert\bar a^{(3)}\rvert]_{k_3} + 3\bar A^{(3)}\varepsilon_k^{(3)}\right]$
$V_{113}^{(3)}$	$3\frac{\alpha_M^{(3)}}{\omega_k^s}$
$V_{133}^{(1)}$	$\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant\lvert k_1\rvert<M}} [\omega_I^{-s}]_{k_1}[\lvert\bar a^{(3)}\rvert]_{k_2}[\lvert\bar a^{(3)}\rvert]_{k_3} + 3(\bar A^{(3)})^2\varepsilon_k^{(3)} + 2\left[\sum_{\substack{k_1+k_2+k_3=k\\ m\leqslant\lvert k_1\rvert<M}} [\omega_I^{-s}]_{k_1}[\lvert\bar a^{(1)}\rvert]_{k_2}[\lvert\bar a^{(3)}\rvert]_{k_3} + 3\bar A^{(1)}\bar A^{(3)}\varepsilon_k^{(3)}\right]$
$V_{133}^{(2)}$	$4\left[\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant\lvert k_i\rvert<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[\lvert\bar a^{(3)}\rvert]_{k_3} + 3\bar A^{(3)}\varepsilon_k^{(3)}\right] + 2\left[\sum_{\substack{k_1+k_2+k_3=k\\ 0\leqslant\lvert k_i\rvert<M}} [\omega^{-s}]_{k_1}[\omega^{-s}]_{k_2}[\lvert\bar a^{(1)}\rvert]_{k_3} + 3\bar A^{(1)}\varepsilon_k^{(3)}\right]$
$V_{133}^{(3)}$	$3\frac{\alpha_M^{(3)}}{\omega_k^s}$

and

$$\left|\sum_{k\geqslant m}[\tilde a_i^{(j)}]_k\frac{(-1)^k}{k^2-1}\right| \leqslant \sum_{k\geqslant m}\frac{1}{k^s(k^2-1)} \leqslant \sum_{k=m}^{M-1}\frac{1}{k^s(k^2-1)} + \frac{1}{(M^2-1)(M-1)^{s-1}(s-1)},$$

we obtain the bounds $z_0^{(1)}$, $z_0^{(2)}$, and $z_0^{(3)}$ satisfying (3.2) for $k=0$. The formulas for $z_0^{(1)}$, $z_0^{(2)}$, and $z_0^{(3)}$ are given in Table A 1 in Appendix A.

<u>Cases $0 < k < m-1$</u>: Similarly as in the case for $k=0$, one gets that

$$\left[Df(\bar x + rw)v - A^\dagger v\right]_k = d\begin{pmatrix} 0 \\ \kappa^2\Big([S_{111}]_{k-1} + [S_{133}]_{k-1} - [S_{111}]_{k+1} - [S_{133}]_{k+1}\Big) \\ 0 \\ [S_{113}]_{k-1} - [S_{113}]_{k+1} \end{pmatrix},$$

and $\lvert\left[Df(\bar x + rw)vr - A^\dagger vr\right]_k\rvert \le z_k^{(1)}r + z_k^{(2)}r^2 + z_k^{(3)}r^3$, where the formulas for $z_k^{(1)}$, $z_k^{(2)}$, and $z_k^{(3)}$ are given in Table A 1 in Appendix A.

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Cases $k = m-1$, $k = m$

$$\left[Df(\bar{x}+rw)v - A^{\dagger}v\right]_k = \\ d\begin{pmatrix} -[\tilde{a}_2^{(2)}]_{k+1} \\ \kappa^2\left([S_{111}]_{k-1} + [S_{133}]_{k-1} - [\tilde{a}_2^{(1)}]_{k-1} - [S_{111}]_{k+1} - [S_{133}]_{k+1} + [\tilde{a}_2^{(1)}]_{k+1}\right) \\ -[\tilde{a}_2^{(4)}]_{k+1} \\ [S_{113}]_{k-1} - [S_{113}]_{k+1} \end{pmatrix}$$

and $|\left[Df(\bar{x}+rw)vr - A^{\dagger}vr\right]_k| \preceq z_k^{(1)}r + z_k^{(2)}r^2 + z_k^{(3)}r^3$, where the formulas for $z_k^{(1)}$, $z_k^{(2)}$, and $z_k^{(3)}$ are given in Table A 1 in Appendix A.

