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CONDITIONS FOR SOLVABILITY OF THE HARTMAN–WINTNER PROBLEM IN TERMS OF COEFFICIENTS

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Abstract The Equation (1) $(r(x)y')' = q(x)y(x)$ is regarded as a perturbation of (2) $(r(x)z'(x))' = q_1(x)z(x)$. The functions $r(x)$, $q_1(x)$ are assumed to be continuous real valued, $r(x) > 0$, $q_1(x) \geqslant 0$, whereas $q(x)$ is continuous complex valued. A problem of Hartman and Wintner regarding the asymptotic integration of (1) for large x by means of solutions of (2) is studied. Sufficiency conditions for solvability of this problem expressed by means of coefficients $r(x)$, $q(x)$, $q_1(x)$ of Equations (1) and (2) are obtained.

Keywords: asymptotics of solutions; differential equations of second order; Hartman–Wintner problem

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1. Introduction

In this paper we consider Equations (1.1) and (1.2) with coefficients satisfying conditions (1.3):

$$(r(x)y'(x))' = q(x)y(x), \quad x \in \mathbb{R}_+, \tag{1.1}$$

$$(r(x)z'(x))' = q_1(x)z(x), \quad x \in \mathbb{R}_+, \tag{1.2}$$

$$0 < r(x) \in C^{\text{loc}}(\mathbb{R}_+) \quad \text{for } x \in \mathbb{R}_+, \qquad q(x), q_1(x) \in C^{\text{loc}}(\mathbb{R}_+). \tag{1.3}$$

(The symbol $C^{\text{loc}}(\mathbb{R}_+)$ stands for the set of functions that are continuous for $x \in \mathbb{R}_+$.) In (1.3), $q_1(x)$ is a real-valued function and $q(x)$ may be complex valued. We also assume that Equation (1.2) does not oscillate at infinity. It is known (see [**9**]) that under these assumptions there exist solutions $u_1(x)$ (the principal solution) and $v_1(x)$ (the non-principal solution) and a point $x_0 \in \mathbb{R}_+$ such that $u_1(x) > 0$, $v_1(x) > 0$ for $x \geqslant x_0$, and the following relations hold:

$$r(x)[v_1'(x)u_1(x) - u_1'(x)v_1(x)] = 1, \quad x \geqslant x_0, \tag{1.4}$$

$$\lim_{x\to\infty} \frac{u_1(x)}{v_1(x)} = 0, \qquad \int_{x_0}^{\infty} \frac{\mathrm{d}t}{r(t)u_1^2(t)} = \infty, \qquad \int_{x_0}^{\infty} \frac{\mathrm{d}t}{r(t)v_1^2(t)} < \infty. \tag{1.5}$$

Recently, the following problem has been actively studied. One has to determine under what conditions there exists a fundamental system of solutions (FSS) $\{u(x), v(x)\}$ of Equation (1.1) for which the following relations hold:

$$\lim_{x\to\infty} \frac{u(x)}{u_1(x)} = \lim_{x\to\infty} \frac{v(x)}{v_1(x)} = 1, \tag{1.6}$$

$$r(x)\left[\frac{u'(x)}{u(x)} - \frac{u_1'(x)}{u_1(x)}\right] = o\left(\frac{1}{u_1(x)v_1(x)}\right), \quad x \to \infty, \tag{1.7}$$

$$r(x)\left[\frac{v'(x)}{v(x)} - \frac{v_1'(x)}{v_1(x)}\right] = o\left(\frac{1}{u_1(x)v_1(x)}\right), \quad x \to \infty. \tag{1.8}$$

This type of problem was first studied by Hartman and Wintner (see [**9**, **10**]) and we therefore name it after them (we denote it by problem (1.6)–(1.8)). We say that a Hartman–Wintner problem is solvable if Equation (1.1) has an FSS $\{u(x), v(x)\}$ satisfying (1.6)–(1.8). The study in [**9**, **10**] was continued in [**1**, **2**, **4**, **12**, **14**]. The latest survey of results related to the problem (1.6)–(1.8) can be found in [**4**]. In particular, we need the following assertion from [**4**].

Theorem 1.1. *Problem (1.6)–(1.8) is solvable if any one of the following three conditions hold.*

(1) *The integral $J(x)$ absolutely converges. Here*

$$J(x) = \int_x^{\infty} \Delta q(t)\rho_1(t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+, \tag{1.9}$$

$$\Delta q(x) = q(x) - q_1(x), \quad \rho_1(x) = u_1(x)v_1(x), \quad x \in \mathbb{R}_+. \tag{1.10}$$

(2) *The integral $J(x)$ converges (at least conditionally) and*

$$\int_0^{\infty} |\Delta q(t)|\rho_1(t)A(t)\,\mathrm{d}t < \infty, \quad A(t) = \sup_{s\geqslant t} |J(s)|, \quad t \in \mathbb{R}_+. \tag{1.11}$$

(3) *The integral $J(x)$ converges (at least conditionally) and*

$$\int_0^{\infty} \frac{|J(t)|^2\,\mathrm{d}t}{r(t)\rho_1(t)} < \infty. \tag{1.12}$$

Remark 1.2. In [**4**] there is precise information on the solvability of problem (1.6)–(1.8).

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To state the present problem, we need some preliminary comments. It is convenient to start with some well-known facts (see [**8**, **11**]). The Hartman–Wintner problem is a formalization of one of the methods of investigating an FSS $\{u(x), v(x)\}$ of Equation (1.1) at infinity (for the method of sample or model equations (see [**8**])). In the framework of this method, Equation (1.2) is chosen in such a way that, on the one hand, it would be 'close' to Equation (1.1) (in our case this means that equalities (1.7), (1.8) hold) and, on the other hand, one must show an FSS $\{u_1(x), v_1(x)\}$ for Equation (1.2). Then, if problem (1.6)–(1.8) turns out to be solvable, equalities (1.6) imply asymptotic properties of an FSS of Equation (1.1). We want to emphasize that knowledge of the FSS $\{u_1(x), v_1(x)\}$ is an *a priori* requirement which significantly restricts possibilities for choosing Equation (1.2) (because an FSS is not known for every Equation (1.2)).

