ABSTRACT EVOLUTION EQUATIONS OF PARABOLIC TYPE IN BANACH AND HILBERT SPACES

TOSIO KATO

Introduction

1. The object of the present paper is to prove some theorems concerning the existence and the uniqueness of the solution of the initial value problem for the *evolution equation*

$$(E) \qquad du/dt + A(t)u = f(t), \qquad 0 \leq t \leq T.$$

Here the unknown $u = u(t)$ as well as the inhomogeneous term $f(t)$ is a function on the closed interval $[0, T]$ to a Banach space $\mathfrak{X}$, whereas $A(t)$ is a function on $[0, T]$ to the set of (in general unbounded) linear operators acting in $\mathfrak{X}$. Before stating the results to be proved in the present paper, it is convenient to give a brief survey of the results so far obtained on this "abstract Cauchy or mixed problem".

In this survey we restrict ourselves to abstract theories, disregarding results obtained primarily for the case in which $A(t)$ is a concrete *differential* operator. Nor shall we mention the results for the case in which $A(t)$ does not depend on t; such a case belongs properly to the theory of generating one-parameter semigroups of operators (Hille-Yosida theory).

With these reservations, the first author to discuss an equation of the form (E) was Phillips [25]; he assumes that the main part A of $A(t)$ is independent of t, $-A$ being the infinitesimal generator of a strongly continuous, bounded semigroup $\exp(-tA)$, while the variable part of $A(t)$ belongs to $\mathbf{B}[\mathfrak{X}]$.[1] A more general case was considered by the present author [3]. Here the essential assumptions are that

1) $-A(t)$ is for each t the infinitesimal generator of a contraction semigroup, and

Received April 20, 1961.

[1] We denote by $\mathbf{B}[\mathfrak{X}]$ the set of all bounded linear operators with domain $\mathfrak{X}$ and range in $\mathfrak{X}$.

2) the domain $\mathfrak{D}[A(t)] = \mathfrak{D}$ of $A(t)$ is independent of t,

with some auxiliary assumptions implying the smoothness of $A(t)$ and $f(t)$ as functions of t. One of the merits of this result is that it is applicable to parabolic differential equations (such as the heat equation), to the Schrödinger equation (in which $iA(t)$ is selfadjoint) and to certain kinds of hyperbolic equations (written in the form of a first order system). But it was felt that the assumptions 1), 2) were still too restrictive, and an attempt was made (Kato [4]) to remove these restrictions, with a not very satisfactory result. See also Mizohata [21] for the relaxation of the assumption 1).

Recently an essential improvement on 1) was attained by a series of papers by Tanabe [31, 32, 33]. Here a new assumption is introduced that

3) $-A(t)$ is for each t the infinitesimal generator of an *analytic* semigroup. (That is, $\exp(-sA(t))$ has an analytic continuation to a sector including the positive s-axis.) But this new assumption is compensated for by the following relaxation of other assumptions: the semigroup $\exp(-sA(t))$ need not be contraction (even for $s>0$), and the smoothness assumption on $A(t)$ as function of t can be weakened considerably. Owing to the new assumption 3), the Tanabe theory is not applicable to the Schrödinger equation or to hyperbolic differential equations, but it is more powerful than the result of [3] when applied to parabolic equations. In particular, the initial value $u(0)$ is allowed to be an arbitrary element of $\mathfrak{X}$, whereas $u(0) \in \mathfrak{D}$ was assumed in [3]. On the basis of the Tanabe theory, Komatsu [7] was able to prove that the solution $u(t)$ of (E) is *analytic* in t if $A(t)$ depends on t analytically in a certain sense and if $f(t)$ is analytic. This result of Komatsu has an interesting application on the unique continuation property of solutions of parabolic differential equations (see also Yosida [38]).

The condition 2) that $\mathfrak{D}[A(t)]$ be independent of t is retained by Tanabe and Komatsu. This assumption was weakened to a great extent by Sobolevskii [29]. Here $A(t)$ are restricted to be positive definite, selfadjoint operators in a Hilbert space, a very strong assumption, but 2) is replaced by the assumption that

4) $\mathfrak{D}[A(t)^h]$ is independent of t for some constant h such that $0<h\leq 1$.

As long as $A(t)$ is positive selfadjoint, this condition is weaker than 2) in virtue of the Heinz inequality. One of the objects of the present paper is to generalize the result of Sobolevskii to the case in which $A(t)$ operate in a Banach space.

2. The works described above are concerned with *strict solutions* of (E), in the sense that the solution $u(t)$ constructed satisfies (E) everywhere in the semi-open interval $(0, T]$. More precisely, $u(t)$ belongs to $\mathfrak{D}[A(t)]$ for each $t \in (0, T]$, has a strong derivative $du(t)/dt \in \mathfrak{X}$ which is strongly continuous in $(0, T]$ and (E) holds true. On the other hand, there are a large number of works which are aimed at obtaining *generalized solutions* of (E).

Višik and Ladyženskaia [13, 14, 15, 34, 35] write (E) in the form

$$Su = f, \qquad Su(t) = du(t)/dt + A(t)\,u(t),$$

and consider the problem as an operator equation in a larger space $\tilde{\mathfrak{X}}$ consisting of functions $u(t)$ on $[0, T]$ to $\mathfrak{X}$. (Actually they assume $\mathfrak{X}$ to be a Hilbert space and $\tilde{\mathfrak{X}}$ is the set of $u(t)$ with integrable $\|u(t)\|^2$; as regards $A(t)$, they assume that the dominant part of $A(t)$ is positive selfadjoint.) To this end they extend S to a closed operator and prove that this closed extension has a bounded inverse in an appropriate topology.[2] The solutions obtained by them are generalized or weak solutions of (E). The degree of weakness denends on the assumptions made; they may be "almost everywhere" solutions, satisfying (E) for almost every $t \in (0, T]$; they may be weak or "hyperweak" solutions. "Almost everywhere" solutions could be regarded as not much different from strict solutions, but it is interesting to note that, in constructing such solutions, these authors also assume that $\mathfrak{D}[A(t)]$ is independent of t. Weak and hyperweak solutions are constructed under weaker assumptions, but it is not clear how near they are to strict solutions.[3]

Generalized solutions of (E) have also been considered extensively by Lions [16, 17, 18, 19, 20]. In principle the method of Lions is similar to that of Višik

[2] Their method depends essentially on the "energy principle." For applications to hyperbolic differential equations and to the Schrödinger equation, they also consider the abstract equations of the forms $d^2u/dt^2 + A(t)u = f(t)$ and $du/dt + iA(t)u = 0$.

[3] It may be said that the distinction between strict and generalized solutions of (E) is not very important if one is exclusively concerned with the application of (E) to linear partial differential equations, where a weak solution can often be proved to be a strict solution. Since, however, we are primarily interested in the abstract equation (E), whose application is not restricted to partial differential equations, it seems rather important to investigate under what conditions (E) has strict solutions.

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

and Ladyženskaia, but Lions start from a *sesquilinear form* $\phi_t[u, v]$ in a Hilbert space $\mathfrak{X}$, to which $A(t)$ is formally related by $(A(t)u, v) = \phi_t[u, v]$. The basic assumption in his earlier work is that the domain of ϕ_t is independent of t; if $A(t)$ is positive selfadjoint, this is equivalent to that $\mathfrak{D}[A(t)^{1/2}]$ is independent of t and, therefore, weaker than the assumption 2) stated above but is a special case of the assumption 4) of Sobolevskii. In a recent paper of Lions [18] this assumption is eliminated, and the existence and the uniqueness of a generalized solution (almost everywhere solution) are proved under very general conditions. In most of these papers, however, the initial value $u(0)$ is subjected to more or less severe restrictions.[4)]

3. In the present paper we want to deduce several results that comprise and strengthen most of the results stated above, except those of [3] and [18]. More precisely, we shall prove the existence and the uniqueness of a strict solution to the initial value problem for (E) for any initial value $u(0) \in \mathfrak{X}$, under the assumptions that

i) for each t, $-A(t)$ is the infinitesimal generator of an analytic semigroup of operators in a Banach space

(as in Tanabe's theory) and

ii) $\mathfrak{D}[A(t)^h]$ is independent of t for some $h = 1/m$ where m is a positive integer

(as in Sobolevskii's theory, though h was arbitrary in the latter theory), with certain additional smoothness assumptions on $A(t)$ and $f(t)$ as functions of t.

Thus our results generalize the results of both Tanabe and Sobolevskii. Furthermore, it will be shown that the solution of (E) is analytic in t if, in addition to i) and ii), $f(t)$ and $A(t)^{-h}$ are analytic in t. This generalizes the result of Komatsu stated above.

It should be noted, however, that these results may not necessarily be stronger for smaller values of h, for the Heinz inequality has not been proved for non-selfadjoint operators. Furthermore, the question arises *how the assumption ii) can be verified in applications.* In fact, the fractional power A^h of an operator A is defined in an abstract fashion, which will be quite complicated

4) For other results on the equation (E), see Foiaş *et al.* [1, 2], Krasnosel'skii *et al.* [8, 9, 10], Krein [11], Krein and Sobolevskii [12], Mlak [22, 23, 24], Sobolevskii [26, 27, 28], Solomiak [30].

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

when applied to, say, a differential operator. At present the author has no answer to this question that is general enough to be useful in the case of Banach spaces, but it can be shown that the operator function $A(t)$ defined in terms of a sesquilinear form $\phi_t[u, v]$ with constant domain (as in Lions' theory) satisfies the condition ii) with $h = 1/3$. In this way we are able to prove the existence and uniqueness of the strict solution of (E) in this case. By making use of the uniqueness of the generalized solution, it can then be shown that the generalized solutions in the earlier works of Lions, and those of Visik and Ladyženskaia, are actually strict solutions, provided the inhomogeneous term $f(t)$ is smooth, at least for certain initial values $u(0) \in \mathfrak{X}$.

Furthermore, it can be shown that $A(t)^{-1/3}$ is analytic in t if $\phi_t[u, u]$ is analytic in t for each u of the constant domain of ϕ_t. It follows that the solution of (E) is even analytic if ϕ_t is analytic in the stated sense and if $f(t)$ is also analytic.

The specific results for the case in which $A(t)$ is defined in terms of the sesquilinear form ϕ_t depend on the results of a separate paper of the author [6], in which the fractional powers of dissipative operators in a Hilbert space are studied in detail.

4. The content of the present paper is as follows. In §1 we state the assumptions and the main theorems (Theorems I and II), together with some inequalities required in the following sections. The proof of these inequalities is given in Appendix at the end of the paper. As a preliminary step in the proof of the main theorems, we consider in §2 the special case in which $A(t)$ belongs to $\mathbf{B}[\mathfrak{X}]$; the evolution operator $U(t, s)$, which plays a central role in the theorems, is constructed and a number of estimates on it are deduced. The general case is treated in §3, where the evolution operator $U(t, s)$ is constructed as the limit of a sequence $U_n(t, s)$ corresponding to bounded $A_n(t)$ that approximates $A(t)$ in an appropriate sense, and where the proof of Theorems I, II is completed. In §4 the main theorems are applied to the case in which $A(t)$ is defined through a sesquilinear form ϕ_t in the way described above, the results being contained in Theorems III and IV. Here an essential use is made of the results of [6]. The final section §5 is devoted to some remarks and examples, with discussions on possible (and impossible) extensions of the results of this paper.