Cases $m < k \leqslant M$

$$\left[Df(\bar{x}+rw)v - A^{\dagger}v\right]_k = \\ d\begin{pmatrix} [\tilde{a}_2^{(2)}]_{k-1} - [\tilde{a}_2^{(2)}]_{k+1} \\ \kappa^2\left([S_{111}]_{k-1} + [S_{133}]_{k-1} - [\tilde{a}_2^{(1)}]_{k-1} - [S_{111}]_{k+1} - [S_{133}]_{k+1} + [\tilde{a}_2^{(1)}]_{k+1}\right) \\ [\tilde{a}_2^{(4)}]_{k-1} - [\tilde{a}_2^{(4)}]_{k+1} \\ [S_{113}]_{k-1} - [S_{113}]_{k+1} \end{pmatrix}$$

and $|\left[Df(\bar{x}+rw)vr - A^{\dagger}vr\right]_k| \leqslant z_k^{(1)}r + z_k^{(2)}r^2 + z_k^{(3)}r^3$, where the formulas for $z_k^{(1)}$, $z_k^{(2)}$, and $z_k^{(3)}$ are given in Table A 1 in Appendix A.

We can finally combine all the above bounds and let

$$\begin{aligned} Z_F^{(i)} &\stackrel{\text{def}}{=} |A_m|\, z_F^{(i)}, \text{ for } i = \{1,2,3\} \\ Z_k^{(i)} &\stackrel{\text{def}}{=} \frac{1}{2k} z_k^{(i)}, \text{ for } i = \{1,2,3\}, k \geqslant m \\ Z_F^{(0)} &\stackrel{\text{def}}{=} \left|I - A_m Df^{(m)}(\bar{x}_F)\right| \tilde{\omega}_F^{-s} \\ Z_k^{(0)} &\stackrel{\text{def}}{=} 0 \text{ for } k \geqslant m, \end{aligned}$$

where $\tilde{\omega}_F^{-s} \stackrel{\text{def}}{=} ((\omega_k^{-s})^{\zeta(k)})_{k\geqslant -1}$, with $(\omega_k^{-s})^{\zeta(k)} = \overbrace{(\omega_k^{-s}, \ldots, \omega_k^{-s})}^{\zeta(k)\ \text{times}}$. We can finally define, for $k = -1, \ldots, M$,

$$Z_k(r) \stackrel{\text{def}}{=} \left(Z_k^{(0)} + Z_k^{(1)}\right) r + Z_k^{(2)} r^2 + Z_k^{(3)} r^3. \tag{3.4}$$

For $k > M$, we need the tail radii polynomial (2.19) to ensure that $Y_k + Z_k(r) - \frac{r}{k^s}\mathbf{1}_4 = \frac{1}{2k}(z_k^{(1)}r + z_k^{(2)}r^2 + z_k^{(3)}r^3) - \frac{r}{k^s}\mathbf{1}_4 \prec 0$ for all $k \geqslant M+1$. Using Lemma 3.1, consider asymptotic bounds $\tilde{z}_{M+1}^{(i)}$ such that $z_k^{(1)} \preceq \frac{\tilde{z}_{M+1}^{(i)}}{k^s}$ for all $k > M$ and for $i = 1,2,3$. The bounds can be

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found at the end of Table A 1. Hence, for all $k > M$, one has that

$$\begin{aligned} Z_k(r) &= \frac{1}{2k}\left(z_k^{(1)} r + z_k^{(2)} r^2 + z_k^{(3)} r^3\right) \\ &\leq \frac{1}{2k}\left(\frac{\tilde{z}_{M+1}^{(1)}}{k^s} r + \frac{\tilde{z}_{M+1}^{(2)}}{k^s} r^2 + \frac{\tilde{z}_{M+1}^{(3)}}{k^s} r^3\right) \\ &\leq \frac{1}{k^s}\left(\frac{\tilde{z}_{M+1}^{(1)}}{2(M+1)} r + \frac{\tilde{z}_{M+1}^{(2)}}{2(M+1)} r^2 + \frac{\tilde{z}_{M+1}^{(3)}}{2(M+1)} r^3\right). \end{aligned}$$

