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ATTACHING GRAPHS TO PSEUDO-SIMILAR VERTICES

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Abstract

Vertices u and v of a graph G are pseudo-similar if $G - u \cong G - v$, but no automorphisms of G maps u to v. Let H be a graph with a distinguished vertex a. Denote by $G(u.H)$ and $G(v.H)$ the graphs obtained from G and H by identifying vertex a of H with pseudo-similar vertices u and v, respectively, of G. Is it possible for $G(u.H)$ and $G(v.H)$ to be isomorphic graphs? We answer this question in the affirmative by constructing graphs G for which $G(u.H) \cong G(v.H)$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 C 25, 05 C 60, 20 B 15.

1. Introduction

At the recent 13th southeastern conference on Combinatorics, Graph Theory, and Computing, B. D. McKay asked the following question, due originally to E. Farrell.

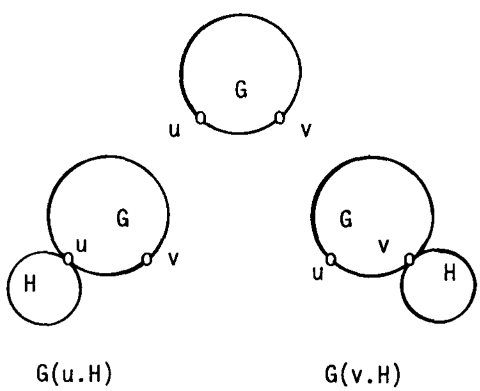

FIGURE 1.1

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Let G be a graph with pseudo-similar vertices u and v, and let H be a graph with a distinguished vertex a. Two graphs $G(u.H)$ and $G(v.H)$ are formed from G and H by identifying vertex a of H with u and v, respectively. Is it possible for $G(u.H)$ and $G(v.H)$ to be isomorphic graphs? This is illustrated in Figure 1.1.

We answer this question in the affirmative, by constructing graphs G for which $G(u.H) \cong G(v.H)$.

2. 2-connected graphs

2.1 DEFINITION. Let G be a graph with vertex set $V(G)$. Vertices $u, v \in V(G)$ are *pseudo-similar* if $G - u \cong G - v$, but there is no automorphism of G mapping u to v.

Let u and v be pseudo-similar vertices in G, and let a graph H be given, with a distinguished vertex a. We denote by $G(u.H)$ and $G(v.H)$ the graphs obtained from G and H by identifying vertex a of H with vertices u and v, respectively. We say we have *attached* the graph H to G at u (or v).

It would seem that if $G(u.H) \cong G(v.H)$, then this isomorphism would induce an automorphism of G mapping u to v. Indeed we have the following.

2.2 THEOREM. *Let G, u, v, and H be as above. If G is 2-connected, then $G(u.H) \not\cong G(v.H)$.*

PROOF. Since G is 2-connected, u and v are cut-vertices of $G(u.H)$ and $G(v.H)$, respectively, and G is an end-block of both these graphs. Let p: $G(u.H) \to G(v.H)$ be an isomorphism. Notice that G is a subgraph of both $G(u.H)$ and $G(v.H)$.

The image $p(G)$ of G is an end-block of $G(v.H)$ isomorphic to G. The image $p(u)$ of u is a cut-vertex of $p(G)$. If in fact $p(G) = G$, then $p(u) = v$, since v is the only cut-vertex of $G(v.H)$ in G. This induces an automorphism of G mapping u to v, a contradiction. Therefore $p(G) \neq G$. It follows that $p(G)$ is an end-block of H isomorphic to G, and that $p(u)$ is an isomorphic image of u in $p(G)$.

Therefore $p(G)$ is an end-block of $G(u.H)$. Consider $p^2(G)$. Again, $p^2(G)$ is an end-block of $G(v.H)$. Either $p^2(G) = G$ or we can find $p^3(G)$, etc. We continue like this until we have $p^k(G) = G$ for some positive integer k. Thus must occur since G and H are finite graphs. But then $p^k(u) = v$, and this induces an automorphism of G mapping u to v, a contradiction.