§ 1. Assumptions and results

In what follows we consider the evolution equation (E) in a Banach space $\mathfrak{X}$. Our main purpose is to construct the *evolution operator* (or *fundamental solution*) $U(t, s)$, defined for $0 \leq s \leq t \leq T$, such that the solution of (E) can be expressed in the form

$$u(t) = U(t, 0)u(0) + \int_0^t U(t, s)f(s)ds. \tag{1.1}$$

To this end we make two assumptions on $A(t)$. Roughly speaking, the first assumption requires that $-A(t)$ be the infinitesimal generator of an analytic semigroup of operators, and the second that $A(t)$ change smoothly with t. More precisely,

i) *For each* $t \in [0, T]$, $A(t)$ *is a densely defined, closed linear operator in* $\mathfrak{X}$ *with its spectrum contained in a fixed sector* S_0: $|\arg z| < \theta < \pi/2$. *The resolvent of* $A(t)$ *satisfies the inequality*

$$\|[z - A(t)]^{-1}\| \leq M_0/|z| \qquad \textit{for } z \notin S_\theta, \tag{1.2}$$

where M_0 *is a constant independent of* t. *Furthermore*, $z = 0$ *also belongs to the resolvent set of* $A(t)$ *and*

$$\|A(t)^{-1}\| \leq M_1, \tag{1.3}$$

M_1 *being independent of* t.

As is well known (see, for example, Yosida [36]), (1.2) implies that $-A(t)$ generates a semigroup $\exp(-sA(t))$ which is holomorphic in a sector containing the positive s-axis. When (1.2) is satisfied, the two conditions (1.2) and (1.3) are satisfied if $A(t)+1$ is replaced by $A(t)$ (possibly with a different M_0). Therefore, (1.3) is not an essential assumption as long as one is concerned with the evolution equation (E); we assume it only for convenience.

Furthermore, (1.2) implies that the fractional powers $A(t)^h$ can be defined and have a similar property as $A(t)$, with θ replaced by $h\theta$ (see Appendix). Our second assumption is concerned with such a fractional power:

ii) *For some* $h = 1/m$, *where* m *is a positive integer*, $\mathfrak{D}[A(t)^h] = \mathfrak{D}$ *is independent of* t, *and there are constants* k, M_2 *and* M_3 *such that*

$$\|A(t)^h A(s)^{-h}\| \leq M_2, \qquad 0 \leqq t \leqq T, \tag{1.4}$$

$$\|A(t)^h A(s)^{-h} - 1\| \leq M_3 |t - s|^k, \qquad 0 \leqq s \leqq T, \tag{1.5}$$

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

$$1-h<k\leq 1. \tag{1.6}$$

As is well known, the independence of $\mathfrak{D}[A(t)^h]$ of t implies that $A(t)^h A(s)^{-h}\in \mathbf{B}[\mathfrak{X}]$; (1.4) means that it is uniformly bounded. Note also that (1.4) and (1.5) are equivalent to that $A(t)^h A(0)^{-h}$ is uniformly bounded and is Hölder continuous in norm with the exponent k.

Our main theorem now reads

THEOREM I. *Let the conditions i) and ii) be satisfied. Then there exists a unique evolution operator $U(t, s)\in \mathbf{B}[\mathfrak{X}]$ defined for $0\leq s\leq t\leq T$, with the following properties. $U(t, s)$ is strongly continuous for $0\leq s\leq t\leq T$ and*

$$U(t, r)=U(t, s)U(s, r), \qquad r\leq s\leq t, \tag{1.7}$$

$$U(t, t)=1. \tag{1.8}$$

For $s<t$, the range of $U(t, s)$ is a subset of $\mathfrak{D}[A(t)]$ and

$$A(t)U(t, s)\in \mathbf{B}[\mathfrak{X}], \qquad \|A(t)U(t, s)\|\leq M|t-s|^{-1}, \tag{1.9}$$

where M is a constant depending only on θ, h, k, T, M_0, M_1, M_2 and M_3. Furthermore, $U(t, s)$ is strongly continuously differentiable in t for $t>s$ and

$$\partial U(t, s)/\partial t+A(t)U(t, s)=0. \tag{1.10}$$

If $u\in\mathfrak{D}$, $U(t, s)u$ is strongly continuously differentiable in s for $s<t$. If in particular $u\in \mathfrak{D}[A(s_0)]$, then

$$(\partial U(t, s)u/\partial s)_{s=s_0}=U(t, s_0)A(s_0)u. \tag{1.11}$$

If $f(t)$ is continuous in t, any strict solution of (E) must be expressible in the form (1.1). Conversely, the $u(t)$ given by (1.1) is a strict solution of (E) if $f(t)$ is Hölder continuous on $[0, T]$; here $u(0)$ may be an arbitrary element of $\mathfrak{X}$.

Here we mean by a *strict solution* of (E) a function $u(t)$ such that $u(t)$ is strongly continuous on $[0, T]$, strongly continuously differentiable in $(0, T]$ and (E) is satisfied for $t\in(0, T]$.

We have also a number of estimates concerning $U(t, s)$, which will not be listed here but which can easily be read off the proof of the theorem given below.

Our second theorem is concerned with the case in which $A(t)$ is analytic in t in a certain generalized sense.

THEOREM II. *Assume that* $A(t)$ *can be continued to a complex neighborhood* Δ *of the interval* $[0, T]$ *in such a way that the conditions* i), ii) *are satisfied for* $t, s \in \Delta$. *Furthermore, let* $A(t)^{-h}$ *be holomorphic for* $t \in \Delta$. *Then the evolution operator* $U(t, s)$ *exists for* $s \leq t$, *satisfies the assertions of Theorem I and is holomorphic in* s *and* t *for* $s < t$. (*Here* "$s < t$" *should be interpreted as meaning* "$t - s \in \Sigma$", *where* Σ *is the sector* $|\arg t| < \frac{\pi}{2} - \theta$ *of the* t*-plane, and* "$s \leq t$" *as* "$s < t$ *or* $s = t$".) *If* $f(t)$ *is holomorphic for* $t \in \Delta$, $t > 0$, *and Hölder continuous at* $t = 0$, *every solution of* (E) *has a continuation holomorphic for* $t \in \Delta$, $t > 0$.

The proof of these theorems will be given in the following sections. Their applications to dissipative evolution equations in a Hilbert space will be given in § 4 (Theorems III, IV).

We collect here some immediate consequences of the assumptions i) and ii). As noted above, i) implies that

$$\|[z - A(t)^h]^{-1}\| \leq M_4/|z| \qquad \text{for } z \notin S_{h\theta}. \tag{1.12}$$

Here and in the following, the constants M_4, M_5, . . . are determined by θ, h, k, T and the preceding ones M_0, M_1, ... It follows from (1.3) that $z = 0$ also belongs to the resolvent set of $A(t)^h$, with

$$\|A(t)^{-h}\| \leq M_5. \tag{1.13}$$

Furthermore, it follows from i) and ii) that

$$\|A(t)^\alpha \exp(-\tau A(t))\| \leq M_6 |\tau|^{-\alpha}, \tag{1.14}$$

$$\|A(t)^\alpha \exp(-\tau A(t)) - A(s)^\alpha \exp(-\tau A(s))\| \leq M_7 |\tau|^{-\alpha} |t - s|^k. \tag{1.15}$$

Here $|\arg \tau| \leq \frac{\pi}{2} - \theta$, $0 \leq \alpha \leq \alpha_0$; α_0 is a constant and it suffices for our purpose to take $\alpha_0 = 2$. The constants M_6, M_7 can be chosen independent of α and τ. Another inequality to be useful is

$$\|A(t)^\alpha [\exp(-\tau A(t)) - \exp(-\sigma A(t))]\| \leq M_8 \beta^{-1} |\sigma|^{-\alpha-\beta} |\tau - \sigma|^\beta, \tag{1.16}$$

$$0 \leq \alpha \leq \alpha_0, \qquad 0 < \beta \leq 1, \qquad |\arg \sigma| \leq \frac{\pi}{2} - \theta,$$

$$|\arg \tau| \leq \frac{\pi}{2} - \theta, \qquad |\arg(\tau - \sigma)| \leq \frac{\pi}{2} - \theta.$$

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

The proofs of these inequalities are given in Appendix.

§ 2. The case of bounded $A(t)$

In the present section we construct the evolution operator $U(t, s)$ and deduce various estimates on it, under the additional assumption that $A(t) \in \mathbf{B}[\mathfrak{X}]$. The results will be used, in the following section, to construct the $U(t, s)$ in the general case as the limit of a sequence $U_n(t, s)$ corresponding to an approximating sequence $A_n(t) \in \mathbf{B}[\mathfrak{X}]$ for $A(t)$.

1. Construction of the evolution operator

If $A(t)$ is assumed to be bounded for a single $t \in [0, T]$, in addition to the assumptions i) and ii), it follows that $A(t) \in \mathbf{B}[\mathfrak{X}]$ for all t. In fact, the boundedness of $A(t)$ implies that of $A(t)^h$, so that the constant domain $\mathfrak{D} = \mathfrak{D}[A(t)^h]$ must coincide with $\mathfrak{X}$. Thus $A(t)^h \in \mathbf{B}[\mathfrak{X}]$ and hence $A(t) \in \mathbf{B}[\mathfrak{X}]$ for all t. Furthermore, it follows from (1.5) that

$$\begin{aligned}\|A(t)^h - A(s)^h\| &\le \|A(t)^h A(s)^{-h} - 1\| \|A(s)^h A(0)^{-h}\| \|A(0)^h\| \\ &\le M_2 M_3 \|A(0)^h\| |t-s|^k.\end{aligned}$$

This shows that $A(t)^h$ is Hölder continuous in the normed topology. Then the same is true with $A(t)$ itself in virtue of $A(t) = [A(t)^h]^m$.

Under these circumstances, the construction of the evolution operator is quite simple. It suffices to solve the integral equation

$$U(t, r) = 1 - \int_r^t A(s) U(s, r) ds. \tag{2.1}$$

Since (2.1) is of Volterra type, it can be solved by successive approximation. The solution obtained satisfies

$$\partial U(t, r)/\partial t = -A(t) U(t, r), \qquad U(t, t) = 1. \tag{2.2}$$

Let us consider another integral equation

$$V(t, r) = 1 - \int_r^t V(t, s) A(s) ds; \tag{2.3}$$

the solution of (2.3) gives an operator function $V(t, r)$ with the properties

$$\partial V(t, r)/\partial r = V(t, r) A(r), \qquad V(t, t) = 1. \tag{2.4}$$

It follows that

$$(\partial/\partial s)V(t, s)U(s, r) = V(t, s)[A(s) - A(s)]U(s, r) = 0,$$

so that $V(t, s)U(s, r)$ is independent of s. This gives immediately the following results. $V(t, s) = U(t, s)$ and

$$\begin{aligned} U(t, s)U(s, r) &= U(t, r), \qquad U(t, t) = 1, \\ \partial U(t, s)/\partial t &= -A(t)U(t, s), \qquad \partial U(t, s)/\partial s = U(t, s)A(s). \end{aligned} \tag{2.5}$$

At the same time the uniqueness of $U(t, s)$ follows.