Hence, we set

$$\bar{Z}_{M+1}(r) \stackrel{\text{def}}{=} \frac{\tilde{z}_{M+1}^{(3)}}{2(M+1)} r^3 + \frac{\tilde{z}_{M+1}^{(2)}}{2(M+1)} r^2 + \frac{\tilde{z}_{M+1}^{(1)}}{2(M+1)} r. \tag{3.5}$$

Combining the bounds (3.1), (3.4) and (3.5), we have the radii polynomials

$$p_k(r) = \begin{cases} Y_k + Z_k(r) - \frac{r}{\omega_k^s} \mathbf{1}_{\zeta(k)}, & -1 \leqslant k \leqslant M, \\ \bar{Z}_{M+1}(r) - r\mathbf{1}_4, & k = M+1. \end{cases} \tag{3.6}$$

We are now ready to present the proofs of the theorems in Section 1.

4 Proofs of the theorems

The computer-assisted proof of each theorem is done using MATLAB and the package IntLab [31]. The proofs can be reproduced by running the program $G\_L\_PROOF\_INTVAL$ for each of the four partitions considered. As mentioned previously, the idea of the proofs is to construct the radii polynomials (3.6), verify the hypothesis of Theorem 2.1, and use Lemma 2.1 to conclude that $u(t) = (u_1, u_2, u_3, u_4) = (\phi, \phi', \psi, \psi') = a_0 + 2\sum_{k\geqslant 1} a_k T_k(t)$ is a solution of the Ginzburg-Landau boundary value problem (1.2). All codes can be found at [33]. The data of the global diagram for each case were computed separately and are provided for the proof to be reproduced. The main program loads these data such as the approximate solution and the values of each parameter (κ, d, h_e). It then computes the bounds Y, Z satisfying (3.1), (3.4), (3.5), and determines the positive interval on which the radii polynomials are negative. The solution is in a ball centred around the approximate solution. Then, the program attests that all solutions ϕ are distinct by verifying rigorously that each value of $\phi(d)$ is distinct. Each proof is done in the space X^2, that is in the space of sequences decaying at least as fast as k^{-2}. For each proof, the computational parameter M has been chosen large enough for the proof to be successful.

Proof of Theorem 1.1 The proof can be reproduced by running the program $G\_L\_PROOF\_INTVAL$ from the folder $S3A2$ available at [33]. In this case $M = 6m - 1$

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and the value of m for each solution is given by the following table

$\mathscr{S}_3 \cap \mathscr{A}_2$	$x_s^{(1)}$	$x_s^{(2)}$	$x_s^{(3)}$	$x_a^{(1)}$	$x_a^{(2)}$	$x_a^{(3)}$	$x_a^{(4)}$
m	260	210	330	280	190	190	280

.

□

Proof of Theorem 1.2 The proof can be reproduced by running the program $G\_L\_PROOF\_INTVAL$ from the folder $S1A2$ available at [33]. In this case $M = 4m - 1$ and the value of m for each solution is given by the following table.

$\mathscr{S}_1 \cap \mathscr{A}_2$	$x_s^{(1)}$	$x_a^{(1)}$	$x_a^{(2)}$	$x_a^{(3)}$	$x_a^{(4)}$
m	190	150	180	180	150

.

□

Proof of Theorem 1.3 The proof can be reproduced by running the program $G\_L\_PROOF\_INTVAL$ from the folder $S2A1$ available at [33]. In this case $M = 5m - 1$ and the value of m for each solution is given by the following table.

$\mathscr{S}_2 \cap \mathscr{A}_1$	$x_s^{(1)}$	$x_s^{(2)}$	$x_a^{(1)}$	$x_a^{(2)}$
m	250	80	100	100

.