Therefore, the following question arises: can one decide whether a Hartman–Wintner problem is solvable looking only at the coefficients $r(x)$, $q(x)$ and $q_1(x)$ of Equations (1.1), (1.2), i.e. without knowing an FSS $\{u_1(x), v_1(x)\}$ of Equation (1.2)? An answer to this question could be useful, for example, for an *a priori* description of the set of equations that make up Equation (1.1), among which one might find an equation convenient for asymptotic integration and similar (in the sense of (1.6)–(1.8)) to Equation (1.1). Then, since the solvability of the Hartman–Wintner problem has already been established *a priori*, looking at the coefficients $r(x)$, $q(x)$ and $q_1(x)$, one can derive from (1.5) asymptotic formulae (as $x \to \infty$) for an FSS $\{u(x), v(x)\}$ of Equation (1.1). That is just what we need.

Note an additional advantage of this approach: we only need to know asymptotics (as $x \to \infty$) of solutions $\{u_1(x), v_1(x)\}$ of Equation (1.2) without knowing their precise values required by Theorem 1.1 (and all other results from [**4**]).

To solve the stated problem, we impose an additional requirement on $q_1(x)$ and add one more assumption in (1.6)–(1.8). We take into account the following well-known assertions (see [**9**]).

(a) The principal solution $u_1(x)$ of Equation (1.2) is determined uniquely up to a constant factor.

(b) All non-principal solutions of (1.2) are asymptotically equivalent as $x \to \infty$. This means that if $v_1(x)$, $v_2(x)$ are non-principal solutions of (1.2), then

$$\frac{v_2(x)}{v_1(x)} \to \text{const.} \quad \text{as } x \to \infty.$$

Therefore, below in a slightly different statement of the Hartman–Wintner problem, we replace an FSS $\{u_1(x), v_1(x)\}$ by another one (we keep the notation $\{u_1(x), v_1(x)\}$ for it) where $u_1(x)$ is a principal solution, and a non-principal solution is chosen with the help of a special procedure. Such a modification does not change our main goal, obtaining asymptotics as $x \to \infty$ of an FSS $\{u(x), v(x)\}$ (this follows from (1.6) and the above arguments).

We turn to precise statements. Let us continue $r(x)$ and $q_1(x)$ to $(-\infty, 0]$ as even functions and keep old notations for them. Throughout what follows we assume that in addition to requirements on $r(x)$, $q(x)$ and $q_1(x)$ given at the beginning of the paper, conditions (1.13) and (1.14) hold:

$$q_1(x) \geqslant 0, \quad x \in \mathbb{R}, \tag{1.13}$$

$$\lim_{|d|\to\infty} \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-d}^{x} q_1(t)\,\mathrm{d}t = \infty. \tag{1.14}$$

We will comment on conditions (1.13)–(1.14) later.

We will systematically use the following main lemma.

Lemma 1.3 (see [6]). *There is an FSS $\{u_1(x), v_1(x)\}$ of Equation (1.2) such that the following relations hold:*

$$v_1(x) > 0, \quad u_1(x) > 0, \quad v_1'(x) > 0, \quad u_1'(x) < 0 \quad \text{for } x \in \mathbb{R}, \tag{1.15}$$

$$r(x)[v_1'(x)u_1(x) - u_1'(x)v_1(x)] \equiv 1 \quad \text{for } x \in \mathbb{R}, \tag{1.16}$$

$$\lim_{x\to-\infty} \frac{v_1(x)}{u_1(x)} = \lim_{x\to\infty} \frac{u_1(x)}{v_1(x)} = 0, \tag{1.17}$$

$$\int_{-\infty}^{0} \frac{\mathrm{d}t}{r(t)v_1^2(t)} = \int_{0}^{\infty} \frac{\mathrm{d}t}{r(t)u_1^2(t)} = \infty, \tag{1.18}$$

$$\int_{0}^{\infty} \frac{\mathrm{d}t}{r(t)v_1^2(t)} < \infty, \qquad \int_{-\infty}^{0} \frac{\mathrm{d}t}{r(t)u_1^2(t)} < \infty. \tag{1.19}$$

From Lemma 1.1 it follows (see [**6**]) that $v_1(x)$ is a principal solution of (1.2) on $(-\infty, 0]$, $u_1(x)$ is a principal solution on $[0, \infty)$, $u_1(x)$ is a non-principal solution on $(-\infty, 0]$, and $v_1(x)$ is a non-principal solution on $[0, \infty)$.

We now can formulate our main problem. Let $\{u_1(x), v_1(x)\}$ be an FSS of Equation (1.2) with properties (1.15)–(1.19). One has to determine under what requirements on $r(x)$, $q(x)$ and $q_1(x)$ Equation (1.1) has an FSS $\{u(x), v(x)\}$ satisfying relations (1.6)–(1.8). (Note that a similar problem (with $r(x) \equiv 1$) was first studied in [**3**], and the present paper can be viewed as a continuation of [**3**].)

We now return to conditions (1.13), (1.14). These requirements were introduced without necessary comments because they appeared for purely technical and not conceptual reasons. To be more precise, under assumptions (1.13), (1.14) we know sharp-by-order two-sided estimates for the function $\rho_1(x) = u_1(x)v_1(x)$, $x \in \mathbb{R}$, in terms of some auxiliary functions in $r(x)$ and $q_1(x)$ (see [**6**] and § 2 below). Note that to apply Theorem 1.1 (and any other statement from [**4**] concerning the solvability of problem (1.6)–(1.8)), one has to know the function $\rho_1(x)$ (estimates are not sufficient).

Thus inequalities for $\rho_1(x)$ given in [**4**] play an essential role in solving our problem. To conclude, note that in final asymptotic formulae for an FSS $\{u(x), v(x)\}$ of Equation (1.1) we do not present estimates for remainder terms since our goal is to obtain assertions which might be useful for *a priori* analysis of an FSS of (1.1) as $x \to \infty$. However, one can derive such estimates from [**4**] if needed.

2. Preliminaries

In this section we present some assertions which will be used in the proofs of our main results (see § 3). Below c stands for absolute positive constants which are not essential for exposition and may be different even within a single chain of computations.