2. First estimates on $U(t, s)$

We shall now deduce several estimates on $U(t, s)$ in terms of the constants T, θ, h, k, M_0, . . . that appear in our fundamental assumptions i), ii). We still keep the additional assumption that $A(t) \in \mathbf{B}[\mathfrak{X}]$, but it is our object to obtain such estimates as have no direct reference to this fact.

To this end, we construct several integral equations satisfied by $U(t, s)$. First we note that by (2.5)

$$\begin{aligned} &(\partial/\partial s)U(t, s)\exp(-(s-r)A(r)) \\ &\qquad = U(t, s)[A(s) - A(r)]\exp(-(s-r)A(r)). \end{aligned} \tag{2.6}$$

Now we make use of a simple identity, due to Sobolevskii [29],

$$A(s) - A(r) = \sum_{p=1}^{m} A(s)^{1-ph} D(s, r) A(r)^{ph} \qquad (mh = 1) \tag{2.7}$$

with

$$D(s, r) = A(s)^h A(r)^{-h} - 1. \tag{2.8}$$

Integration of (2.6) with respect to s on (r, t) gives, after substitution of (2.7),

$$\begin{aligned} &U(t, r) - \exp(-(t-r)A(r)) \\ &\quad = -\sum_{p=1}^{m}\int_r^t U(t, s)A(s)^{1-ph}D(s, r)A(r)^{ph}\exp(-(s-r)A(r))ds. \end{aligned} \tag{2.9}$$

Set

$$\begin{aligned} X_q(t, s) &= U(t, s)A(s)^{1-qh}, \\ X_{q0}(t, s) &= A(s)^{1-qh}\exp(-(t-s)A(s)), \end{aligned} \qquad q = 1, \ldots, m. \tag{2.10}$$

Multiplying (2.9) from right by $A(r)^{1-qh}$, we obtain a *system of integral equations* satisfied by X_q, $q = 1, \ldots, m$.

In writing down these integral equations, we find it convenient to introduce the following notation. For any two operator-valued functions $K'(t, s)$, $K''(t, s)$ defined for $0 < s < t < T$, we define their *convolution* by

$$K = K' * K'', \qquad K(t, r) = \int_r^t K'(t, s) K''(s, r) ds.$$

Then the system of integral equations for X_q has the form

$$X_q = X_{q0} + \sum_{p=1}^{m} X_p * K_{pq}, \qquad q = 1, \ldots, m, \tag{2.11}$$

where the kernels K_{pq} are given by

$$K_{pq}(s, r) = -D(s, r) A(r)^{1+ph-qh} \exp(-(s-r) A(r)). \tag{2.12}$$

Suppose that the system (2.11) has been solved for X_q by successive approximation in the form

$$X_q(t, s) = \sum_{i=0}^{\infty} X_{qi}(t, s), \tag{2.13}$$

$$X_{q, i+1} = \sum_{p=1}^{m} X_{pi} * K_{pq}. \tag{2.14}$$

We shall show that the series (2.13) are in fact convergent, with the rate of convergence determined by the constants T, θ, h, k, $M_0, \ldots$ alone. For convenience in this estimation, we further introduce the following notation. We denote by $\mathbf{Q}(a, M)$ the set of all operator-valued functions $K(t, s)$, defined and strongly continuous for $0 \leq s < t \leq T$, such that

$$\|K(t, s)\| \leq M(t-s)^{a-1}.$$

In particular, $K \in \mathbf{Q}(a, M)$ with $a > 1$ implies that $K(t, s)$ is continuous even for $s = t$ and $K(t, t) = 0$. The following lemma is a direct consequence of the definition.

LEMMA 1. *If $K' \in \mathbf{Q}(a', M')$ and $K'' \in \mathbf{Q}(a'', M'')$ with a' and a'' positive, then $K' * K'' \in \mathbf{Q}(a' + a'', B(a', a'') M' M'')$. Here B denotes the beta function.*[5]

Now we have from (1.5) and (1.14)

$$K_{pq} \in \mathbf{Q}(k - ph + qh, M_3 M_6), \tag{2.15}$$

$$X_{q0} \in \mathbf{Q}(qh, M_6); \tag{2.16}$$

in applying (1.14) note that $0 < h \leq 1 + ph - qh < 2$ and $1 - qh \geq 0$. (2.15) and (2.16) lead to the following estimates on X_{qi}:

5) The proof that $(K' * K'')(t, s)$ is strongly continuous for $s < t$ is not quite trivial, but we may omit it since there is no particular difficulty.

$$X_{qi} \in \mathbf{Q}(qh+ik,\ L_i M_6 (mM_3M_6)^i), \tag{2.17}$$

where $\{L_i\}$ is a sequence defined successively by

$$L_0 = 1, \qquad L_{i+1}/L_i = B(h+ik,\ h+k-1). \tag{2.18}$$

(2.17) can be proved by mathematical induction. For $i=0$, it coincides with (2.16). Assuming that it was proved for i, we have from (2.14) and (2.15), using Lemma 1,

$$\begin{aligned} X_{pi} * K_{pq} &\in \mathbf{Q}(qh+(i+1)k,\ C_{pqi}),\\ C_{pqi} &= L_i M_6 m^i (M_3M_6)^{i+1} B(ph+ik,\ k-ph+qh)\\ &\le L_i M_6 m^i (M_3 M_6)^{1+i} B(h+ik,\ h+k-1) \end{aligned}$$

from which (2.17) follows for i replaced by $i+1$ in virtue of (2.18). Here it should be noted that $ph+ik \ge h+ik > 0$, $k-ph+qh \ge h+k-1 > 0$, see (1.6).

It follows from (2.18) that L_{i+1}/L_i is of the order $i^{-(h+k-1)}$ for $i \to \infty$. Since $h+k-1>0$, we see from (2.17) that the series in (2.13) are absolutely convergent for $s<t$, the convergence being uniform for $t-s \ge a > 0$. Noting that the first term in each of these series is estimated by (2.16), we thus obtain the estimates

$$X_q \in \mathbf{Q}(qh,\ M_9), \qquad q=1,\ldots,m, \tag{2.19}$$

where M_9 may depend, among others, on T.

We shall now extend (2.19) to *non-integral values* of q. To this end we multiply (2.9) from right with $A(r)^{1-qh}$ as before, but this time with not necessarily an integral value of q; we assume only that $1-qh \ge 0$. The result is formally the same equation (2.11), where the summation is to be taken over $p=1,\ldots,m$ as before. But it is easily seen that the estimates (2.15) and (2.16) remain true for such non-integral q. Since X_p for integral p have been estimated by (2.19), we can estimate X_q from (2.11) by using Lemma 1:

$$X_q \in \mathbf{Q}(qh,\ (k-1+qh)^{-1} M_{10}) \qquad \text{for } k-1+qh>0, \tag{2.20}$$

where M_{10} does not depend on q.[6] Writing $\alpha = 1-qh$, this gives

$$\|U(t,s)A(s)^\alpha\| \le (k-\alpha)^{-1} M_{10} (t-s)^{-\alpha}, \qquad 0 \le \alpha < k. \tag{2.21}$$

[6] The factor $(k-1+qh)^{-1}$ has been taken out of $B(ph,\ k-ph+qh)$ for $p=m$, which appears on the application of Lemma 1 to the right member of (2.11), since this factor tends to infinity for $1-qh \to k$.

We further note that the X_q with non-integral q can also be written explicitly in an absolutely convergent series if $k-1+qh>0$; this series is obtained by substituting (2.13) for X_p of (2.11). We do not write down the result; it will only be remarked that the integrals involved exist if $k-1+qh>0$.

3. Second estimates on $U(t, s)$

We next form another system of integral equations, starting from the identity

$$(\partial/\partial s)\exp(-(t-s)A(t))U(s, r) = \exp(-(t-s)A(t))[A(t)-A(s)]U(s, r) \tag{2.22}$$

that follows from (2.5). Substituting the identity (2.7), integrating with respect to s on (r, t) and multiplying from left by $A(t)^{qh}$, we obtain from (2.22) a system of integral equations

$$Y_q = Y_{q0} + \sum_{p=1}^{m} H_{qp} * Y_p, \qquad q = 1, \ldots, m, \tag{2.23}$$

where

$$Y_q(t, s) = A(t)^{qh} U(t, s),$$

$$Y_{q0}(t, s) = A(t)^{qh} \exp(-(t-s)A(t)), \tag{2.24}$$

$$H_{qp}(t, s) = A(t)^{1+qh-ph} \exp(-(t-s)A(t))D(t, s). \tag{2.25}$$

For the kernels H_{qp} and the inhomogeneous terms Y_{q0} of (2.23), we have the following estimates analogous to (2.15) and (2.16):

$$H_{qp} \in \mathrm{Q}(k-qh+ph, M_3M_6), \qquad Y_{q0} \in \mathrm{Q}(1-qh, M_6). \tag{2.26}$$

An essential difference of (2.26) from (2.16) is that $1-qh$ in (2.26) becomes 0 for $q=m$ (whereas qh in (2.16) was >0). This makes it difficult to deduce estimates on the series obtained by solving the system (2.23) by successive approximation in as simple a fashion as in the preceding paragraph.

Thus we have to deal with (2.23) more carefully. We set

$$Y_q = Y_{q0} + Y'_q \tag{2.27}$$

and transform (2.23) into a system of integral equations for Y'_q:

$$Y'_q = Y'_{q0} + \sum_{p=1}^{m} H_{qp} * Y'_p \tag{2.28}$$

where

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

$$(2.29) \qquad Y'_{q0} = \sum_{p=1}^{m} H_{qp} * Y_{p0}.$$

Let us now estimate Y'_{q0}. For $p<m$, we have by (2.26) and Lemma 1

$$(2.30) \qquad H_{qp} * Y_{p0} \in \mathbf{Q}(1+k-qh,\ M_{11}), \qquad p<m.$$

To estimate $H_{qm} * Y_{m0}$, we write it as the sum of the following three integrals (note that $mh=1$);

$$I_1(t,\ r) = \int_r^t H_{qm}(t,\ s)[A(s)\exp(-(s-r)A(s)) - A(r)\exp(-(s-r)A(r))]ds,$$

$$I_2(t,\ r) = \int_r^t H_{qm}(t,\ r)A(r)\exp(-(s-r)A(r))ds$$
$$= H_{qm}(t,\ r)[1-\exp(-(t-r)A(r))],$$

$$I_3(t,\ r) = \int_r^t [H_{qm}(t,\ s) - H_{qm}(t,\ r)]A(r)\exp(-(s-r)A(r))ds.$$

I_1 and I_2 can be estimated by (2.26), (1.14) and (1.15) with $\alpha=0$, and Lemma 1:

$$I_1 \in \mathbf{Q}(1+2k-qh,\ M_3M_6M_7), \qquad I_2 \in \mathbf{Q}(1+k-qh,\ M_3M_6(1+M_6)).$$

I_3 is further divided into two integrals:

$$I_3(t,\ r) = I_3' + I_3'' = \int_r^S + \int_S^t \qquad \text{with } S=(r+t)/2.$$