□

Proof of Theorem 1.4 The proof can be reproduced by running the program $G\_L\_PROOF\_INTVAL$ from the folder $S2A2$ available at [33]. In this case $M = 5m - 1$ and the value of m for each solution is given by the following table.

$\mathscr{S}_2 \cap \mathscr{A}_2$	$x_s^{(1)}$	$x_s^{(2)}$	$x_a^{(1)}$	$x_a^{(2)}$	$x_a^{(3)}$	$x_a^{(4)}$
m	150	70	90	120	90	120

.

□

5 Conclusion

In this paper, we introduced a rigorous computational method using Chebyshev series to compute solutions of the Euler-Lagrange equations for the one-dimensional Ginzburg-Landau model of superconductivity. Our approach used analytic estimates, the radii polynomials and the contraction mapping theorem to show existence of solutions near numerical approximations. Coexistence of as many as seven nontrivial solutions was proved. This result is new and prior to this paper has been open for more than thirty years.

Finally, let us briefly mention that none of the apparent bifurcations that appeared in our diagrams have been proved rigorously. However, we believe that the method introduced in [32] could be applied to prove that the bifurcations are there, especially since many bifurcations seem to involve the breaking of some symmetry.

Appendix A Estimates and bounds

This appendix provides the necessary convolution estimates required and the final bounds z_k to construct the radii polynomials constructed in Section 3. All proofs can be found in [23,28]. Consider a decay rate $s \geqslant 2$, a computational parameter $M \geqslant 6$ and define, for $k \geqslant 3$,

$$\gamma_k = \gamma_k(s) \stackrel{\text{def}}{=} 2\left[\frac{k}{k-1}\right]^s + \left[\frac{4\ln(k-2)}{k} + \frac{\pi^2-6}{3}\right]\left[\frac{2}{k}+\frac{1}{2}\right]^{s-2}. \tag{A 1}$$

Lemma A.1 *For $s \geqslant 2$ and $k \geqslant 4$ we have*

$$\sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s(k-k_1)^s} \leqslant \gamma_k.$$

Lemma A.2 (Quadratic estimates) *Given a decay rate $s \geqslant 2$ and $M \geqslant 6$. For $k \in \mathbb{Z}$, define the quadratic asymptotic estimates $\alpha_k^{(2)} = \alpha_k^{(2)}(s, M)$ by*

$$\alpha_k^{(2)} \stackrel{\text{def}}{=} \begin{cases} 1 + 2\sum_{k_1=1}^{M} \dfrac{1}{\omega_{k_1}^{2s}} + \dfrac{2}{M^{2s-1}(2s-1)}, & \text{for } k = 0 \\ \sum_{k_1=1}^{M} \dfrac{2\omega_k^s}{\omega_{k_1}^s \omega_{k+k_1}^s} + \dfrac{2\omega_k^s}{(k+M+1)^s M^{s-1}(s-1)} & \\ \qquad\qquad + 2 + \sum_{k_1=1}^{k-1} \dfrac{\omega_k^s}{\omega_{k_1}^s \omega_{k-k_1}^s}, & \text{for } 1 \leqslant k \leqslant M-1 \\ 2 + 2\sum_{k_1=1}^{M} \dfrac{1}{\omega_{k_1}^s} + \dfrac{2}{M^{s-1}(s-1)} + \gamma_M, & \text{for } k \geqslant M, \end{cases} \tag{A 2}$$

and for $k < 0$,

$$\alpha_k^{(2)} \stackrel{\text{def}}{=} \alpha_{|k|}^{(2)}.$$

Then, for any $k \in \mathbb{Z}$,

$$\sum_{\substack{k_1+k_2=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s} \leqslant \frac{\alpha_k^{(2)}}{\omega_k^s}.$$

Lemma A.3 *For any $k \in \mathbb{Z}$ with $|k| \geqslant M \geqslant 6$, we have that $\alpha_k^{(2)} \leqslant \alpha_M^{(2)}$.*