Thus if G is 2-connected, it is impossible for $G(u.H)$ and $G(v.H)$ to be isomorphic. If G is separable, though, the situation is different.

We construct several graphs G with pseudo-similar vertices u and v, so that $G(u.H) \cong G(v.H)$. The construction requires a graph with a given permutation group acting on a subset of the vertices. The methods of either Bouwer ([3], [4]) or Babai [1] will construct such a graph. The method we use is based on Bouwer's method, but is a considerable simplification of it. We give a brief description of it in the next section.

3. Bouwer's method

Let P be a permutation group acting on a set X. We construct a graph Z such that $X \subseteq V(Z)$, the automorphism group of Z is abstractly isomorphic to P, and furthermore the restriction of Aut Z to X is equal to P.

Given P, form a Cayley colour-graph Y for P and label the vertices of Y with the elements of P. Y will have coloured and directed edges. Let P have orbits $X_1, X_2, \ldots, X_n$ on X. Then P is a subdirect product of transitive permutation groups acting on the sets $X_1, X_2, \ldots, X_n$ (see [7]).

Choose a representative $x_1 \in X_1$. Denote by Stab(x_1) the stabilizer subgroup of x_1 in P. Now Aut Y is abstractly isomorphic to P. Moreover, every representation of Aut Y by a transitive permutation group is isomorphic to a representation by cosets of some subgroup (see [7]). Thus if we join x_1 to Stab$(x_1) \subset V(Y)$, and then join the remaining vertices of X_1 to their respective cosets of Stab (x_1) in P, the action of Aut Y on the resulting graph will induce a permutation group P_1 acting on X_1. P_1 will be equal to the transitive constituent of P on X_1, since the representation by cosets of a point-stabilizer subgroup is always faithful (see [7]). Whether right or left cosets are used will depend on how the Cayley colour-graph is constructed.

If we now do the same for the remaining orbits $X_2, X_3, \ldots, X_n$, then Aut Y will induce on X a permutation group exactly equal to P.

This is the essence of Bouwer's method.

We now alter the coloured and directed edges of Y, replacing them with "gadgets" (see [2]), using Frucht's method, so that no new automorphisms are introduced.

Finally, to ensure that the only automorphisms of the resultant graph are those arising from P, we adjust the degrees of the vertices, if necessary (by adding "tails", say, as in [3]), to distinguish the set X from the remaining vertices of the graph. Call the resultant graph Z. It has the required properties.

4. The main construction

Let a graph H with a distinguished vertex a be given.

Let A_4 denote the alternating group acting on $X = \{u, v, w, x\}$. Use the method of Section 3 to construct a graph Z such that $X \subseteq V(Z)$ and the

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restriction of Aut Z to X is equal to A_4. Form $Z(x.H)$ by attaching H to x. Let $G = Z(x.H) - w$. This is illustrated in Figure 4.1.

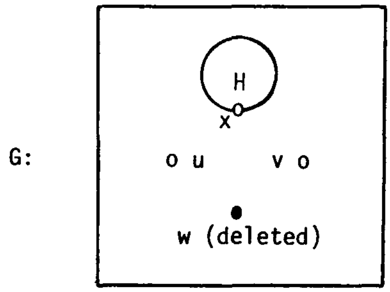

FIGURE 4.1

To ensure that the only automorphisms of G are those arising from A_4, it may be necessary to adjust the construction of Section 3 somewhat, since attaching H and deleting w may conceivably introduce new automorphisms. This can be done by adding "tails" to some of the vertices to distinguish them by their degree.