I_3'' can be estimated by (2.26):

$$\|I_3''\| \le M_3M_6^2\int_S^t [(t-s)^{k-qh} + (t-r)^{k-qh}](s-r)^{-1}ds \le M_{12}(t-r)^{k-qh}.$$

In I_3' we make use of the identity

$$(2.31) \qquad H_{qm}(t,\ s) - H_{qm}(t,\ r)$$
$$= A(t)^{qh}[\exp(-(t-s)A(t)) - \exp(-(t-r)A(t))]D(t,\ s)$$
$$- A(t)^{qh}\exp(-(t-r)A(t))A(t)^{h}A(s)^{-h}D(s,\ r).$$

This gives, in virtue of (1.4), (1.5), (1.14) and (1.16) with $\beta=k$,

$$\|H_{qm}(t,\ s) - H_{qm}(t,\ r)\|$$
$$\le k^{-1}M_3M_8(t-s)^{-k-qh}(s-r)^k(t-s)^k + M_2M_3M_6(t-r)^{-qh}(s-r)^k$$
$$\le M_{13}(t-r)^{-qh}(s-r)^k \qquad \text{for } r\le s\le S.$$

Hence

$$\|I_3'\| \le M_6M_{13}\int_r^S (t-r)^{-qh}(s-r)^k(s-r)^{-1}ds \le M_{14}(t-r)^{k-qh}.$$

Collecting the above results, we have

(2.32) $$H_{qm} * Y_{m0} = I_1 + I_2 + I_3' + I_3'' \in \mathbf{Q}(1+k-qh,\ M_{15}).$$

Combined with (2.30), we have by (2.29)

(2.33) $$Y_{q0}' \in \mathbf{Q}(1+k-qh,\ M_{16}), \qquad q=1, \ldots, m.$$

In (2.33) we have $1+k-qh \geq k > 0$. Since we have also $k-qh+ph \geq k-1+h > 0$ in the estimate of H_{qp} (see (2.26)), it is now easy to solve the system (2.28) for Y_q' by successive approximation as in the preceding paragraph. In particular, we thus arrive at the estimates

(2.34) $$Y_q' \in \mathbf{Q}(1+k-qh,\ M_{17}), \qquad q=1, \ldots, m.$$

We can deduce from (2.34) an estimate on Y_q with not necessarily integral q. To see this, we note that (2.27) is valid also for non-integral $q \geq 0$, where Y_{q0}' is again given by (2.29) and Y_q' is to be defined by (2.28) (in which p takes integral values as before). Since Y_q' in (2.28) has been estimated by (2.34), (2.27) and (2.28) give an estimate of Y_q. In this way we obtain an estimate $Y_q \in \mathbf{Q}(1-qh,\ (k-qh+h)^{-1}M_{18})$; the factor $(k-qh+h)^{-1}$ comes from the application of Lemma 1, the condition $k-qh+h>0$ being required.[7] On writing $qh=\alpha$, we thus arrive at the estimate

(2.35) $$\|A(t)^{\alpha} U(t,\ s)\| \leq (h+k-\alpha)^{-1} M_{18}(t-s)^{-\alpha}, \qquad 0 \leq \alpha < h+k.$$

Note that the range of α for which (2.35) is valid is slightly larger than in (2.21); in particular (2.35) is true for $\alpha = 1$.

4. Third estimates on $U(t,\ s)$

We also need estimates on the operators of the form $A(t)^{\alpha+h} U(t,s) A(s)^{-h}$. To obtain such estimates, we multiply (2.23) (written for the arguments $t,\ r$) from right with $A(r)^{-h}$. On setting

(2.36) $$Z_q(t,\ r) = A(t)^{qh} U(t,\ r) A(r)^{-h},$$

we obtain a system of integral equations satisfied by Z_q:

(2.37) $$Z_q = Z_{q0} + \sum_{p=1}^{m} H_{qp} * Z_p, \qquad q=1, \ldots, m,$$

where the inhomogeneous term is

[7] See footnote [6].

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

$$(2.38)\qquad Z_{q0}(t, r) = A(t)^{qh}\exp(-(t-r)A(t))A(r)^{-h}$$
$$= A(t)^{qh-h}\exp(-(t-r)A(t))A(t)^{h}A(r)^{-h}.$$

We have by (1.4) and (1.14)

$$(2.39)\qquad Z_{q0}\in \mathbf{Q}(1+h-qh,\ M_2M_6).$$

Since $1+h-qh\geq h>0$, there is no difficulty in solving the system (2.37) for Z_q by successive approximation as before, obtaining estimates on Z_q. Then, again considering non-integral values of q and proceeding as before, we have[8]

$$(2.40)\qquad \|A(t)^{\alpha+h}U(t, s)A(s)^{-h}\|\leq (k-\alpha)^{-1}M_{19}(t-s)^{-\alpha},\qquad 0\leq\alpha<k.$$

5. Degree of continuity of $U(t, s)$

We next estimate the degree of continuity of $U(t, s)$ in terms of the basic constants. For $s_1\leq t_1$, $s_2\leq t_2$ and $t_1\leq t_2$ we have

$$(2.41)\qquad U(t_2, s_2)-U(t_1, s_1)$$
$$= U(t_2, s_2)-U(t_2, s_1)+U(t_2, s_1)-U(t_1, s_1)$$
$$= \int_{s_1}^{s_2}[\partial U(t_2, s)/\partial s]ds+\int_{t_1}^{t_2}[\partial U(t, s_1)/\partial t]dt$$
$$= \int_{s_1}^{s_2}U(t_2, s)A(s)ds-\int_{t_1}^{t_2}A(t)U(t, s_1)dt.$$

Hence

$$(2.42)\qquad [U(t_2, s_2)-U(t_1, s_1)]A(0)^{-h}$$
$$= \int_{s_1}^{s_2}U(t_2, s)A(s)^{1-h}A(s)^{h}A(0)^{-h}ds$$
$$-\int_{t_1}^{t_2}A(t)U(t, s_1)A(s_1)^{-h}A(s_1)^{h}A(0)^{-h}dt.$$

Making use of (1.4), (2.21) and (2.40) with $\alpha=1-h<k$, we obtain

$$(2.43)\qquad \|[U(t_2, s_2)-U(t_1, s_1)]A(0)^{-h}\|$$
$$\leq (h+k-1)^{-1}\Big[M_2M_{10}\int_{s_1}^{s_2}(t_2-s)^{h-1}ds+M_2M_{19}\int_{t_1}^{t_2}(t-s_1)^{h-1}dt\Big]$$
$$\leq M_{20}[\,|(t_2-s_1)^{h}-(t_2-s_2)^{h}|+|(t_2-s_1)^{h}-(t_1-s_1)^{h}|\,]$$
$$\leq M_{20}(|s_2-s_1|^{h}+|t_2-t_1|^{h}).$$

Here we have assumed that $t_1\leq t_2$, but the final result of (2.43) is obviously true without this assumption, provided that $s_1\leq t_1$ and $s_2\leq t_2$.

[8] Set $\alpha=qh-h$.

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

§ 3. The general case

1. The approximating sequence $A_n(t)$.

We now turn to the general case in which $A(t)$ is not necessarily bounded. According to the program stated in the beginning of § 2, we first construct a sequence of bounded operators $A_n(t)$ that approximate $A(t)$ in a certain sense. We set

$$(3.1)\qquad A_n(t)=A(t)J_n(t),\qquad J_n(t)=[1+n^{-1}A(t)^h]^{-m},\qquad n=1,2,\ldots.$$

This is equivalent to

$$(3.2)\qquad A_n(t)^h=A(t)^hJ_n(t)^h=n[1-J_n(t)^h],\quad J_n(t)^h=[1+n^{-1}A(t)^h]^{-1}.$$

The equivalence of (3.1) and (3.2) is a simple consequence of the "operational calculus", which can be justified with the aid of Dunford integrals representing various "functions" of the operator $A(t)$.[9] Obviously $J_n(t)^h$, $A_n(t)^h$, $J_n(t)$ and $A_n(t)$ belong to $\mathrm{B}[\mathfrak{X}]$, and it follows from (1.12) that (set $z=-n$)

$$(3.3)\qquad \|J_n(t)^h\|\le M_4,\qquad \|A_n(t)^h\|\le n(M_4+1).$$

An important property of $A_n(t)$ is that they satisfy the conditions i), ii) for $A(t)$, with possibly different constants. In other words, there exist positive constants N_0 to N_3, independent of t and n, such that

$$(3.4)\qquad \|[z-A_n(t)]^{-1}\|\le N_0/|z|,\qquad z\notin S_\theta,$$

$$(3.5)\qquad \|A_n(t)^{-1}\|\le N_1,$$

$$(3.6)\qquad \|A_n(t)^hA_n(s)^{-h}\|\le N_2,$$

$$(3.7)\qquad \|A_n(t)^hA_n(s)^{-h}-1\|\le N_3|t-s|^k.$$

To show this, we first note that a simple calculation based on (3.2) gives (for simplicity we write A in place of $A(t)$ when there is no ambiguity)

$$(3.8)\qquad (z-A_n^h)^{-1}=-(n-z)^{-1}+n^2(n-z)^{-2}[nz(n-z)^{-1}-A^h]^{-1}.$$

But it is easily seen that $z\notin S_{h\theta}$ implies $nz(n-z)^{-1}\notin S_{h\theta}$ and $|n-z|\ge(n+|z|)\sin(h\theta/2)$. Hence we have, noting (1.12),

$$(3.9)\qquad \begin{aligned}\|(z-A_n^h)^{-1}\|&\le|n-z|^{-1}+nM_4|n-z|^{-1}|z|^{-1}\\ &\le(n+|z|)M_4/|z||n-z|\le M_4/|z|\sin(h\theta/2)=N_4/|z|\qquad\text{for } z\notin S_{h\theta}\end{aligned}$$

---

[9] An example of such a formula is (A 2) of Appendix, which represents $A^{-\alpha}$ if $z=0$.

with $N_4 = M_4/\sin(h\theta/2)$; here we have used the fact that $M_4 \geq 1$ (which is a necessary consequence of (1.12)). (3.9) is an inequality corresponding to (1.12). From this (3.4) follows directly. In fact, let $z \notin S_\theta$ and set $\omega = \exp(2\pi ih)$. Then $\omega^p z^h \notin S_{h\theta}$ for $p = 0, 1, \ldots, m-1$, so that $\|(\omega^p z^h - A_n^h)^{-1}\| \leq N_4 |z|^{-h}$ by (3.9). Since $(z - A_n)^{-1} = -\prod_{p=0}^{m-1} (A_n^h - \omega^p z^h)^{-1}$, (3.4) follows with $N_0 = N_4^m$.

To prove (3.5), we note that (3.2) implies

$$A_n(t)^{-h} = A(t)^{-h} + n^{-1}. \tag{3.10}$$

Since $\|A(t)^{-h}\| \leq M_5$ by (1.13), we have $\|A_n(t)^{-h}\| \leq 1 + M_5$ and (3.5) holds with $N_1 = (1 + M_5)^m$. Again, (3.2) and (3.10) give

$$A_n(t)^h A_n(s)^{-h} = A_n(t)^h [A(s)^{-h} + n^{-1}] = 1 + J_n(t)^h [A(t)^h A(s)^{-h} - 1]. \tag{3.11}$$

Hence $\|A_n(t)^h A_n(s)^{-h}\| \leq 1 + M_4(1 + M_2)$ by (3.3) and (1.4), and $\|A_n(t)^h A_n(s)^{-h} - 1\| \leq M_3 M_4 |t - s|^k$ by (3.3) and (1.5). This proves (3.6) and (3.7) with $N_2 = 1 + M_4(1 + M_2)$ and $N_3 = M_3 M_4$.