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Lemma A.4 (**Cubic estimates**) *Given $s \geqslant 2$ and $M \geqslant 6$. Let*

$$\Sigma_a^* \stackrel{\text{def}}{=} \sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(2)} M^s}{\omega_{k_1}^s \left(M-k_1\right)^s} + \alpha_M^{(2)} \left(\gamma_M - \sum_{k_1=1}^{M-1} \frac{1}{\omega_{k_1}^s} \right),$$

$\tilde{\alpha}_M^{(2)} \stackrel{\text{def}}{=} \max\{\alpha_k^{(2)} \mid k = 0,\dots,M\}$, $\Sigma_b^* \stackrel{\text{def}}{=} \tilde{\alpha}_M^{(2)} \gamma_M$ *and* $\Sigma^* \stackrel{\text{def}}{=} \min\{\Sigma_a^*, \Sigma_b^*\}$. *Define the cubic asymptotic estimates* $\alpha_k^{(3)} = \alpha_k^{(3)}(s, M)$ *by*

$$\alpha_k^{(3)} \stackrel{\text{def}}{=} \begin{cases} \alpha_0^{(2)} + 2 \displaystyle\sum_{k_1=1}^{M-1} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^{2s}} + \frac{2\alpha_M^{(2)}}{(M-1)^{2s-1}(2s-1)}, & \text{for } k = 0 \\ \displaystyle\sum_{k_1=1}^{M-k} \frac{\alpha_{k+k_1}^{(2)} \omega_k^s}{\omega_{k_1}^s \omega_{k+k_1}^s} + \frac{\alpha_M^{(2)} \omega_k^s}{(M+1)^s (M-k)^{s-1} (s-1)} + \sum_{k_1=1}^{k-1} \frac{\alpha_{k_1}^{(2)} \omega_k^s}{\omega_{k_1}^s \omega_{k-k_1}^s} & \\ \quad + \displaystyle\sum_{k_1=1}^{M} \frac{\alpha_{k_1}^{(2)} \omega_k^s}{\omega_{k_1}^s \omega_{k+k_1}^s} + \frac{\alpha_M^{(2)} \omega_k^s}{(M+k+1)^s M^{s-1} (s-1)} + \alpha_k^{(2)} + \alpha_0^{(2)}, & \\ & \text{for } 1 \leqslant k \leqslant M-1 \\ \alpha_M^{(2)} \displaystyle\sum_{k_1=1}^{M} \frac{1}{\omega_{k_1}^s} + \frac{2\alpha_M^{(2)}}{M^{s-1}(s-1)} + \Sigma^* + \sum_{k_1=1}^{M} \frac{\alpha_{k_1}^{(2)}}{\omega_{k_1}^s} + \alpha_M^{(2)} + \alpha_0^{(2)}, & \text{for } k \geqslant M \end{cases} \tag{A 3}$$

and for $k < 0$,

$$\alpha_k^{(3)} \stackrel{\text{def}}{=} \alpha_{|k|}^{(3)}.$$

Then, for any $k \in \mathbb{Z}$,

$$\sum_{\substack{k_1+k_2+k_3=k \\ k_j \in \mathbb{Z}}} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s \omega_{k_3}^s} \leqslant \frac{\alpha_k^{(3)}}{\omega_k^s}.$$

Moreover, $\alpha_k^{(3)} \leqslant \alpha_M^{(3)}$, for all $k \geqslant M$.

Lemma A.5 *For any $k \in \mathbb{Z}$ with $|k| \geqslant M \geqslant 6$, we have that $\alpha_k^{(3)} \leqslant \alpha_M^{(3)}$.*