4.1 THEOREM. *Vertices u and v are pseudo-similar in G.*

PROOF. Notice that $(x)(uvw) \in A_4$. This permutation takes $\{v, w\}$ to $\{u, w\}$. Therefore $G - u \cong G - v$. However u and v are not similar in G; for the only possible automorphism mapping u to v would be $(x)(w)(uv)$, which is not in A_4. Therefore u and v are pseudo-similar.

Now form $G(u.H)$ and $G(v.H)$, as illustrated in Figure 4.2.

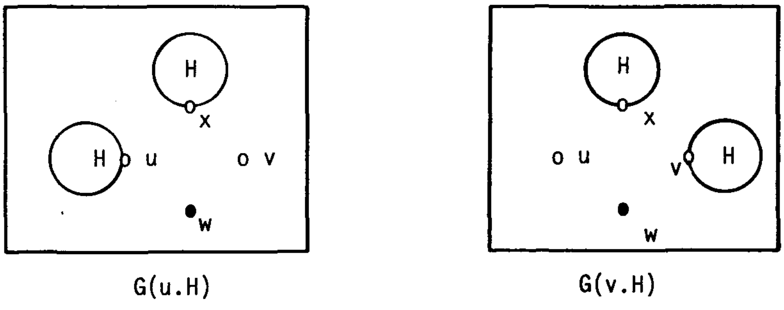

FIGURE 4.2

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4.2 THEOREM. $G(u.H) \cong G(v.H)$.

PROOF. The permutation $(uxy)(w) \in A_4$ maps $G(u.H)$ to $G(v.H)$.

5. Further examples

The graph arising from the alternating group A_4 is not the only graph with this property. We demonstrate a second example.

Let $\theta = (1234)(56)$ and $\phi = (1456)(23)$ be permutations acting on $X = \{1,2,3,4,5,6\}$. Let $P = \langle \theta, \phi \rangle$. The order of P is 36. (The author found the CAMAC [10] group theory computer program useful for these calculations.)

Let Z be the graph formed in Section 3, so that Aut $Z \cong P$ and $(\text{Aut } Z)_X = P$, and let H be given.

We form two graphs G and G' from Z and H. Let $G = Z(3.H, 4.H) - 1$, that is, we attach copies of H to vertices 3 and 4 of $X \subseteq V(Z)$, and delete vertex 1. Similarly, let $G' = Z(5.H, 6.H) - \{1, 2\}$. Graphs G and G' are illustrated in Figure 5.1.

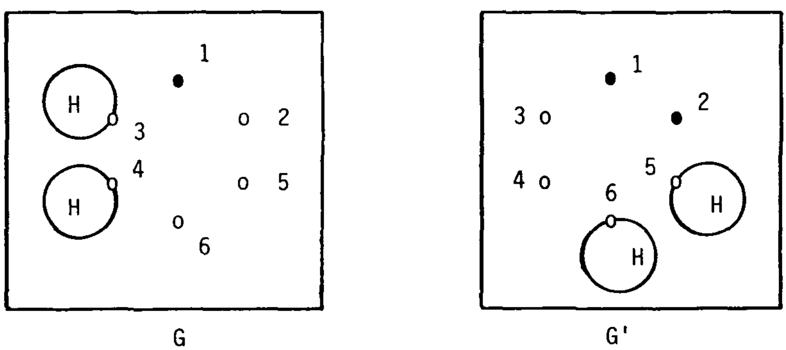

FIGURE 5.1

Again we adjust the construction of Section 3 if necessary so that the only automorphisms of G and G' are those arising from P. As in Section 4, the properties of the group P give the following results.

5.1 THEOREM. *Vertices* 2 *and* 6 *are pseudo-similar in* G, *but* $G(2.H) \cong G(6.H)$.

5.2 THEOREM. *Vertices* 3 *and* 4 *are pseudo-similar in* G', *but* $G'(3.H) \cong G'(4.H)$.

Pseudo-similar vertices are studied in some detail in [5], [6], [8], and [9]. Interest in them arose from attempts to settle the reconstruction conjecture.

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