The approximating property of $A_n(t)$ is based on the relation[10]

$$\text{s-lim}\, J_n(t)^h = 1 \qquad \text{for } n \to \infty, \tag{3.12}$$

the convergence being uniform in t.[11] This follows from

$$\|(1 - J_n(t)^h) A(0)^{-h}\| = n^{-1} \|J_n(t)^h A(t)^h A(0)^{-h}\| \leq n^{-1} M_2 M_4 \to 0,$$

considering that $\mathfrak{D}[A(0)^h] = \mathfrak{D}$ is dense in $\mathfrak{X}$.

2. Construction of $U(t, s)$.

We have shown above that $A_n(t)$ satisfy the fundamental assumptions i), ii) and, moreover, that $A_n(t) \in \mathbf{B}[\mathfrak{X}]$. Therefore, there exist the associated evolution operstors $U_n(t, s)$ satisfying the fundamental equations (2.5) with $A(t)$ replaced by $A_n(t)$, and all the estimates deduced in the preceding sections are valid for $U_n(t, s)$, provided we replace the constants M_0 to M_3 by N_0 to N_3 and, accordingly, the auxiliary constants $M_4, M_5, \ldots$ by the corresponding ones $N_4, N_5, \ldots$. It is important to notice that these constants are determined by the fundamental constants $T, \theta, h, k, M_0, M_1, M_2, M_3$ alone

---

[10] s-lim denotes strong limit.

[11] The convergence s-lim $B_n(t) = B(t)$ is *uniform* in t if $\|B_n(t)u - B(t)u\| \to 0$ uniformly in t for every $u \in \mathfrak{X}$.

and do not depend on n. For example we have, corresponding to (2.21), (2.35) and (2.40)

$$\|U_n(t, s)A_n(s)^\alpha\| \leq (k-\alpha)^{-1}N_{10}(t-s)^{-\alpha}, \qquad 0 \leq \alpha < k, \tag{3.13}$$

$$\|A_n(t)^\alpha U_n(t, s)\| \leq (h+k-\alpha)^{-1}N_{18}(t-s)^{-\alpha}, \qquad 0 \leq \alpha < h+k, \tag{3.14}$$

$$\|A_n(t)^{\alpha+h}U_n(t, s)A_n(s)^{-h}\| \leq (k-\alpha)^{-1}N_{19}(t-s)^{-\alpha}, \qquad 0 \leq \alpha < k. \tag{3.15}$$

Also we have the estimate which corresponds to (2.43), but it is not necessary to write it explicitly.

We shall now show that s-lim $U_n(t, s) = U(t, s)$ exists. To this end we start from the expression $X_q^{(n)}$ given by (2.13), where $X_q^{(n)}$ are the X_q of §2 associated with $A_n(t)$, and show that s-lim $X_q^{(n)}$ exist for $q = 1, \ldots, m$. Since $U_n = X_m^{(n)}$, this will give the desired result for $q = m$. Now the series on the right of (2.13) has a majorizing series by (2.17) (where X_{qi} should be replaced by $X_{qi}^{(n)}$ and M_3, M_6 by N_3, N_6 respectively), which is independent of n. Therefore, it suffices to show that each term of this series has a limit for $n \to \infty$. But these terms are determined successively by a formula corresponding to (2.14), where we have the estimates (2.15) and (2.17) independent of n. In view of the principle of dominated convergence, it then suffices to show that

$$\text{s-lim}\, X_{q0}^{(n)}(t, s) = X_{q0}(t, s), \tag{3.16}$$
$$\text{s-lim}\, K_{pq}^{(n)}(t, s) = K_{pq}(t, s)$$

for $s < t$ and $p, q = 1, \ldots, m$. Recalling the definition of $X_{q0}^{(n)}$ and $K_{pq}^{(n)}$ (replace A by A_n in (2.10) and (2.12)), the problem is finally reduced to the verification of the two relations

$$\text{s-lim}\, A_n(t)^\alpha \exp(-\tau A_n(t)) = A(t)^\alpha \exp(-\tau A(t)), \tag{3.17}$$

$$\text{s-lim}\, A_n(t)^h A_n(s)^{-h} = A(t)^h A(s)^{-h}. \tag{3.18}$$

But these are simple consequences of the definition of $A_n(t)$. (3.18) follows immediately from (3.11) and (3.12). (3.17) also follows from (3.12); this is a simple fact related to the approximation of semigroups and the proof will be given in Appendix (see (A 8)).

Let us note that the convergence in (3.18) is uniform in s and t and that in (3.17) is uniform in t and τ as long as τ is bounded from below. This is due to the uniform convergence in t of (3.12). Hence it follows that (3.16) holds uniformly for $t-s\geq a>0$. In virtue of Lemma 2 stated below, we see that the convergence of $X_{qi}^{(n)}(t, s)$ and, consequently, of $X_q^{(n)}$ is uniform for $t-s\geq a>0$. Since $X_q^{(n)}(t, s)$ are strongly continuous in s and t, we conclude that s-lim $X_q^{(n)}(t, s)$ are also strongly continuous for $s<t$.

LEMMA 2.[12] *Let* $H, H_n\in\mathbf{Q}(a, M)$ *and* $K, K_n\in\mathbf{Q}(b, N)$ *with* $a, b>0$. *Let* $H_n(t, s)\to H(t, s)$ *and* $K_n(t, s)\to K(t, s)$, $n\to\infty$, *strongly and uniformly for* $t-s\geq c$ *for any* $c>0$. *Then* $(H_n*K_n)(t, s)\to(H*K)(t, s)$ *strongly and uniformly for* $t-s\geq c$ *for any* $c>0$.

A similar argument shows, more generally, that the same is true with non-integral q(see §2.2). In other words,

$$\text{s-lim}\, U_n(t.\ s)A_n(s)^\alpha = V(t, s;\ \alpha),\qquad s<t,\qquad 0\leq\alpha<k, \tag{3.19}$$

exists, the convergence being uniform for $t-s\geq a>0$, so that $V(t, s;\ \alpha)$ is strongly continuous in s, t for $s<t$. In particular, we set

$$U(t, s) = V(t, s;\ 0). \tag{3.20}$$

It will be seen that $U(t, s)$ is the required evolution operator.

Since N_{10} is independent of n, it follows from (3.13) that

$$\|V(t, s;\ \alpha)\|\leq(k-\alpha)^{-1}N_{10}(t-s)^{-\alpha}. \tag{3.21}$$

In particular $U(t, s)$ is uniformly bounded by

$$\|U(t, s)\|\leq k^{-1}N_{10}. \tag{3.22}$$

Next we see from (3.19) that

$$\begin{aligned} U(t, s) &= \text{s-lim}\, U_n(t, s) = \text{s-lim}\, U_n(t, s)A_n(s)^\alpha A_n(s)^{-\alpha} \\ &= V(t, s;\ \alpha)A(s)^{-\alpha}, \end{aligned} \tag{3.23}$$

where we used the fact that s-lim $A_n(t)^{-\alpha} = A(t)^{-\alpha}$ (for the proof see (*A* 7) of Appendix). (3.23) shows that

$$U(t, s)A(s)^\alpha\subset V(t, s;\ \alpha)\in\mathbf{B}[\mathfrak{X}]; \tag{3.24}$$

in other words the opeartor $U(t, s)A(s)^\alpha$, defined with domain $\mathfrak{D}[A(s)^\alpha]$, is

[12] The proof of this lemma may be omitted since there is no difficulty. Only it should be remarked that $H_n(t, s)K_n(s, r)\to H(t, s)K(s, r)$ strongly and uniformly for $r+c\leq s\leq t-c$ for any $c>0$. This is seen, for example, by Lemma 4 of [3].

bounded if $\alpha<k$, the bound satisfying the inequality (3.21).

In quite the same way, it can be proved that s-lim $Y_q^{(n)}(t, s)$ exists for $s<t$, uniformly for $t-s\geq a>0$, where $Y_q^{(n)}$ are the Y_q of §2.3 associated with $A_n(t)$. Again considering non-integral q, we have that

$$\text{(3.25)} \qquad \text{s-lim}\, A_n(t)^\alpha U_n(t, s) = W(t, s;\ \alpha), \qquad s<t, \qquad 0\leq\alpha<h+k,$$

exists, the convergence being uniform[13] for $t-s\geq a>0$, and is strongly continuous in s, t for $s<t$. Then we see as in (3.23) that $U(t, s) = A(t)^{-\alpha}W(t, s;\ \alpha)$. This shows that

$$\text{(3.26)} \qquad A(t)^\alpha U(t, s) = W(t, s;\ \alpha) \in \mathbf{B}[\mathfrak{X}], \qquad 0\leq\alpha<h+k,$$

which implies that $\mathfrak{R}[U(t, s)]$[14] $\subset\mathfrak{D}[A(t)^\alpha]$ for $\alpha<h+k$. Since $h+k>1$, $\alpha=1$ is permitted and, therefore, $A(t)U(t, s)$ beolngs to $\mathbf{B}[\mathfrak{X}]$ and is strongly continuous in s, t for $s<t$. Furthermore, (3.14) gives

$$\text{(3.27)} \qquad \|A(t)^\alpha U(t, s)\|\leq(h+k-\alpha)^{-1}N_{18}(t-s)^{-\alpha}.$$

This proves (1.9).

Now we have by (2.5)

$$U_n(t_2, r) - U_n(t_1, r) = -\int_{t_1}^{t_2} A_n(s)U_n(s, r)ds,$$

for $r<t_1\leq t_2$. Going to the limit $n\to\infty$, we obtain

$$\text{(3.28)} \qquad U(t_2, r) - U(t_1, r) = -\int_{t_1}^{t_2} A(s)U(s, r)ds;$$

taking the limit under the integral sign is justified, for example, by the principle of dominated convergence. Since $A(s)U(s, r)$ has been shown to be strongly continuous in s for $s>r$, (3.28) shows that $U(t, s)$ is strongly continuously differentiable for $t>s$ and $\partial U(t, s)/dt = -A(t)U(t, s)$. This proves (1.10).

---

13) Actually the direct proof of this result would not be very easy, for the proof of the uniformity of the convergence $Y_{0q}^{\prime(n)}(t, s)\to Y_{q0}'(t, s)$ requires rather complicated estimates (just as this was the case with the estimates of Y_{q0}' themselves, see §2.3). But this difficulty can be avoided by first considering Z_q instead of Y_q. In fact, there is no such difficulty in proving that $Z_q^{(n)}(t, s)\to Z_q(t, s)$ strongly and uniformly in s, t for $t-s\geq a>0$. As before, the consideration of non-integral q leads to the result that $A_n(t)^\alpha U_n(t, s)A_n(0)^{-h}\to W(t, s;\ \alpha) = W(t, s;\ \alpha)\ A(0)^{-h}$, $0\leq\alpha<h+k$, strongly and uniformly for $t-s\geq a>0$. Since we know that $A_n(t)^\alpha U_n(t, s)$ are uniformly bounded for $t-s\geq a>0$, it follows that $A_n(t)^\alpha U_n(t, s)\to W(t, s;\ \alpha)\supset W'(t, s;\ \alpha)A(0)^h$ uniformly for $t-s\geq a>0$ (note that $\mathfrak{D}=\mathfrak{D}[A(0)^h]$ is dense in $\mathfrak{X}$).