Lemma A.6 *Given $s \geqslant 2$ and $M \geqslant 6$, define for $0 \leqslant k \leqslant M-1$*

$$\begin{aligned} \varepsilon_k^{(3)} = \varepsilon_k^{(3)}(s, M) \stackrel{\text{def}}{=} & \sum_{k_1=M}^{M-k} \frac{\alpha_{k+k_1}^{(2)}}{\omega_{k_1}^s \omega_{k+k_1}^s} \\ & + \sum_{k_1=M}^{M+k} \frac{\alpha_{k_1-k}^{(2)}}{\omega_{k_1}^s \omega_{k_1-k}^s} + \frac{\alpha_M^{(2)}}{(M+1)^s (s-1)} \left[\frac{1}{(M-k)^{s-1}} + \frac{1}{(M+k)^{s-1}} \right] \end{aligned} \tag{A 4}$$

and for $k < 0$

$$\varepsilon_k^{(3)}(s, M) \stackrel{\text{def}}{=} \varepsilon_{|k|}^{(3)}(s, M).$$

Fix $0 \leqslant |k| \leqslant M-1$ and $\ell \in \{1, 2, 3\}$. Then, we have that

$$\sum_{\substack{k_1+k_2+k_3=k \\ \max\{|k_1|,\dots,|k_\ell|\} \geqslant M}} \frac{1}{\omega_{k_1}^s \omega_{k_2}^s \omega_{k_3}^s} \leqslant \ell \varepsilon_k^{(3)}.$$

Table A 1. *Formulas for* z_k

	$k=-1$
$z_{-1}^{(1)}$	$\left(\sum_{k=m}^{M-1}\frac{2}{k^s}+\frac{2}{(M-1)^{s-1}(s-1)}\right)\mathbf{1}_2$
$z_{-1}^{(2)}$	0
$z_{-1}^{(3)}$	0
	$k=0$
$z_0^{(1)}$	$d\begin{pmatrix} 2\sum_{k=m}^{M-1}\frac{1}{k^s(k^2-1)}+\frac{2}{(M^2-1)(M-1)^{s-1}(s-1)} \\ \kappa^2\Big([V_{111}^{(1)}]_0+[V_{133}^{(1)}]_0+\frac{[V_{111}^{(1)}]_1+[V_{133}^{(1)}]_1}{2}+2\sum_{k=m}^{M-1}\frac{[V_{111}^{(1)}]_k+[V_{133}^{(1)}]_k}{k^2-1} \\ +\frac{2\alpha_M^{(3)}\left(W_{111}^{(1)}+W_{133}^{(1)}\right)}{(M^2-1)(M-1)^{s-1}(s-1)}+\frac{1}{(M^2+2M)M^{s-1}(s-1)}\Big) \\ 2\sum_{k=m}^{M-1}\frac{1}{k^s(k^2-1)}+\frac{2}{(M^2-1)(M-1)^{s-1}(s-1)} \\ [V_{113}^{(1)}]_0+\frac{[V_{113}^{(1)}]_1}{2}+2\sum_{k=m}^{M-1}\frac{[V_{113}^{(1)}]_k}{k^2-1}+\frac{2\alpha_M^{(3)}W_{113}^{(1)}}{(M^2-1)(M-1)^{s-1}(s-1)} \end{pmatrix}$
$z_0^{(2)}$	$d\begin{pmatrix} 0 \\ \kappa^2\Big([V_{111}^{(2)}]_0+[V_{133}^{(2)}]_0+\frac{[V_{111}^{(2)}]_1+[V_{133}^{(2)}]_1}{2} \\ +2\sum_{k=m}^{M-1}\frac{[V_{111}^{(2)}]_k+[V_{133}^{(2)}]_k}{k^2-1}+\frac{2\alpha_M^{(3)}\left(W_{111}^{(2)}+W_{133}^{(2)}\right)}{(M^2-1)(M-1)^{s-1}(s-1)}\Big) \\ 0 \\ [V_{113}^{(2)}]_0+\frac{[V_{113}^{(2)}]_1}{2}+2\sum_{k=m}^{M-1}\frac{[V_{113}^{(2)}]_k}{k^2-1}+\frac{2\alpha_M^{(3)}W_{113}^{(2)}}{(M^2-1)(M-1)^{s-1}(s-1)} \end{pmatrix}$
$z_0^{(3)}$	$d\begin{pmatrix} 0 \\ \kappa^2\Big([V_{111}^{(3)}]_0+[V_{133}^{(3)}]_0+\frac{[V_{111}^{(3)}]_1+[V_{133}^{(3)}]_1}{2} \\ +2\sum_{k=m}^{M-1}\frac{[V_{111}^{(3)}]_k+[V_{133}^{(3)}]_k}{k^2-1}+\frac{2\alpha_M^{(3)}\left(W_{111}^{(3)}+W_{133}^{(3)}\right)}{(M^2-1)(M-1)^{s-1}(s-1)}\Big) \\ 0 \\ [V_{113}^{(3)}]_0+\frac{[V_{113}^{(3)}]_1}{2}+2\sum_{k=m}^{M-1}\frac{[V_{113}^{(3)}]_k}{k^2-1}+\frac{2\alpha_M^{(3)}W_{113}^{(3)}}{(M^2-1)(M-1)^{s-1}(s-1)} \end{pmatrix}$
	$0<k<m-1$
$z_k^{(1)}$	$d\begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(1)}]_{k-1}+[V_{133}^{(1)}]_{k-1}+[V_{111}^{(1)}]_{k+1}+[V_{133}^{(1)}]_{k+1}\right) \\ 0 \\ [V_{113}^{(1)}]_{k-1}+[V_{113}^{(1)}]_{k+1} \end{pmatrix}$
$z_k^{(2)}$	$d\begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(2)}]_{k-1}+[V_{133}^{(2)}]_{k-1}+[V_{111}^{(2)}]_{k+1}+[V_{133}^{(2)}]_{k+1}\right) \\ 0 \\ [V_{113}^{(2)}]_{k-1}+[V_{113}^{(2)}]_{k+1} \end{pmatrix}$
$z_k^{(3)}$	$d\begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(3)}]_{k-1}+[V_{133}^{(3)}]_{k-1}+[V_{111}^{(3)}]_{k+1}+[V_{133}^{(3)}]_{k+1}\right) \\ 0 \\ [V_{113}^{(3)}]_{k-1}+[V_{113}^{(3)}]_{k+1} \end{pmatrix}$