Lemma 2.1 (see [7]). *For an FSS $\{u_1(x), v_1(x)\}$ of Equation (1.2) (see Lemma 1.3), the following relations hold for $x \in \mathbb{R}$ (see (1.10)):*

$$\frac{v_1'(x)}{v_1(x)} = \frac{1 + r(x)\rho_1'(x)}{2r(x)\rho_1(x)}, \quad \frac{u_1'(x)}{u_1(x)} = -\frac{1 - r(x)\rho_1'(x)}{2r(x)\rho_1(x)}, \quad x \in \mathbb{R}, \tag{2.1}$$

$$r(x)|\rho_1'(x)| < 1, \quad x \in \mathbb{R}. \tag{2.2}$$

Fix $x \in \mathbb{R}$ and consider the following equations in $d \geqslant 0$:

$$1 = \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-d}^{x} q(t)\,\mathrm{d}t, \qquad 1 = \int_{x}^{x+d} \frac{\mathrm{d}t}{r(t)} \int_{x}^{x+d} q(t)\,\mathrm{d}t. \tag{2.3}$$

Lemma 2.2 (see [6]). *For $x \in \mathbb{R}$ each of the Equations (2.1) has a unique finite positive solution.*

Denote by $d_1(x)$, $d_2(x)$ the respective solutions of Equations (2.3). We introduce functions

$$\left.\begin{aligned} \varphi(x) &= \int_{x-d_1(x)}^{x} \frac{\mathrm{d}t}{r(t)}, \quad \psi(x) = \int_{x}^{x+d_2(x)} \frac{\mathrm{d}t}{r(t)}, \quad x \in \mathbb{R}, \\ h(x) &= \frac{\varphi(x)\psi(x)}{\varphi(x) + \psi(x)} \equiv \left(\int_{x-d_1(x)}^{x+d_2(x)} q(t)\,\mathrm{d}t\right)^{-1}, \quad x \in \mathbb{R}. \end{aligned}\right\} \tag{2.4}$$

Theorem 2.3 (see [6]). *The function $\rho_1(x) = u_1(x)v_1(x)$, $x \in \mathbb{R}$, satisfies the inequalities*

$$2^{-1}h(x) \leqslant \rho_1(x) \leqslant 2h(x), \quad x \in \mathbb{R}. \tag{2.5}$$

Theorem 2.4 (see [7]). *Denote*

$$m = \sup_{x \in \mathbb{R}} r(x)|\rho'(x)|. \tag{2.6}$$

The inequality $m < 1$ holds if and only if

$$c^{-1} \leqslant \frac{\varphi(x)}{\psi(x)} \leqslant c, \quad x \in \mathbb{R}. \tag{2.7}$$

Fix $x \in \mathbb{R}$ and consider the following equation in $d \geqslant 0$:

$$1 = \int_{x-d}^{x+d} \frac{\mathrm{d}t}{r(t)h(t)}. \tag{2.8}$$

Lemma 2.5 (see [5]). *For every $x \in \mathbb{R}$, (2.8) has a unique finite positive solution. Denote it by $d(x)$. The function $d(x)$ is continuous for $x \in \mathbb{R}$. Moreover, for every $x \in \mathbb{R}$ and $\varepsilon \in [0, 1]$, we have*

$$(1 - \varepsilon)d(x) \leqslant d(t) \leqslant (1 + \varepsilon)(d(x) \quad \text{for } |t - x| \leqslant \varepsilon d(x). \tag{2.9}$$

Lemma 2.6 (see [6]). *An FSS* $\{u_1(x), v_1(x)\}$ *of (1.2) (see Lemma 1.3) satisfies the following inequalities for every* $x \in \mathbb{R}$*:*

$$\left.\begin{aligned} &\mathrm{e}^{-2} \leqslant \frac{v_1(t)}{v_1(x)} \leqslant \mathrm{e}^2, \qquad \mathrm{e}^{-2} \leqslant \frac{u_1(t)}{u_1(x)} \leqslant \mathrm{e}^2 \qquad \text{for } |t-x| \leqslant d(x), \\ &\mathrm{e}^{-2} \leqslant \frac{\rho_1(t)}{\rho_1(x)} \leqslant \mathrm{e}^2, \qquad (2\mathrm{e})^{-2} \leqslant \frac{h(t)}{h(x)} \leqslant (2\mathrm{e})^2 \qquad \text{for } |t-x| \leqslant d(x). \end{aligned}\right\} \tag{2.10}$$

Definition 2.7 (see [6]). We say that segments $\{\Delta_n\}_{n=1}^{\infty}$ centred at $\{x_n\}_{n=1}^{\infty}$ form an $R(x, d(\cdot))$-covering of $[x, \infty)$ if relations (1)–(3) hold.

(1) $\Delta_n = [\Delta_n^-, \Delta_n^+] \overset{\text{def}}{=} [x_n - d(x_n), x_n + d(x_n)]$, $n = 1, 2, \dots$.

(2) $\Delta_{n+1}^- = \Delta_n^+$, $n = 1, 2, \dots$; $\Delta_1^- = x$.

(3) $\bigcup_{n=1}^{\infty} \Delta_n = [x, \infty)$.

Lemma 2.8 (see [6]). *For every* $x \in \mathbb{R}$ *there exists an* $R(x, d(\cdot))$*-covering of* $[x, \infty)$.

3. Statement of results

In this section we present our main results. The simplest *a priori* condition for the solvability of problem (1.6)–(1.8) is given by Theorem 3.1.

Theorem 3.1. *Consider the following integral (see (1.9) and (2.4)):*

$$H(x) = \int_x^{\infty} |\Delta q(t)| h(t)\, \mathrm{d}t. \tag{3.1}$$

If the integral $H(x)$ *converges, problem (1.6)–(1.8) is solvable.*

Remark 3.2. It is easy to see that Theorem 3.1 in fact repeats condition (1) of Theorem 1.1, and the integral $J(x)$ absolutely converges if and only if $H(x)$ converges (see (2.5)). (By the way, this completes the proof of Theorem 3.1.) Here the assumption on absolute convergence of the integral $J(x)$ is too restrictive, whereas conditional convergence $J(x)$ allows much more freedom in choosing (1.2) corresponding to a given Equation (1.1). See [**12, 14**].