14) $\mathfrak{R}[U]$ denotes the range of the operator U.

Similarly we have from (2.5)

$$[U_n(t, r_2) - U_n(t, r_1)]A_n(0)^{-h} = \int_{r_1}^{r_2} U_n(t, s) A_n(s)^{1-h} A_n(s)^h A_n(0)^{-h} ds$$

for $r_1 \leq r_2 < t$. Taking the limit $n \to \infty$, we have as above

$$[U(t, r_2) - U(t, r_1)]A(0)^{-h} = \int_{r_1}^{r_2} V(t, s; 1-h) A(s)^h A(0)^{-h} ds. \tag{3.29}$$

Since $V(t, s; 1-h)$ is strongly continuous for $s < t$ (note that $1-h<k$), it follows that $U(t, s)u$ is strongly continuously differentiable in s for $s<t$ for each $u \in \mathfrak{D}$, with

$$\partial U(t, s)u/\partial s = V(t, s; 1-h) A(s)^h u. \tag{3.30}$$

If, in particular, $u \in \mathfrak{D}[A(s)] \subset \mathfrak{D}$, then $A(s)^h u \in \mathfrak{D}[A(s)^{1-h}]$ and, noting that $1-h<k$, we see from (3.24) that the right member of (3.30) is equal to $U(t, s)A(s)u$. This proves (1.11).

It remains to show that $U(t, s)$ is strongly continuous in s, t for $s \leq t$. To this end we recall (2.43), in which U and A should be replaced by U_n and A_n, respectively, and M_{20} by N_{20}. On letting $n \to \infty$, we obtain the same estimate (2.43) with M_{20} replaced by N_{20}. This inequality shows that $U(t, s)A(0)^{-h}$ is Hölder continuous in s, t in norm for $s \leq t$. This implies that $U(t, s)u$ is Hölder continuous in t, s for $u \in \mathfrak{D} = \mathfrak{D}[A(0)^h]$. Since $U(t, s)$ is uniformly bounded by (3.22) and $\mathfrak{D}$ is dense in $\mathfrak{X}$, it follows that $U(t, s)$ is strongly continuous for $s \leq t$. The relations (1.7) and (1.8) then follows from the corresponding ones for $U_n(t, s)$ (see (2.5)) by taking the limit $n \to \infty$.

This completes the proof of the first part of Theorem I. (The uniqueness of $U(t, s)$ can be proved as in §2.1.)

3. Solution of the inhomogeneous equation.

Suppose that $u(t)$ is a strict solution of (E) in the sense stated in Theorem I. Then we have, in virtue of (1.11),

$$(\partial/\partial s) U(t, s) u(s) = U(t, s) du(s)/ds + U(t, s) A(s) u(s) = U(t, s) f(s).$$

Integration of this equation with respect to s on $(0, t)$ gives (1.1).

Finally let us prove the last assertion of Theorem I. Suppose $u(t)$ is defined by (1.1); we may assume that $u(0)=0$, for $U(t, 0)\, u(0)$ satisfies the homogeneous equation $du/dt + A(t)u(t) = 0$ by the properties of $U(t, s)$ already proved. Set

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

$$u_n(t)=\int_0^t U_n(t,\ s)f(s)ds. \tag{3.31}$$

Since $U_n(t,\ s)\to U(t,\ s)$ boundedly, this gives

$$\lim_{n\to\infty} u_n(t)=\int_0^t U(t,\ s)f(s)ds=u(t). \tag{3.32}$$

In virtue of (2.5), (3.31) implies that $du_n/dt=-A_n(t)u_n+f(t)$ and $u_n(0)=0$, so that we have, after integration,

$$u_n(t)=\int_0^t [f(s)-A_n(s)u_n(s)]ds. \tag{3.33}$$

We shall now show that

$$\lim A_n(t)u_n(t)=A(t)u(t) \qquad \text{uniformly in } t, \tag{3.34}$$

so that (3.33) gives by (3.32)

$$u(t)=\int_0^t [f(s)-A(s)u(s)]ds.$$

Differentiation with respect to t then gives (E), thereby completing the proof of Theorem I (note that (3.34) implies that $A(t)u(t)$ is strongly continuous in t).

To prove (3.34), we write

(3.35)
$$A_n(t)u_n(t)=\int_0^t A_n(t)U_n(t,\ s)[f(s)-f(t)]ds+\left[\int_0^t A_n(t)U_n(t,\ s)ds\right]f(t).$$

We recall that $A_n(t)U_n(t,\ s)\to W(t,\ s;\ 1)=A(t)U(t,s)$ strongly and uniformly for $t-s\geq a>0$, being dominated by const. $(t-s)^{-1}$ by (3.14). Since $\|f(s)-f(t)\|\leq$ const. $(t-s)^{-\varepsilon}$, $\varepsilon>0$, by the assumed Hölder continuity of $f(t)$, it follows that

$$\int_0^t A_n(t)U_n(t,\ s)[f(s)-f(t)]ds\to\int_0^t A(t)U(t,\ s)[f(s)-f(t)]ds$$

uniformly in t.

To deal with the second term on the right of (3.35), we note that (2.27) gives for $q=m$

$$A_n(t)U_n(t,\ s)=A_n(t)\exp(-(t-s)A_n(t))+Y_m^{(n)\prime}(t,\ s)$$

where $Y_m^{(n)\prime}(t, s)$ is majorized by const. $(t-s)^{k-1}$ by (2.34). Hence

$$\int_0^t A_n(t)\, U_n(t, s)ds = 1 - \exp(-tA_n(t)) + \int_0^t Y_m^{(n)\prime}(t, s)ds$$

and

$$\text{s-lim}\int_0^t A_n(t)\, U_n(t, s)ds = 1 - \exp(-tA(t)) + \int_0^t Y_m'(t, s)ds$$

uniformly in t.[15)]

Thus we have shown that

$$\lim A_n(t)\, u_n(t) = v(t) \tag{3.36}$$

exists, the convergence being uniform in t. This implies, in particular, that $v(t)$ is continuous in t. Then

$$u(t) = \lim u_n(t) = \lim A_n(t)^{-1} A_n(t)\, u_n(t) = A(t)^{-1} v(t),$$

so that $A(t)u(t)$ exists and is equal to $v(t)$. Thus (3.36) is equivalent to the required result (3.34). This completes the proof of Theorem I.

4. Proof of Theorem II.

If the assumptions of Theorem II are satisfied, the operators $J_n(t)^h$ are holomorphic for $t \in \Delta$. This is a direct consequence of a general theorem that the resolvent $(z - T(t))^{-1}$ of a closed linear operator depending on t is holomorphic in t for every z belonging to the resolvent set, if this is the case for some particular value of z; in the present case this particular value is $z = 0$. (3.2) then shows that $A_n(t)^h$ and, consequently, $A_n(t) = (A_n(t)^h)^m$ are also holomorphic in Δ. It is now obvious from the construction of $U_n(t, s)$ described in § 2.1 that $U_n(t, s)$ are holomorphic for $s, t \in \Delta \times \Delta$.

The various estimates deduced in § 3.2 remain true for these $U_n(t, s)$ with complex s, t provided that $s < t$ in the sense stated in Theorem II. To see this, it suffices to note that these estimates can be deduced by considering the integral equations satisfied by $U_n(t, s)$ (such as considered in § 2.2 to § 2.4) in which s, t are restricted to lie on a straight line in Δ that has an angle smaller than $\frac{\pi}{2} - \theta$ with the real axis. For such s, t the estimates can be deduced by

15) Here we have set $Y_m'(t, s) = \text{s-lim}\, Y_m'^{(n)}(t, s)$. The existence of this limit and the uniformity of the convergence for $t - s \geq a > 0$ follows from that of $\lim Y_m(t, s) = W(t, s; 1)$. See footnote.[13)]

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

making use of the inequalities (1.2) to (1.6) and (1.12) to (1.16) in which τ may be any complex number with $|\arg \tau| \leq \frac{\pi}{2} - \theta$. Also the convergence $U_n(t, s) \to U(t, s)$ can be proved in the same way for $s \leq t$.

Thus $U(t, s)$ is the limit of a sequence $\{U_n(t, s)\}$ where $U_n(t, s)$ are holomorphic and uniformly bounded in each compact subset of $\Delta \times \Delta$ with $s < t$. Hence $U(t, s)$ must be holomorphic for $s < t$. This proves the first part of Theorem II.

To prove the second part, we note that the solution of (E) is given by (1.1) as shown by Theorem I. The first term on the right of (1.1) is holomorphic in t by what is proved above. Hence we may hereafter assume that $u(0) = 0$. Then we have $u(t) = \lim u_n(t)$ by (3.32). But $u_n(t)$ has a continuation holomorphic in $\Sigma \cap \Delta$, for it is given by (3.31) where $U_n(t, s)$ is holomorphic in $\Delta \times \Delta$ and $f(s)$ is holomorphic in $\Sigma \cap \Delta$. Now it is obvious that $u_n(t) \to u(t) = \int_0^t U(t, s) f(s) ds$ boundedly for t belonging to any compact subset of $\Sigma \cap \Delta$. Hence $u(t)$ is holomorphic for $t \in \Sigma \cap \Delta$.

§ 4. Application to dissipative evolution equations in a Hilbert space

The object of the present secton is to show that Theorems I and II are applicable with satisfactory results to the evolution equation (E) in which $-A(t)$ are *closed, maximal dissipative operators* in a Hilbert space $\mathfrak{X}$ defined in terms of certain *sesquilinear forms*. The contents of this section depend on the results of a separate paper of the author [6] devoted to the study of the fractional powers of dissipative operators in a Hilbert space.

A linear operator A in a Hilbert space is said to be *accretive* (and $-A$ *dissipative*) if $\mathrm{Re}(Au, u) \geq 0$ for $u \in \mathfrak{D}[A]$. A closed, maximal dissipative operator $-A$ is the infinitesimal generator of a contraction semigroup $\exp(-tA)$ (that is, $\|\exp(-tA)\| \leq 1$). The fractional powers A^α can be defined for such operators. An important class of dissipative operators are defined in terms of certain sesquilinear forms $\phi[u, v]$ (linear in u, conjugate-linear in v). Suppose that $\phi[u, v]$ is defined for $u, v \in \mathfrak{D} = \mathfrak{D}[\phi]$, the domain of ϕ, which is dense in $\mathfrak{X}$, and that $f[u] = \mathrm{Re}\,\phi[u] \geq 0$ (we write $\phi[u] = \phi[u, u]$). Suppose further that the "quadratic form" $f[u]$ is *closed* and $|g[u]| \leq \beta f[u]$ for $u \in \mathfrak{D}$, where $g[u] = \mathrm{Im}\,\phi[u]$ and β is a constant. Then we shall say that the sesquilnear form

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

ϕ is *regular*. It can be shown that to each regular sesquilinear form ϕ is associated a closed, maximal accretive operator A such that $\mathfrak{D}[A] \subset \mathfrak{D}[\phi]$ and $(Au, v) = \phi[u,v]$ for $u \in \mathfrak{D}[A]$ and $v \in \mathfrak{D}[\phi]$. Such an operator A will be said to be *regularly* accretive.