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Table A 1. *Continued*

$k = m-1,\ k = m$

$$z_k^{(1)} \quad d \begin{pmatrix} \frac{1}{(k+1)^s} \\ \kappa^2\left([V_{111}^{(1)}]_{k-1} + [V_{133}^{(1)}]_{k-1} + [V_{111}^{(1)}]_{k+1} + [V_{133}^{(1)}]_{k+1} + \frac{1}{(k+1)^s}\right) \\ \frac{1}{(k+1)^s} \\ [V_{113}^{(1)}]_{k-1} + [V_{113}^{(1)}]_{k+1} \end{pmatrix}$$

$$z_k^{(2)} \quad d \begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(2)}]_{k-1} + [V_{133}^{(2)}]_{k-1} + [V_{111}^{(2)}]_{k+1} + [V_{133}^{(2)}]_{k+1}\right) \\ 0 \\ [V_{113}^{(2)}]_{k-1} + [V_{113}^{(2)}]_{k+1} \end{pmatrix}$$

$$z_k^{(3)} \quad d \begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(3)}]_{k-1} + [V_{133}^{(3)}]_{k-1} + [V_{111}^{(3)}]_{k+1} + [V_{133}^{(3)}]_{k+1}\right) \\ 0 \\ [V_{113}^{(3)}]_{k-1} + [V_{113}^{(3)}]_{k+1} \end{pmatrix}$$