To introduce a condition guaranteeing conditional convergence of $J(x)$ (and to obtain its estimate), we need Definition 3.3 and Lemma 3.4 and Lemma 3.6.

Definition 3.3. We say that a pair of functions $\{r(x), q_1(x)\}$ belongs to a class $\mathcal{K}(\gamma)$ (and write $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$) if there exist positive numbers a, b such that for all $x \gg 1$ the inequalities

$$a^{-1} \leqslant \frac{d(t)}{d(x)} \leqslant a \tag{3.2}$$

hold for $|t - x| \leqslant b d(x)$. Here $\gamma = ab^{-1}$.

Lemma 3.4. *If* $\gamma \geqslant 4$*, for any of pair of functions we have* $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$.

Remark 3.5. Below we systematically use the condition $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma \leqslant \frac{1}{2}$. By Lemma 3.4, such a requirement means that it is not satisfied by all pairs $\{r(x), q_1(x)\}$ but only by a part of the class $\mathcal{K}(4)$. Nevertheless, the class $\mathcal{K}(\frac{1}{2})$ is 'rich' enough and includes pairs $\{r(x), q_1(x)\}$ consisting of smooth as well as of non-differentiable functions (see Theorem 3.3 below and the example in § 5). We also note that by Lemma 3.4 the condition $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma \leqslant \frac{1}{2}$, is neither restrictive nor artificial.

Let us introduce the function $\omega(x, b)$. For a given $b > 0$, we set for every $x \in \mathbb{R}$,

$$\omega(x, b) = \sup_{\xi,\eta\in\mathcal{D}_b(x)} \left| \int_\xi^\eta \Delta q(t)\,\mathrm{d}t \right|, \quad \mathcal{D}_b(x) = [x - bd(x), x + bd(x)]. \tag{3.3}$$

Lemma 3.6. *Let $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma = (a/b) \leqslant \frac{1}{2}$. If the integral*

$$I(x) = \int_x^\infty \frac{\omega(t, b)}{r(t)}\,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{3.4}$$

converges, then the integral $J(x)$ (see (1.9)) converges (at least conditionally), and

$$|J(x)| \leqslant cI(x), \quad x \in \mathbb{R}. \tag{3.5}$$

The next theorem is our main result.

Theorem 3.7. *Let $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma = (a/b) \leqslant \frac{1}{2}$. Then problem (1.6)–(1.8) is solvable provided either of the following two conditions holds:*

$$I(0) < \infty, \quad \int_0^\infty |\Delta q(q)| h(t) I(t)\,\mathrm{d}t < \infty, \tag{3.6}$$

$$I(0) < \infty, \quad \int_0^\infty \frac{I(t)^2\,\mathrm{d}t}{r(t)h(t)} < \infty. \tag{3.7}$$

Here is one more definition.

Definition 3.8. Let $\{u(x), v(x)\}$ and $\{u_1(x), v_1(x)\}$ be FSSs of Equations (1.1) and (1.2), respectively. We say that these FSSs are fully asymptotically equivalent as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{v^{(i)}(x)}{v_1^{(i)}(x)} = \lim_{x\to\infty} \frac{u^{(i)}(x)}{u_1^{(i)}(x)} = 1, \quad i = 0, 1 \quad (f^{(0)}(x) := f(x)). \tag{3.8}$$

Theorem 3.9. *Let problem (1.6)–(1.8) be solvable. Then, if $m < 1$ (see (2.6)), the FSSs of Equations (1.1) and (1.2) are fully asymptotically equivalent as $x \to \infty$.*

Remark 3.10. All the above statements are given in terms of functions $d_1(x)$, $d_2(x)$, $d(x)$, $\psi(x)$, $\varphi(x)$ and $h(x)$. Usually it is impossible to find precise values of these functions. However, to apply Theorems 3.1–3.9, it is enough to have sharp-by-order two-sided estimates of these functions. Usually one can easily obtain such estimates (see [**6**, **7**]) because all the functions are given locally, and to obtain the needed inequalities one can use various local tools of analysis. We give an example of such estimates in the following theorem. We emphasize that this theorem gives only one of the possible and simplest ways for getting the needed inequalities. See [**5**–**7**] for more general examples of this type.

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Theorem 3.11. *Let functions $r(x)$, $q_1(x)$ be continuous and positive for all $x \in \mathbb{R}$, and suppose that there are constants $\beta > 0$, $\alpha \geqslant 1$, $\beta \geqslant (3\alpha)^3$ such that for $x \gg 1$, the inequalities*

$$\frac{1}{\alpha} \leqslant \frac{r(t)}{r(x)}, \qquad \frac{q_1(t)}{q_1(x)} \leqslant \alpha \tag{3.9}$$

hold for

$$|t-x| \leqslant \beta \hat{d}(x), \qquad \hat{d}(x) = \sqrt{\frac{r(x)}{q_1(x)}}.$$

Then for $x \gg 1$, the following estimates hold:

$$\alpha^{-1}\hat{d}(x) \leqslant d_1(x), \qquad d_2(x) \leqslant \alpha \hat{d}(x), \tag{3.10}$$

$$\frac{\alpha^{-2}}{\sqrt{r(x)q_1(x)}} \leqslant \varphi(x), \qquad \psi(x) \leqslant \frac{\alpha^2}{\sqrt{r(x)q_1(x)}}, \tag{3.11}$$

$$\frac{1}{2\alpha^2}\frac{1}{\sqrt{r(x)q_1(x)}} \leqslant h(x) \leqslant \frac{\alpha^2}{2\sqrt{r(x)q_1(x)}}, \tag{3.12}$$

$$\frac{1}{300}\frac{1}{\alpha^3}\hat{d}(x) \leqslant d(x) \leqslant (3\alpha)^3\hat{d}(x). \tag{3.13}$$

4. Proofs

In this section we prove Theorems 3.7–3.11 and Lemmas 3.4 and 3.6. Recall that the proof of Theorem 3.1 is given in Remark 3.2.

