ON A CONJECTURE OF G. HAJÓS

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(Received 6 March, 1973)

1. Introduction. The purpose of this note is to provide by means of an example a negative answer to a conjecture of Hajós [3] concerning the factorization of finite abelian groups. This question is also raised as Problem 81 in Fuchs [2].

If S, T are subsets of an additive abelian group G their sum $S+T$ is said to be *direct* if $s_1+t_1 = s_2+t_2$ implies $s_1 = s_2$, $t_1 = t_2$, where $s_i \in S$, $t_i \in T$. If the sum is direct and $S+T = G$, then we have a factorization of G. All sums considered in this note are direct. A subset S of G is said to be periodic if there exists $h \in G$, $h \neq 0$, with $S+h = S$. If $H = \{h \in G \mid S+h = S\}$, then H is a subgroup of G and we have $S = H+S_1$ for some subset S_1. When Hajós discovered that neither factor in a factorization of certain finite abelian groups G need be periodic he asked the following weaker question. Is every factorization $G = S+T$ of a finite abelian group G *quasi-periodic* in the sense that one factor, say T, is a disjoint union of subsets T_i $(1 \leqq i \leqq m,\ m > 1)$, such that there is a subgroup H of G of order m with $S+T_1 = S+T_i+h_i$, where $H = \{h_i \mid 1 \leqq i \leqq m\}$? Clearly, if T is periodic, the factorization is quasi-periodic with the set of periods of T, including 0, as the subgroup H.

2. Example. We give the following example of a non-quasi-periodic factorization. The construction is provided by a special case of a technique of de Bruijn [1], despite the closing remark of that paper.

Let p be a prime, $p > 3$. Let G be the direct sum of cyclic groups of orders p^2 and p. Let a and b of orders p^2 and p generate G. We take

$$S = \{0, pa+2b, 2pa+b, 3(pa+b), 4(pa+b), \ldots, (p-1)(pa+b)\},$$
$$T = V \cup W,$$

where

$$V = \{0, pa, 2pa, \ldots, (p-1)pa\},$$
$$W = \{0, b, 2b, \ldots, (p-1)b\} + \{a, 2a, \ldots, (p-1)a\}.$$

Then $G = S+T$, as is easily verified, and neither S nor T is periodic. This is essentially de Bruijn's construction of Theorem 2 of [1]. His notation is multiplicative and we have also multiplied his first factor by st in order to put the identity into it, before changing to additive notation, replacing s by pa and t by b and using the particular set of coset representatives $0, a, \ldots, (p-1)a$, for $c_1, \ldots, c_m$.

If the factorization is quasi-periodic, one factor will split as a disjoint union of m subsets of equal order, $m > 1$. These subsets can have order one only if one factor is periodic. Since neither S nor T is periodic and S has prime order we see that the only possibility is that T splits as a union of p subsets each of order p. Let such a splitting occur and let $S+T_1 = S+T_i+h_i$. Then the subgroup H has order p. Hence H must be contained in the subgroup K generated

https://doi.org/10.1017/S0017089500002202 Published online by Cambridge University Press

by pa and b. The sum $S+H$ is direct and so has order p^2. Now S, $H \subset K$. It follows that $S+H = K$. From $S+T = G$ we have

$$G = S+(\bigcup T_i) = \bigcup(S+T_i) = S+T_1+H = S+T_i+h_i+H = S+T_i+H = K+T_i.$$

Hence each set T_i must be a set of coset representatives for G modulo K. Therefore each set T_i contains one element from V and one element from $\{0, b, \ldots, (p-1)b\}+ra$, for each r such that $1 \leqq r \leqq p-1$. Let $x_1 pa, y_1 b+a \in T_1$ and $x_2 pa, y_2 b+a \in T_2$. Then $S+T_1 = S+T_2+h_2$ implies that

$$(S+T_1) \cap K = (S+T_2+h_2) \cap K.$$

Therefore $S+x_1 pa = S+x_2 pa+h_2$. Since S is not periodic, we have $h_2 = (x_1-x_2)pa$. Similarly $(S+T_1) \cap (K+a) = (S+T_2+h_2) \cap (K+a)$ implies that $S+y_1 b+a = S+y_2 b+a+h_2$. Thus $h_2 = (y_1-y_2)b$. This gives $(x_1-x_2)pa = (y_1-y_2)b$. As G is a direct sum of the subgroups generated by a and b it follows that $x_1 pa = x_2 pa$. This is impossible as T_1 and T_2 have empty intersection. Therefore the factorization $G = S+T$ is not quasi-periodic.

3. Other related conjectures. Under certain conditions a factorization must be quasi-periodic. For example, let us assume that the factor S is contained in a proper subgroup K of G such that G is the direct sum of K and a subgroup H. Then letting $T_i = T \cap (K+h_i)$ for each $h_i \in H$, from $S+T = G$ and $S \subset K$ we find that $S+T_i = K+h_i$. If H is listed so that $h_1 = 0$, then $S+T_1 = K$ and so $S+T_i = S+T_1+h_i$ and the factorization is quasi-periodic. As we have seen, it need not be the case that such subgroups K and H exist. However the following weaker question is still open:

"If G is a nonzero additive finite abelian group and $G = S+T$, where $0 \in S$, $0 \in T$, must one of the factors be contained in some proper subgroup K of G?"

There is another open question, which is weaker than the quasi-periodicity conjecture. If the factorization $G = S+T$ is quasi-periodic, as above, then $G = S+T_1+H$ and T has been replaced by the periodic factor T_1+H. So we have the question as to whether it is always possible to replace one factor by a periodic factor. This question has already been suggested, in a letter to Fuchs, when a counterexample to problem 77 of **[2]** was given (see **[5]**), and is quoted by Fuchs in **[4]**, p. 364. Thus this question is a possible replacement for both Problems 77 and 81 of **[2]**.

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