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COUNTEREXAMPLE TO A CONJECTURE ON POSITIVE DEFINITE FUNCTIONS

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1. Introduction. Cooper [**1**] called a complex-valued function f on the real line $\mathbf{R}$ *positive definite for* F, where F is a set of complex-valued functions on $\mathbf{R}$, if the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\phi(x)\overline{\phi(y)}dxdy$$

exists as a Lebesgue integral and is nonnegative for every ϕ in F. Let us denote by $P(F)$ the set of functions positive definite for F, by L_c^p the set of functions in $L^p(\mathbf{R})$ with compact support, and by $L^p{}_{\text{loc}}$ the set of functions which are locally in $L^p(\mathbf{R})$, i.e., $f \in L^p{}_{\text{loc}} \Leftrightarrow f \in L^p(K)$ for every compact subset K of $\mathbf{R}$. Cooper showed that $P(L_c^p) = P(L_c^2)$ for any $p \geqq 2$ and that each function in $P(L_c^1)$ is essentially bounded and hence equal almost everywhere to an ordinary continuous positive definite function in the sense of Bochner. For $1 < p < 2$ he showed that if $q = p/2(p-1)$ then

$$P(L_c^2) \cap L^q{}_{\text{loc}} \subset P(L_c^p) \tag{1}$$

and he conjectured that equality holds in (1). In § 2 we give a counterexample to this conjecture, i.e., we construct a function $f \in P(L_c^p)$ which is not in $L^q{}_{\text{loc}}$.

The definition of $P(F)$ makes sense on any locally compact abelian group G and the inclusion (1) also holds in that situation (see [**4**, Theorem 2.3] and [**5**, Theorem 2.2]). If G is not a discrete group it seems unlikely that equality holds but the only group, apart from $\mathbf{R}$, for which we have discovered a counterexample is $G = T$, the circle group, and this appears in § 3.

In constructing the examples we shall need to use some of the theory of the Lorentz spaces $L(p, q)$, and we list here the relevant facts. If f is a measurable function defined on the measure space (X, μ) we define

$$m(f, y) = \mu\{x \in X : |f(x)| > y\} \text{ and } f^*(x) = \inf\{y > 0 : m(f, y) \leqq x\}.$$

For $1 \leqq p < \infty$, $1 \leqq q < \infty$, we let

$$L(p, q) = \left\{f : \int_0^{\infty} [x^{1/p} f^*(x)]^q x^{-1} dx < \infty\right\}.$$

It is well known that $L(p, p) = L^p$, that $L(p, q_1) \subset L(p, q_2)$ for $q_1 \leqq q_2$, and that

$$\int_X |f(x)g(x)|d\mu(x) \leqq \int_0^{\mu(X)} f^*(t)g^*(t)dt. \tag{2}$$

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It was proved by O'Neil [**3**, Theorem 2.6] that if $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$, where $1/p_1 + 1/p_2 > 1$, then the convolution $f * g \in L(r, s)$ where $1/r = 1/p_1 + 1/p_2 - 1$ and $1/s \leqq 1/q_1 + 1/q_2$.

2. The counterexample for $G = \mathbf{R}$. Let $1 < p < 2$, $q = p/2(p-1)$, and $s = q^{-1}$. If $\rho(t) = t^s$ for $t \geqq 0$ and $\rho(t) = 0$ for $t < 0$, we define

$$f(x) = \int_{-\infty}^{\infty} e^{ixt} d\rho(t) = s \int_0^{\infty} e^{ixt} t^{s-1} dt$$
$$= s\Gamma(s) e^{i\pi s/2} x^{-s} \qquad \text{if } x > 0.$$

If $c = s\Gamma(s)e^{i\pi s/2}$ then

$$f(x) = \begin{cases} cx^{-s} & \text{if } x > 0 \\ \bar{c}(-x)^{-s} & \text{if } x < 0. \end{cases}$$

Clearly $f \notin L^q{}_{\text{loc}}$. We now show that the integral

$$(3) \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\phi(x)\overline{\phi(y)} dx dy = \int_{-\infty}^{\infty} f(x)\phi * \tilde{\phi}(x) dx$$

exists for all $\phi \in L_c^p(\tilde{\phi}(x) = \overline{\phi(-x)})$. Since ϕ and $\tilde{\phi}$ are in $L^p = L(p, p)$ we have, using the theorem of O'Neil,

$$g \equiv \phi * \tilde{\phi} \in L(r, 1) \quad \text{where} \quad r = p/(2-p) = q',$$

and so

$$\int_0^{\infty} x^{-s} g^*(x) dx = \int_0^{\infty} x^{1/q'-1} g^*(x) dx < \infty.$$