Proof of Lemma 3.4. From (2.9) it follows that for any $\varepsilon \in [0,1]$ we have

$$1-\varepsilon \leqslant \frac{d(t)}{d(x)} \leqslant 1+\varepsilon \leqslant \frac{1}{1-\varepsilon}, \quad |t-x| \leqslant \varepsilon d(x), \quad x \in \mathbb{R}. \tag{4.1}$$

According to (4.1), we get $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma = a/b$, where $a = (1-\varepsilon)^{-1}$, $b = \varepsilon$. Hence

$$\gamma = \frac{1}{(1-\varepsilon)\varepsilon} = \gamma(\varepsilon), \quad \varepsilon \in [0,1].$$

It is easy to see that on the segment $(0, \frac{1}{2}]$ the function $\gamma(\varepsilon)$ monotonically decreases from ∞ to 4 and is continuous, $\gamma(\varepsilon)$ monotonically increases from 4 to ∞ on the segment $[\frac{1}{2}, 1)$ and is continuous. Hence for any $\gamma_0 \geqslant 4$ there is $\varepsilon = \varepsilon_0 \in (0,1)$ such that $\gamma(\varepsilon_0) = \gamma_0$. This means that for $a = (1-\varepsilon_0)^{-1}$, $b = \varepsilon_0$ the pair $\{r(x), q_1(x)\}$ belongs to $\mathcal{K}(\gamma_0)$, $\gamma_0 = a/b$. □

Proof of Lemma 3.6. To prove the lemma, we need some auxiliary assertions.

Lemma 4.1. *Let $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma = (a/b) \leqslant \frac{1}{2}$. Then for all $x \in \mathbb{R}$ and every $t \in \Delta(x) = [x-d(x), x+d(x)]$ we have an inclusion*

$$\Delta(x) \subseteq \mathcal{D}_b(t) = [t - bd(t), t + bd(t)]. \tag{4.2}$$

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Proof. Let $t \in \Delta(x)$. Then from (3.2) it follows that

$$\frac{d(t)}{d(x)} \geqslant \frac{1}{a} \geqslant \frac{2}{b} \;\Rightarrow\; bd(t) \geqslant 2d(x) \geqslant d(x) + t - x \;\Rightarrow\; x - d(x) \geqslant t - bd(t),$$
$$\frac{d(t)}{d(x)} \geqslant \frac{1}{a} \geqslant \frac{2}{b} \;\Rightarrow\; bd(t) \geqslant 2d(x) \geqslant x - t + d(x) \;\Rightarrow\; t + bd(t) \geqslant x + d(x).$$

□

Lemma 4.2. *Let* $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma = (a/b) \leqslant \frac{1}{2}$. *Then for any* $\xi, \eta, t \in \Delta(x) = [x - d(x), x + d(x)]$ *and every* $x \in \mathbb{R}$, *we have*

$$\left| \int_\xi^\eta \Delta q(s)\,\mathrm{d}s \right| \leqslant \omega(t, b) = \sup_{\xi,\eta \in \mathcal{D}_b(t)} \left| \int_\xi^\eta \Delta q(s)\,\mathrm{d}s \right|. \tag{4.3}$$

Proof. By Lemma 4.1, for $\xi, \eta, t \in \Delta(x)$ we have the inclusions

$$[\xi, \eta] \subseteq [x - d(x), x + d(x)] \subseteq [t - bd(t), t + bd(t)]$$

for any $t \in \Delta(x)$. Hence inequality (4.3) holds. □

Lemma 4.3. *Let* $x \in \mathbb{R}$, $\xi, \eta \in \Delta(x) = [x - d(x), x + d(x)]$ *and* $\{r(x), q_1(x)\} \in \mathcal{K}(\gamma)$, $\gamma = (a/b) \leqslant \frac{1}{2}$. *Then*

$$\left| \int_\xi^\eta \Delta q(s)\rho_1(s)\,\mathrm{d}s \right| \leqslant c \int_{\Delta(x)} \frac{\omega(t, b)}{r(t)}\,\mathrm{d}t. \tag{4.4}$$

Proof. Integrating by parts, we get

$$\begin{aligned} T(\xi, \eta) &\overset{\text{def}}{:=} \int_\eta^\xi \Delta q(t)\rho_1(t)\,\mathrm{d}t \\ &= \rho_1(\xi) \int_\eta^\xi \Delta q(s)\,\mathrm{d}s - \int_\eta^\xi \rho_1'(t)\left(\int_\eta^t \Delta q(s)\,\mathrm{d}s \right)\mathrm{d}t. \end{aligned} \tag{4.5}$$

We estimate the summands in the right-hand side of (4.5) separately. In the following relations we use (2.10), (2.8), (2.5):

$$\begin{aligned} A(\xi, \eta) &\overset{\text{def}}{:=} \rho_1(\xi) \left| \int_\eta^\xi \Delta q(s)\,\mathrm{d}s \right| = \rho_1(\xi) \left| \int_\eta^\xi \Delta q(s)\,\mathrm{d}s \right| \int_{\Delta(x)} \frac{\mathrm{d}t}{r(t)h(t)} \\ &\leqslant c\frac{\rho_1(x)}{h(x)} \left| \int_\eta^\xi \Delta q(s)\,\mathrm{d}s \right| \int_{\Delta(x)} \frac{\mathrm{d}t}{r(t)} \leqslant c \left| \int_\eta^\xi \Delta q(s)\,\mathrm{d}s \right| \int_{\Delta(x)} \frac{\mathrm{d}t}{r(t)}. \end{aligned} \tag{4.6}$$

From (4.3) for $t \in \Delta(x)$, we get

$$\left| \int_\eta^\xi \Delta q(s)\,\mathrm{d}s \right| \frac{1}{r(t)} \leqslant \frac{\omega(t, b)}{r(t)} \;\Rightarrow\; \left| \int_\eta^\xi \Delta q(s)\,\mathrm{d}s \right| \int_{\Delta(x)} \frac{\mathrm{d}t}{r(t)} \leqslant \int_{\Delta(x)} \frac{\omega(t, b)\,\mathrm{d}t}{r(t)}.$$