$m < k < M-1$

$$z_k^{(1)} \quad d \begin{pmatrix} \frac{1}{(k-1)^s} + \frac{1}{(k+1)^s} \\ \kappa^2\left([V_{111}^{(1)}]_{k-1} + [V_{133}^{(1)}]_{k-1} + [V_{111}^{(1)}]_{k+1} + [V_{133}^{(1)}]_{k+1} + \frac{1}{(k-1)^s} + \frac{1}{(k+1)^s}\right) \\ \frac{1}{(k-1)^s} + \frac{1}{(k+1)^s} \\ [V_{113}^{(1)}]_{k-1} + [V_{113}^{(1)}]_{k+1} \end{pmatrix}$$

$$z_k^{(2)} \quad d \begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(2)}]_{k-1} + [V_{133}^{(2)}]_{k-1} + [V_{111}^{(2)}]_{k+1} + [V_{133}^{(2)}]_{k+1}\right) \\ 0 \\ [V_{113}^{(2)}]_{k-1} + [V_{113}^{(2)}]_{k+1} \end{pmatrix}$$

$$z_k^{(3)} \quad d \begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(3)}]_{k-1} + [V_{133}^{(3)}]_{k-1} + [V_{111}^{(3)}]_{k+1} + [V_{133}^{(3)}]_{k+1}\right) \\ 0 \\ [V_{113}^{(3)}]_{k-1} + [V_{113}^{(3)}]_{k+1} \end{pmatrix}$$

$k = M-1,\ k = M$

$$z_k^{(1)} \quad d \begin{pmatrix} \frac{1}{(k-1)^s} + \frac{1}{(k+1)^s} \\ \kappa^2\left([V_{111}^{(1)}]_{k-1} + [V_{133}^{(1)}]_{k-1} + \frac{1}{(k-1)^s} + \frac{\alpha_M^{(3)}}{(k+1)^s}\left(W_{111}^{(1)} + W_{133}^{(1)}\right) + \frac{1}{(k+1)^s}\right) \\ \frac{1}{(k-1)^s} + \frac{1}{(k+1)^s} \\ [V_{113}^{(1)}]_{k-1} + \frac{\alpha_M^{(3)}}{(k+1)^s} W_{113}^{(1)} \end{pmatrix}$$

$$z_k^{(2)} \quad d \begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(2)}]_{k-1} + [V_{133}^{(2)}]_{k-1} + \frac{\alpha_M^{(3)}}{(k+1)^s}\left(W_{111}^{(2)} + W_{133}^{(2)}\right)\right) \\ 0 \\ [V_{113}^{(2)}]_{k-1} + \frac{\alpha_M^{(3)}}{(k+1)^s} W_{113}^{(2)} \end{pmatrix}$$

$$z_k^{(3)} \quad d \begin{pmatrix} 0 \\ \kappa^2\left([V_{111}^{(3)}]_{k-1} + [V_{133}^{(3)}]_{k-1} + \frac{\alpha_M^{(3)}}{(k+1)^s}\left(W_{111}^{(3)} + W_{133}^{(3)}\right)\right) \\ 0 \\ [V_{113}^{(3)}]_{k-1} + \frac{\alpha_M^{(3)}}{(k+1)^s} W_{113}^{(3)} \end{pmatrix}$$

$k > M$

$$\tilde{z}_{M+1}^{(1)} \, d \begin{pmatrix} \left(\frac{M+1}{M}\right)^s + \left(\frac{M+1}{M+2}\right)^s \\ \kappa^2\left[2\alpha_M^{(3)}\left(W_{111}^{(1)} + W_{133}^{(1)}\right) + \left(\frac{M+1}{M}\right)^s + \left(\frac{M+1}{M+2}\right)^s\right] \\ \left(\frac{M+1}{M}\right)^s + \left(\frac{M+1}{M+2}\right)^s \\ 2\alpha_M^{(3)} W_{113}^{(1)} \end{pmatrix}$$

$$\tilde{z}_{M+1}^{(2)} \, d \begin{pmatrix} 0 \\ 2\kappa^2\alpha_M^{(3)}\left(W_{111}^{(2)} + W_{133}^{(2)}\right) \\ 0 \\ 2\alpha_M^{(3)} W_{113}^{(2)} \end{pmatrix}$$

$$\tilde{z}_{M+1}^{(3)} \, d \begin{pmatrix} 0 \\ 2\kappa^2\alpha_M^{(3)}\left(W_{111}^{(3)} + W_{133}^{(3)}\right) \\ 0 \\ 2\alpha_M^{(3)} W_{113}^{(3)} \end{pmatrix}$$

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