If A is regularly accretive, the inequality (1.2) for $A(t)$ is satisfied by A with M_0 depending only on β (see Theorem 2.2 of [6]). If we further assume that $f[u] \geq \delta \|u\|^2$ with a constant $\delta > 0$, then A^{-1} exists and is bounded with $\|A^{-1}\| \leq \delta^{-1}$. Thus (1.3) is satisfied by A with $M_1 = \delta^{-1}$.

Suppose now that $A(t)$ is a family of regularly accretive operators such that the associated regular sesquilinear forms $\phi(t)$ have the constants β and δ independent of t. Then the condition i) is satisfied. Furthermore, suppose that $\phi(t)$ have a domain $\mathfrak{D}$ independent of t and $\phi(t)[u]$ is Hölder continuous in the sense that

$$(4.1) \qquad |\phi(t)[u] - \phi(s)[u]| \leq M|t-s|^k f(s)[u], \qquad u \in \mathfrak{D},$$

where $f(t)[u] = \mathrm{Re}\,\phi(t)[u]$. Then it can be shown (see Theorem 4.2 of [6]) that, for $0 \leq \alpha < 1/2$, $A(t)^\alpha$ have a domain $\mathfrak{D}_\alpha$ independent of t and

$$(4.2) \qquad \|A(t)^\alpha A(s)^{-\alpha} - 1\| \leq M'|t-s|^k.$$

Therefore, the condition ii) is satisfied with $h = 1/3$ provided that $k > 2/3$. In this way we are led to the following theorem.

THEOREM III. *Let $\phi(t)$, $0 \leq t \leq T$, be a family of regular sesquilinear forms in a Hilbert space $\mathfrak{X}$. Let $\phi(t)$ have a constant domain $\mathfrak{D}$, and let*

$$(4.3) \qquad |g(t)[u]| \leq \beta f(t)[u], \qquad f(t)[u] \geq \delta \|u\|^2,$$

for $u \in \mathfrak{D}$, where $f(t)[u] = \mathrm{Re}\,\phi(t)[u]$, $g(t)[u] = \mathrm{Im}\,\phi(t)[u]$ and $\beta \geq 0$ and $\delta > 0$ are constants. Furthermore, let $\phi(t)$ be Hölder continuous in the sense of (4.1) *with $2/3 < k \leq 1$. If $A(t)$ are the regularly accretive operators associated with $\phi(t)$, then the results of Theorem I hold true.*

If, in addition to the assumptions of Theorem III, $\phi(t)$ can be continued to a complex neighborhood of the interval $[0, T]$ in such a way that $\phi(t)[u]$ is holomorphic for every $u \in \mathfrak{D}$, it can be shown that $A(t)^{-\alpha}$ is holomorphic for $0 \leq \alpha < 1/2$ (see Theorem 4.3 of [6]). Thus we have

THEOREM IV. *Let $\phi(t)$ be a family of regular sesquilinear forms defined for*

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

each t of a complex neighborhood Δ of the interval $0 \leq t \leq T$ such that $\mathfrak{D}[\phi(t)] = \mathfrak{D}$ is constant and (4.3) *is satisfied. Furthermore, let $\phi(t)[u]$ be holomorphic for $t \in \Delta$ for each $u \in \mathfrak{D}$. Then the results of Theorem II are true.*

As we have mentioned in Introduction, these theorems strengthen some of the earlier results of Lions and Višik-Ladyženskaia.

§ 5. Remarks and examples

In the theorems proved above, we have been able to extend many of the earlier results on the evolution equation (E) (see Introduction). Yet the results are not satisfactory enough. In particular, the requirement in Theorems I, II that $\mathfrak{D}[A(t)^h]$ be constant for some $h = 1/m$, $m = 1, 2, \ldots$, appears still too restrictive, and similar remark applies to the assumption in Theorems III, IV that $\mathfrak{D}[\phi(t)]$ be constant. The assumption that $k > 2/3$ in Theorem III also appears rather artificial.

It is highly desirable to see whether or not such assumptions on the domains of $A(t)^h$ or of $\phi(t)$ can be eliminated. For example, one could raise the question whether or not the condition ii) can be replaced simply by the smoothness of $A(t)^{-1}$ in t.

In this connection the result of Lions [18] should be mentioned, in which no assumption on $\mathfrak{D}[A(t)^h]$ is made. But he obtains only generalized solutions, and it is not known how close these solutions are to strict solutions.

It should be remarked that the smoothness of $A(t)^{-1}$ in t alone is not sufficient to ensure the existence of strict solutions of (E) unless the condition i) is assumed, even when $A(t)$ are the infinitesimal generators of contraction semi-groups. This was shown by a counter example in an earlier paper of the author [4].

With a slight modification, this example can be written

$$A(t) = i[1 + (t - x)^{-2}], \tag{5.1}$$

$A(t)$ being a family of multiplication operators in the Hilbert space $\mathfrak{X} = L^2(a, b)$. For convenience we assume that $0 < a < b < T$. In this example, there is no strict solution of the homogeneous equation $du/dt = -A(t)u$ other than $u(t) = 0$. Nevertheless, $A(t)$ are the infinitesimal generators of unitary groups $(iA(t)$

are selfadjoint) and, moreover $A(t)^{-1}$ is holomorphic in t in the whole complex t-plane. Of course the condition ii) is not satisfied, for $\mathfrak{D}[A(t)^h]$ is the set of all $u(x)\in L^2(a, b)$ such that $(t-x)^{-2h}u(x)\in L^2(a, b)$ and this set changes with t, at least for $a\le t\le b$, no matter how small h may be.

It is interesting to note that the same problem becomes well posed if the factor i in (5.1) is dropped, namely

$$A(t)=1+(t-x)^{-2}. \tag{5.2}$$

Then $A(t)$ are themselves selfadjoint and positive: $A(t)\ge 1$, so that the condition i) is satisfied. Again $A(t)^{-1}$ is holomorphic in the whole t-plane but $\mathfrak{D}[A(t)^h]$ is not constant for any $h>0$. In this case, however, the evolution equation (E) is solvable. In fact, the evolution operator exists and is given by

$$U(t, s)u(x)=\begin{cases}\exp[(t-x)^{-1}-(s-x)^{-1}]u(x) & \\ & \text{if } x>t \text{ or } x<s, \\ 0 & \text{if } s\le x\le t.\end{cases} \tag{5.3}$$

The difference in behavior of the evolution equation in these two apparently similar examples lies in the fact that $A(t)$ are the infinitesimal generators of *analytic* semigroups (that is, the condition i) is satisfied) in the second example, while this is not the case with the first. This suggests the possibility that strict solutions of (E) could be obtained without assuming that $\mathfrak{D}[A(t)^h]$ is constant.

A remarkable feature of this example (5.2) is that

$$U(t, s)=0 \qquad \text{if } s\le a \text{ and } t\ge b \tag{5.4}$$

and $U(t, s)\neq 0$ otherwise. This implies that $U(t, s)$ is *not* holomorphic in s or in t. Thus it appears that the constancy of $\mathfrak{D}[A(t)^h]$ is rather essential at least in Theorem II.

Appendix

In this appendix we prove several inequalities needed in the text. These are mainly concerned with the infinitesimal generators $-A$ of analytic semigroups and their fractional powers A^α. Assume that A satisfies the condition i) for $A(t)$ stated in §1. Then A^α can be defined indirectly by

$$A^{-\alpha}=\frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda+A)^{-1}d\lambda, \qquad 0<\alpha<1, \tag{A 1}$$

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

see Kato [5, 6]. The integral on the right of (A1) is absolutely convergent by (1.2) and (1.3), so that $A^{-\alpha} \in \mathbf{B}[\mathfrak{X}]$. $\|A^{-\alpha}\|$ can be estimated easily by α, M_0 and M_1. This proves (1.13).

For the resolvent of A^{α}, we have the expression

$$(\mathrm{A}2) \qquad (z-A^{\alpha})^{-1} = -\frac{1}{2\pi i}\int_C (z-\zeta^{\alpha})^{-1}(\zeta-A)^{-1}d\zeta, \qquad z \notin S_\theta,$$

see [5]. The integration path C can be chosen so as to run in the resolvent set of A from infinity in the lower halfplane to infinity in the upper halfplane, *just inside* the boundary of S_θ (note that $\theta < \pi/2$). This is permitted since (1.2) implies that the resolvent set of A penetrates into S_0 at least by a definite angle determined by M_0, and that an inequality similar to (1.2) holds for $z \notin S_{\theta}$, with a $\theta' < \theta$. This path C can then be deformed to the union C' of the two rays $re^{\pm i\theta'}$, $0 \leq r < \infty$. Then the inequality (1.12) can be proved by using a similarity transformation $\zeta = |z|^{1/\alpha}\zeta'$ of the integration variable.

We next prove the inequality (1.14). To this end we make use of the formula

(A3)

$$A^{\alpha}\exp(-\tau A) = -\frac{1}{2\pi i}\int_C z^{\alpha}e^{-\tau z}(z-A)^{-1}dz, \quad \alpha \geq 0, \quad |\arg\tau| \leq \frac{\pi}{2}-\theta,$$

which is also a simple consequence of the operational calculus with Dunford integrals. Here the path C may be chosen as above; then we have $|\arg \tau z| < \pi/2$ for $z \in C$ and the integral converges. On changing the integration variable by $\tau z = \zeta$, we obtain

$$A^{\alpha}\exp(-\tau A) = -\frac{\tau^{-\alpha-1}}{2\pi i}\int_{C'} \zeta^{\alpha}e^{-\zeta}\left(\frac{\zeta}{\tau}-A\right)^{-1}d\zeta.$$

Here it is convenient to choose the path C' in the following way as the sum of four parts $C_1, \ldots, C_4$. C_1 is the straight line from $\infty e^{-i(\pi/2-\varepsilon)}$ to -1, where ε is a small positive number. C_2 and C_3 are the segments from -1 to 0 and back, making a turn of $+\pi$ at 0. C_4 is the straight line from -1 to $\infty e^{i(\pi/2-\varepsilon)}$. Since $|\arg\tau| \leq \frac{\pi}{2}-\theta$, we have $|\arg \zeta/\tau| \geq \theta - \varepsilon > \theta'$ if ε is sufficiently small, so that $\left\|\left(\frac{\zeta}{\tau}-A\right)^{-1}\right\| \leq \text{const.}\,|\tau/\zeta|$ by the extension of (1.2) stated above. In this way we obtain the estimate (1.14). The choice of the special path C' has the advantage that a *uniform* estimate valid for $0 \leq \alpha \leq \alpha_0$ is thereby

https://doi.org/10.1017/S0027763000002415 Published online by Cambridge University Press

obtained (α_0 is any fixed positive number), for the contributions to the integral from the parts C_2 and C_3 tend to cancel each other for $\alpha \to 0$, which compensates for the factor $|\zeta|^{-1}$ that arises from the use of the estimate $\left\|\left(\frac{\zeta}{\tau} - A\right)^{-1}\right\| \leq \text{const.} |\tau/\zeta|$.