We have thus obtained an estimate of $A(\xi,\eta)$ for $\xi,\eta\in\Delta(x)$:

$$A(\xi,\eta)=\rho_1(\xi)\biggl|\int_\eta^\xi \Delta q(s)\,\mathrm{d}s\biggr|\leqslant c\int_{\Delta(x)}\frac{\omega(t,b)\,\mathrm{d}t}{r(t)}. \tag{4.7}$$

In the following relations, we use (2.2) and (4.3):

$$\begin{aligned}B(\xi,\eta)&\overset{\text{def}}{:=}\biggl|\int_\eta^\xi \rho_1'(t)\biggl(\int_\eta^t \Delta q(s)\,\mathrm{d}s\biggr)\mathrm{d}t\biggr|\leqslant\int_\eta^\xi|\rho_1'(t)|\biggl|\int_\eta^t\Delta q(s)\,\mathrm{d}s\biggr|\,\mathrm{d}t\\&\leqslant\int_\eta^\xi\frac{1}{r(t)}\biggl|\int_\eta^t\Delta q(s)\,\mathrm{d}s\biggr|\,\mathrm{d}t\leqslant\int_\eta^\xi\frac{\omega(t,b)\,\mathrm{d}t}{r(t)}\leqslant\int_{\Delta(x)}\frac{\omega(t,b)\,\mathrm{d}t}{r(t)}.\end{aligned} \tag{4.8}$$

From (4.7) and (4.8) we obtain (4.4). □

We now turn to the proof of Lemma 3.6. To check the convergence of the integral $J(0)$, for given $\varepsilon>0$, we have to find $N_0(\varepsilon)$ such that for all $M_1,M_2\geqslant N_0(\varepsilon)$ and $M_2\geqslant M_1$ the following inequality holds:

$$\biggl|\int_{M_1}^{M_2}\Delta q(t)\rho_1(t)\,\mathrm{d}t\biggr|\leqslant\varepsilon. \tag{4.9}$$

Let the segments $\{\Delta_n\}_{n=1}^\infty$ form an $R(M_1,d(\cdot))$-covering of $[M_1,\infty)$. Then $M_1=\Delta_1^-$, and there is n such that $M_2\in\Delta_n$. If $n=1$, then $M_2\in\Delta_1$, and from (4.4) it follows that

$$\biggl|\int_{M_1}^{M_2}\Delta q(t)\rho_1(t)\,\mathrm{d}t\biggr|\leqslant c\int_{\Delta_1}\frac{\omega(t,b)}{r(t)}\,\mathrm{d}t\leqslant c\int_{M_1}^\infty\frac{\omega(t,b)\,\mathrm{d}t}{r(t)}. \tag{4.10}$$

Since $I(x)$ converges, we have $I(x)\leqslant\varepsilon$ for $x\geqslant x_0(\varepsilon)\gg1$. Therefore, for $N_0(\varepsilon)=x_0(\varepsilon)$ we obtain (4.9). Let $M_2\in\Delta_n$, $n\geqslant2$. Then from the properties of $R(M_1,d(\cdot))$-coverings and (4.4) it follows that

$$\begin{aligned}\biggl|\int_{M_1}^{M_2}\Delta q(t)\rho_1(t)\,\mathrm{d}t\biggr|&=\biggl|\sum_{k=1}^{n-1}\int_{\Delta_k}\Delta q(t)\rho_1(t)\,\mathrm{d}t+\int_{\Delta_n^-}^{M_2}\Delta q(t)\rho_1(t)\,\mathrm{d}t\biggr|\\&\leqslant\sum_{k=1}^{n-1}\biggl|\int_{\Delta_k}\Delta q(t)\rho_1(t)\,\mathrm{d}t\biggr|+\biggl|\int_{\Delta_n^-}^{M_2}\Delta q(t)\rho_1(t)\,\mathrm{d}t\biggr|\\&\leqslant c\sum_{k=1}^{n-1}\int_{\Delta_k}\frac{\omega(t,b)}{r(t)}\,\mathrm{d}t+c\int_{\Delta_n}\frac{\omega(t,b)\,\mathrm{d}t}{r(t)}\\&=c\int_{M_1}^{\Delta_n^+}\frac{\omega(t,b)\,\mathrm{d}t}{r(t)}\\&\leqslant c\int_{M_1}^\infty\frac{\omega(t,b)\,\mathrm{d}t}{r(t)}.\end{aligned}$$

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Since $I(x)$ converges, for $N_0(\varepsilon) = x_0(\varepsilon)$ (see above) we get (4.9). Hence $J(x)$ converges (at least conditionally). Let us verify estimate (3.5). Below for a given $x \in \mathbb{R}$ we use the properties of an $R(x, d(\cdot))$-covering of $[x, \infty)$ and (4.4):

$$\begin{aligned}|J(x)| &= \left|\int_x^\infty \Delta q(t)\rho_1(t)\,\mathrm{d}t\right| = \left|\sum_{k=1}^\infty \int_{\Delta_k} \Delta q(t)\rho_1(t)\,\mathrm{d}t\right| \leqslant \sum_{k=1}^\infty \left|\int_{\Delta_k} \Delta q(t)\rho_1(t)\,\mathrm{d}t\right| \\ &\leqslant c\sum_{k=1}^\infty \int_{\Delta_k} \frac{\omega(t,b)\,\mathrm{d}t}{r(t)} = c\int_x^\infty \frac{\omega(t,b)\,\mathrm{d}t}{r(t)} = cI(x).\end{aligned}$$

□

Proof of Theorem 3.7. The assertions of Theorem 3.7 immediately follow from Lemma 3.6 and Theorem 1.1. □

Proof of Theorem 3.9. Let us verify the equalities

$$\lim_{x\to\infty} \frac{v(x)}{v_1(x)} = 1, \qquad \lim_{x\to\infty} \frac{v'(x)}{v_1(x)} = 1. \tag{4.11}$$

Equalities (3.8) for $u(x)$ and $v(x)$ are proved in a similar way. Since problem (1.6)–(1.8) is solvable, the first equality in (4.11) coincides with (1.6). We write (1.8) in the form

$$\frac{v'(x)}{v(x)} = \frac{v_1'(x)}{v_1(x)} + \frac{\varepsilon(x)}{r(x)\rho_1(x)}, \qquad \lim_{x\to\infty} \varepsilon(x) = 0. \tag{4.12}$$