The decreasing rearrangement of f is easily calculated to be

$$f^*(x) = |c|(x/2)^{-s} = Kx^{-s}, \text{ say.}$$

Using (2) we have

$$\int_{-\infty}^{\infty} |f(x)g(x)| dx \leqq \int_0^{\infty} f^*(x) g^*(x) dx = K \int_0^{\infty} x^{-s} g^*(x) dx < \infty.$$

Thus the integral (3) exists.

In order to show that it is non-negative we can consider f as a distribution which is the Fourier transform of the distribution corresponding to the locally summable function $\Psi(t) = st^{s-1}(t > 0)$, $\Psi(t) = 0(t < 0)$. (See [**2**, § 2.3]). Thus we can write, for every $\phi \in C_c^{\infty}$,

$$\int f(x)\phi(x) dx = \int \hat{\phi}(t) d\rho(t)$$

where $\hat{\phi}$ is the Fourier transform of ϕ. Hence

$$\int f(x)\phi * \tilde{\phi}(x) dx = \int |\hat{\phi}(t)|^2 d\rho(t) \geqq 0 \qquad (\phi \in C_c^{\infty}).$$

If $\phi \in L_c^p$ we can find functions $\phi_n \in C_c^\infty$ such that $\phi_n(x) \to \phi(x)$ and $|\phi_n(x)| \leqq |\phi(x)|$. Now $|\phi_n * \tilde{\phi}_n| \leqq |\phi| * |\tilde{\phi}|$, and since $|\phi| \in L_c^p$ we have $\int f(x)(|\phi| * |\tilde{\phi}|)(x)dx < \infty$. Therefore by the dominated convergence theorem we have $\int f(x)\phi * \tilde{\phi}(x)dx \geqq 0$, i.e., $f \in P(L_c^p)$.

3. The counterexample for $G = \mathbf{T}$. By analogy with the example of the previous section we define

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{inx}}{n^\alpha} \left(\alpha = 1 - \frac{1}{q}\right).$$

In view of the formula

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n^\alpha} = i\Gamma(1-\alpha)e^{-i\pi\alpha/2}x^{\alpha-1} + O(1) \quad (0 < x \leqq \pi)$$

(see [**6**, vol. 1, p. 70]) we can write

$$f(x) = \Gamma(q^{-1})e^{i\pi/2q}x^{-1/q} + h(x) = F(x) + h(x)$$

where $h \in L^\infty(T)$. Thus $f \notin L^q(T)$.

As in § 2 we let $\phi \in L^p(T)$ and then $g = \phi * \tilde{\phi} \in L(q', 1)$. Write

$$\int_0^{2\pi} |fg|dx \leqq \int_0^{2\pi} |Fg|dx + \int_0^{2\pi} |hg|dx.$$

Since $h \in L^\infty(T)$ and $g \in L^1(T)$ the second integral exists, whereas

$$\int_0^{2\pi} |Fg|dx \leqq \int_0^{2\pi} F^*(x)g^*(x)dx = K\int_0^{2\pi} x^{-1/q}g^*(x)dx < \infty.$$

Thus the integral

$$\int_0^{2\pi} f(x)\,\phi*\tilde{\phi}(x)dx$$

exists.

To show that it is nonnegative we consider the functions

$$f_N(x) = \sum_{n=1}^{N} \frac{e^{inx}}{n^\alpha}.$$

Since this is a finite trigonometric series with positive coefficients, each f_N is an ordinary positive definite function, and therefore

$$\int_0^{2\pi} f_N(x)\,\phi*\tilde{\phi}(x)dx \geqq 0 \text{ for every } \phi \in L^p.$$

We have $f_N(x) \to f(x)$ and also

$$|f_N(x)| \leqq C_\alpha x^{\alpha-1}$$

where C_α is a constant which depends on α but not on N (see [**6**, vol. I, p. 191]).

Since $\int C_\alpha x^{\alpha-1}\phi * \tilde{\phi}(x)dx < \infty$, we can use the dominated convergence theorem to show that

$$\int_0^{2\pi} f(x)\,\phi*\tilde{\phi}(x)dx \geqq 0.$$

We have thus shown that $f \in P(L^p)$.

References

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