To prove the inequality (1.15), we resort to another expression ($h = 1/m$)

$$A^\alpha \exp(-\tau A) = -\frac{1}{2\pi i}\int_C z^{\alpha m} e^{-\tau z^m}(z - A^h)^{-1} dz, \tag{A 4}$$

which is also a simple consequence of the operational calculus. Here C is similar to the one used above, except that θ' is to be replaced by $h\theta'$. Subtracting from (A 4) a similar expression with A replaced by B and noting that $(z - A^h)^{-1} - (z - B^h)^{-1} = (z - A^h)^{-1}(A^h - B^h)(z - B^h)^{-1}$, we obtain

$$\begin{aligned} A^\alpha \exp(-\tau A) - B^\alpha \exp(-\tau B) \\ = -\frac{1}{2\pi i}\int z^{\alpha m} e^{-\tau z^m}(z - A^h)^{-1}(A^h B^{-h} - 1)B^h(z - B^h)^{-1} dz. \end{aligned} \tag{A 5}$$

Performing the change of the integration variable by $\tau^h z = \zeta$ and then choosing an integration path C' similar to that used above, it is easy to obtain from (A 5) the desired inequality (1.15). Note that $\|(z - A^h)^{-1}\| \leq M_4' |z|^{-1}$ and $\|B^h(z - B^h)^{-1}\| = \|-1 + z(z - B^h)^{-1}\| \leq 1 + M_4'$ for $z \notin S_{h\theta'}$. Again the contributions to the integral from the parts C_2 and C_3 of C' tend to cancel each other for $\alpha \to 0$, so that the estimate is uniform in α for $0 \leq \alpha \leq \alpha_0$.

To prove the inequality (1.16), we start from the identity

$$A^\alpha[\exp(-\tau A) - \exp(-\sigma A)] = -\int_\sigma^\tau A^{\alpha+1} \exp(-zA) dz, \tag{A 6}$$

where the integral may be taken along the segment joining σ and τ. In virtue of (1.14), the norm of the right member of (A 6) is not larger than

$$\begin{aligned} \int_\sigma^\tau \|A^{\alpha+1} \exp(-zA)\| |dz| &\leq M_6 \int_{|\sigma|}^{|\tau|} |z|^{-\alpha-1} d|z| / \sin\theta \\ &= (M_6/\beta \sin\theta) \int_{|\sigma|^\beta}^{|\tau|^\beta} \lambda^{-\alpha/\beta - 1} d\lambda \qquad (|z| = \lambda^{1/\beta}) \\ &\leq M_8 \beta^{-1} |\sigma|^{-\alpha-\beta}(|\tau|^\beta - |\sigma|^\beta) \qquad (M_8 = M_6/\sin\theta) \\ &\leq M_8 \beta^{-1} |\sigma|^{-\alpha-\beta}(|\tau| - |\sigma|)^\beta \\ &\leq M_8 \beta^{-1} |\sigma|^{-\alpha-\beta} |\tau - \sigma|^\beta \end{aligned}$$

with an arbitrary β such that $0 < \beta \leq 1$. This proves (1.16).

Next we prove that

$$(A\,7)\qquad A_n(t)^{-\alpha}\to A(t)^{-\alpha},\qquad \alpha\geq 0,$$

strongly and uniformly in t, where $A_n(t)$ are as in §3. Since $A_n^{-h}\to A^{-h}$ in norm and uniformly in t by (3.10), we have $(\lambda+A_n^h)^{-1}\to(\lambda+A^h)^{-1}$ in the same sense for $\lambda>0$. An application of (A 1) then shows that $A_n(t)^{-\alpha}\to A(t)^{-\alpha}$ at least strongly for $0<\alpha<h$. Hence follows (A 7) for an arbitrary $\alpha\geq 0$ by considering appropriate powers of both sides.

Similarly we can prove

$$(A\,8)\qquad A_n(t)^{\alpha}\exp(-\tau A_n(t))\to A(t)^{\alpha}\exp(-\tau A(t)),$$

$$\alpha\geq 0,\qquad |\arg\tau|\leq\frac{\pi}{2}-\theta,\qquad \tau\neq 0.$$

To this end it is convenient to use (A 5) with $B=A_n$. In view of (3.10), this gives

$$A^{\alpha}\exp(-\tau A)-A_n^{\alpha}\exp(-\tau A_n)$$
$$=-\frac{1}{2\pi in}\int_D z^{\alpha m}e^{-\tau z^m}A^h(z-A^h)^{-1}A_n^h(z-A_n^h)^{-1}dz.$$

Thus (A 8) follows immediately by noting that $\|A^h(z-A^h)^{-1}\|\leq 1+M_4$, $\|A_n^h(z-A_n^h)^{-1}\|\leq 1+N_1$ (see the proof of (1.15) given above). Incidentally, this shows that (A 8) is true in the sense of norm and uniformly in t and τ as long as $|\tau|\geq\delta>0$.

Bibliography

[1] C. Foiaș, G. Gussi and V. Poenaru, L'études de l'équation $du/dt=A(t)u$ pour certaines classes d'opérateurs non bornés de l'espace de Hilbert, Trans. Amer. Math. Soc. **86** (1957), 335-347.

[2] C. Foiaș, G. Gussi and V. Poenaru, Sur les solutions généralisées de certains équations linéaires et quasi linéaires dans l'espace de Banach, Rev. math. pures appl. **3** (1958), 283-304.

[3] T. Kato, Integration of the equation of evolution in a Banach space, J. Math. Soc. Japan **5** (1953), 208-234.

[4] T. Kato, On linear differential equations in Banach spaces, Comm. Pure Appl. Math. **9** (1956), 479-486.

[5] T. Kato, Note on fractional powers of linear operators, Proc. Japan Acad. **36** (1960), 94-96.

[6] T. Kato, Fractional powers of dissipative operators, J. Math. Soc. Japan **13** (1961), in press.

[7] H. Komatsu, Abstract analyticity in time and unique continuation property of solutions of a parabolic equation, J. Fac. Sci. Univ. Tokyo, Section I, Vol 9, Part 1 (1961), 1–11.

[8] M. A. Krasnosel'skii, S. G. Krein and P. E. Sobolevskii, On differential equations with unbounded operators in Banach spaces (Russian), Doklady Akad. Nauk **111** (1956), 19–22.

[9] M. A. Krasnosel'skii, S. G. Krein and P. E. Sobolevskii, On differential equations with unbounded operators in Hilbert space (Russian), Doklady Akad. Nauk **112** (1957), 990–993.

[10] M. A. Krasnosel'skii and P. E. Sobolevskii, Fractional powers of operators acting in Banach spaces (Russian), Doklady Akad. Nauk **129** (1959), 449–502.

[11] S. G. Krein, On correctness classes for certain boundary problems (Russian), Doklady Akad. Nauk **114** (1957), 1162–1165.

[12] S. G. Krein and P. E. Sobolevskii, Differential equation with abstract elliptic operator in Hilbert space (Russian), Doklady Akad. Nauk **118** (1958), 233–236.

[13] O. A. Ladyženskaia, On the solution of operator equations of different types (Russian), Doklady Akad. Nauk **102** (1955), 207-210.

[14] O. A. Ladyženskaia, On the solution of nonstationary operator equations (Russian), Mat. Sb. **39** (81) (1956), 491–524.

[15] O. A. Ladyženskaia, On nonstationary operator equations and their applications to linear problems of mathematical physics (Russian), Mat. Sb. **45** (87) (1958), 123–158.

[16] J. L. Lions, Équations différentielles à coefficients opérateurs non bornés, Bull. soc. math. France **86** (1958), 321–330.

[17] J. L. Lions, Sur certaines équations aux dérivées partielles à coefficients opérateurs non bornés, J. Anal. Math. Israel **6** (1958), 333–355.

[18] J. L. Lions, Équations différentielles du premier ordre dans un espace de Hilbert, Comptes rendus **248** (1959), 1099–1102.

[19] J. L. Lions, Problemi misti nel senso di Hadamard classici e generalizzati, Rendiconti del Seminario Matematico e Fisico di Mirano, Vol. 28, 1959.

[20] J. L. Lions, Problèmes mixtes abstraits, Proc. Internat. Congr. Math. 1958, Cambridge, 1960, pp. 389–397.

[21] S. Mizohata, Le problème de Cauchy pour les équations paraboliques, J. Math. Soc. Japan **8** (1956), 269–299.

[22] W. Mlak, Limitations and dependence on parameter of solutions of non-stationary almost linear differential operator equations, Bull. Acad. Pol. Sci. **6** (1958), 677–682.

[23] W. Mlak, Limitations and dependence on parameter of solutions of non-stationary differential operator equations, Ann. Pol. Math. **6** (1959), 305–322.

[24] W. Mlak, Integration of differential equations with unbounded operators in abstract (L)-spaces, Bull. Acad. Pol. Sci. 8 (1960), 163-168.

[25] R. S. Phillips, Perturbation theory for semi-groups of linear operators, Trans. Amer. Math. Soc. **74** (1954), 199–221.

[26] P. E. Sobolevskii, Approximate methods of solving differential equations in Banach space (Russian), Doklady Akad. Nauk **115** (1957), 240–243.

[27] P. E. Sobolevskii, On equations with operators forming an acute angle (Russian), Doklady Akad. Nauk **116** (1957), 754–757.

[28] P. E. Sobolevskii, Generalized solutions of the first order differential equations in

Hilbert space (Russian), Doklady Akad. Nauk **122** (1958), 994–996.

[29] P. E. Sobolevskii, First order differential equations in Hilbert space with a variable positive-definite self-adjoint operator whose fractional power has a constant domain of definition (Russian), Doklady Akad. Nauk **123** (1958), 984–987.

[30] M. Z. Solomiak, The application of the semigroup theory to the study of differential equations in Banach spaces (Russian), Doklay Akad. Nauk **122** (1958), 766–769.

[31] H. Tanabe, A class of the equations of evolution in a Banach space, Osaka Math. J. **11** (1959), 121-145.

[32] H. Tanabe, Remarks on the equations of evolution in a Banach space, Osaka Math. J. **12** (1960), 145–166.

[33] H. Tanabe, On the equations of evolution in a Banach space, Osaka Math. J. **12** (1960), 363-376.

[34] M. I. Višik, The problem of Cauchy with operators as coefficients, the mixed boundary problem for systems of differential equations and on approximate method for their solution (Russian), Mat. Sb. **39** (81) (1956), 51-148.

[35] M. I. Višik and O. A. Ladyzenskaia, Boundary value problems for partial differential equations and certain classes of operator equations (Russian), Uspehi Mat. Nauk **11**, 6 (72) (1956), 41–97.

[36] K. Yosida, On the differentiabilify of semi-groups of linear operators, Proc. Japan Acad. **34** (1958), 337-340.

[37] K. Yosida, Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them, Proc. Japan Acad. **36** (1960), 86–89.

[38] K. Yosida, On a class of infinitesimal generators and the integration problem of evolution equations, Berkeley Symposium, 1960.

Department of Physics
University of Tokyo