From (4.12), also using Lemma 1.3 and (2.1), (2.2), we obtain

$$\begin{aligned}\frac{v'(x)}{v_1'(x)} &= \frac{v(x)}{v_1(x)} + \frac{v(x)}{v_1'(x)}\frac{\varepsilon(x)}{r(x)\rho_1(x)} = \frac{v(x)}{v_1(x)} + \frac{v_1(x)}{v_1'(x)}\frac{v(x)}{v_1(x)}\frac{\varepsilon(x)}{r(x)\rho_1(x)} \\ &= \frac{v(x)}{v_1(x)}\left[1 + \frac{2r(x)\rho_1(x)}{1+r(x)\rho_1'(x)}\frac{\varepsilon(x)}{r(x)\rho_1(x)}\right] = \frac{v(x)}{v_1(x)}\left[1 + \frac{2\varepsilon(x)}{1+r(x)\rho_1'(x)}\right].\end{aligned} \tag{4.13}$$

Since $m < 1$ (see (2.6)), we have

$$0 \leqslant \frac{2|\varepsilon(x)|}{1+r(x)\rho_1'(x)} \leqslant \frac{2|\varepsilon(x)|}{1-m} \;\Rightarrow\; \lim_{x\to\infty} \frac{\varepsilon(x)}{1+r(x)\rho_1'(x)} = 0. \tag{4.14}$$

The second equality in (4.11) now follows from (4.14), (4.13) and the first equality of (4.11). □

Proof of Theorem 3.11. Below we will need Lemma 4.4

Lemma 4.4. *Let $x \in \mathbb{R}$. The inequality $\eta \geqslant d_1(x)$ $(0 < \eta \leqslant d_1(x))$ holds if and only if*

$$F(\eta) \overset{\text{def}}{=} \int_{x-\eta}^x \frac{\mathrm{d}t}{r(t)} \int_{x-\eta}^x q(t)\,\mathrm{d}t \geqslant 1 \quad (F(\eta) \leqslant 1). \tag{4.15}$$

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Proof.

Necessity. Let $\eta \geqslant d_1(x)$. Then $[x-d_1(x),x] \subseteq [x-\eta,x]$. Since $F(\eta)$ does not decrease, we have

$$F(\eta) = \int_{x-\eta}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-\eta}^{x} q(t)\,\mathrm{d}t \geqslant \int_{x-d_1(x)}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-d_1(x)}^{x} q(t)\,\mathrm{d}t = 1.$$

Sufficiency. Assume the contrary, i.e. (4.15) holds but $0<\eta<d_1(x)$. Then $[x-\eta,x] \subseteq [x-d_1(x),x_1]$ and therefore

$$1 \leqslant \int_{x-\eta}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-\eta}^{x} q(t)\,\mathrm{d}t \leqslant \int_{x-d_1(x)}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-d_1(x)}^{x} q(t)\,\mathrm{d}t = 1.$$

Then $F(\eta)=1$, and by Lemma 2.2 we get $\eta = d_1(x)$, a contradiction. □

Below we prove inequalities (3.10) for $d_1(x)$. Estimates (3.10) for $d_2(x)$ can be proved in a similar way. Let $\eta = \alpha\hat{d}(x)$. From (3.9) it follows that

$$\int_{x-\eta}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-\eta}^{x} q(t)\,\mathrm{d}t = \int_{x-\eta}^{x} \frac{r(x)}{r(t)}\frac{\mathrm{d}t}{r(x)} \int_{x-\eta}^{x} \frac{q_1(t)}{q_1(x)} q_1(x)\,\mathrm{d}t \geqslant \frac{1}{\alpha^2}\frac{q(x)}{r(x)}\eta^2 = 1.$$

By Lemma 4.4, we get $d_1(x) \leqslant \alpha\hat{d}(x)$.

We now set $\eta = \alpha^{-1}\hat{d}(x)$. Using (3.9) once again, we get

$$\int_{x-\eta}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-\eta}^{x} q_1(t)\,\mathrm{d}t = \int_{x-\eta}^{x} \frac{r(x)}{r(t)}\frac{\mathrm{d}t}{r(x)} \int_{x-\eta}^{x} \frac{q(t)}{q(x)} q(x)\,\mathrm{d}t \leqslant \alpha^2\frac{q_1(x)}{r(x)}\eta^2 = 1.$$

Using Lemma 4.4 once more, we obtain $d_1(x) \geqslant \alpha^{-1}\hat{d}(x)$. From (3.9) and (3.10) we now obtain estimates (3.11) for $\varphi(x)$:

$$\varphi(x) = \int_{x-d_1(x)}^{x} \frac{\mathrm{d}t}{r(t)} \leqslant \int_{x-\alpha\hat{d}(x)}^{x} \frac{r(x)}{r(t)}\frac{\mathrm{d}t}{r(x)} \leqslant \alpha^2\frac{\hat{d}(x)}{r(x)} = \frac{\alpha^2}{\sqrt{r(x)q_1(x)}},$$
$$\varphi(x) = \int_{x-d_1(x)}^{x} \frac{\mathrm{d}t}{r(t)} \geqslant \int_{x-\alpha^{-1}(x)}^{x} \frac{r(x)}{r(t)}\frac{\mathrm{d}t}{r(x)} \geqslant \frac{1}{\alpha^2}\frac{\hat{d}(x)}{r(x)} = \frac{\alpha^{-2}}{\sqrt{r(x)q_1(x)}}.$$

From (2.4) and (3.11) we get inequalities (3.12). To prove (3.13), note that (2.10) and (2.8) imply

$$\frac{1}{36}\int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)} \leqslant h(x) \leqslant 36\int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)}. \tag{4.16}$$

Suppose now that

$$d(x) \leqslant \frac{1}{300}\frac{\hat{d}(x)}{\alpha^3} := \gamma\hat{d}(x).$$

Then from (3.12), (3.9) and (4.16) we obtain

$$\begin{aligned}\frac{1}{2\alpha^2}\frac{1}{\sqrt{r(x)q_1(x)}} &\leqslant h(x) \leqslant 36\int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)} \leqslant 36\int_{x-\gamma\hat{d}(x)}^{x+\gamma\hat{d}(x)} \frac{r(x)}{r(t)}\frac{\mathrm{d}t}{r(x)} \\ &\leqslant 72\alpha\gamma\frac{\hat{d}(x)}{r(x)} = \frac{72\alpha\gamma}{\sqrt{r(x)q_1(x)}} \;\Rightarrow\; 1 \leqslant 144\alpha^3\gamma < \tfrac{1}{2},\end{aligned}$$

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a contradiction. Hence

$$d(x) \geqslant \frac{\hat{d}(x)}{300\alpha^3}.$$

Suppose now that $d(x) \geqslant \gamma\hat{d}(x)$, $\gamma = 27\alpha^3$. Then from (3.12), (3.9) and (4.16) we get

$$\frac{\alpha^2}{2}\frac{1}{r(x)q_1(x)} \geqslant h(x) \geqslant \frac{1}{36}\int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)} \geqslant \frac{1}{36}\int_{x-\gamma\hat{d}(x)}^{x+\gamma\hat{d}(x)} \frac{r(x)}{r(t)}\frac{\mathrm{d}t}{r(x)}$$
$$\geqslant \frac{2\gamma}{36\alpha}\frac{\hat{d}(x)}{r(x)} = \frac{\gamma}{18\alpha}\frac{1}{\sqrt{r(x)q_1(x)}} \;\Rightarrow\; 1 \geqslant \frac{\gamma}{9\alpha^3} = 3,$$

a contradiction. Hence $d(x) \leqslant (3\alpha)^3\hat{d}(x)$. □

5. Example

Consider Equations (1.1) and (1.2) with coefficients

$$r(x) = (1+x^2)^2, \quad q(x) = \mathrm{e}^{2x} + (1+x^2)\mathrm{e}^{\theta x}\cos \mathrm{e}^{\gamma x}, \quad q_1(x) = \mathrm{e}^{2x}, \quad x \geqslant 0. \tag{5.1}$$

Our goal is to determine for what $\theta \geqslant 0$, $\gamma \geqslant 0$ problem (1.6)–(1.8) is solvable using the results of § 3. We have to estimate auxiliary functions. In this particular case, such inequalities can be easily obtained from Theorem 3.11. From (5.1) it follows that

$$\hat{d}(x) = \sqrt{\frac{r(x)}{q_1(x)}} = \frac{1+x^2}{\mathrm{e}^x} \to 0 \quad \text{as } x \to \infty. \tag{5.2}$$

Let $\beta = 30$, $\alpha = \frac{1}{3}\sqrt[3]{30}$. Then $\beta = (3\alpha)^3$, and by (5.2) there is $x_0 \gg 1$ such that for $x \geqslant x_0$ we have (3.9). Then by Theorem 3.4 we have estimates (3.10)–(3.13). Since $\alpha < 2$, we can replace these estimates by more rough inequalities (5.3)–(5.6) where the constants are more convenient:

$$\frac{1+x^2}{2\mathrm{e}^x} \leqslant d_1(x), \quad d_2(x) \leqslant 2\frac{1+x^2}{\mathrm{e}^x}, \qquad x \gg 1, \tag{5.3}$$
$$\frac{1}{4(1+x^2)\mathrm{e}^x} \leqslant \varphi(x), \qquad \psi(x) \leqslant \frac{4}{(1+x^2)\mathrm{e}^x}, \quad x \gg 1, \tag{5.4}$$
$$\frac{1}{8(1+x^2)\mathrm{e}^x} \leqslant h(x) \leqslant \frac{2}{(1+x^2)\mathrm{e}^x}, \quad x \gg 1, \tag{5.5}$$
$$\frac{1}{2400}\frac{1+x^2}{\mathrm{e}^x} \leqslant d(x) \leqslant 30\frac{1+x^2}{\mathrm{e}^x}, \qquad x \gg 1. \tag{5.6}$$

It is easy to verify (see below) that the replacement of inequalities (3.10)–(3.13) with more rough estimates (5.3)–(5.6) does not change the final results. Estimate (5.5) makes it possible to apply Theorem 3.1. For $x \gg 1$ we have

$$H(x) = \int_x^\infty |\Delta q(t)|h(t)\,\mathrm{d}t \leqslant c\int_x^\infty (1+t^2)\mathrm{e}^{\theta t}\frac{|\cos(\mathrm{e}^{\gamma t})|}{(1+t^2)\mathrm{e}^t}\,\mathrm{d}t \leqslant c\int_x^\infty \mathrm{e}^{(\theta-1)t}\,\mathrm{d}t.$$

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Hence problem (1.6)–(1.8) is solvable for $\theta < 1$. Theorem 3.7 and condition (1) of Theorem 1.1 give restrictions on θ. Fix $a \geqslant 1$ and $b > 0$. By (5.2), for given a, b there is $x_1(a,b)$ such that for $x \geqslant x_1(a,b)$ we have

$$\frac{1}{a} \leqslant \frac{\hat{d}(t)}{\hat{d}(x)} \leqslant a \quad \text{for } |t-x| \leqslant 30b\hat{d}(x). \tag{5.7}$$

Recall that for $x \geqslant x_0$ estimates (5.3)–(5.6) hold. Then for $x \geqslant \max\{x_0, x_1(a,b)\} \overset{\text{def}}{=} x_2$ we have inclusions

$$\Delta(x) = [x - bd(x), x + bd(x)] \subseteq [x - 30b\hat{d}(x), x + 30b\hat{d}(x)] \overset{\text{def}}{=} \tilde{\mathcal{D}}_b(x). \tag{5.8}$$

Then for $t \in \Delta(x)$ and $x \geqslant x_2$ the following relations hold (see (5.6)–(5.8)):

$$\frac{d(t)}{d(x)} = \frac{d(t)}{\hat{d}(t)} \frac{\hat{d}(t)}{\hat{d}(x)} \frac{\hat{d}(x)}{d(x)} \leqslant (ca)^{\pm 1}, \quad |t-x| \leqslant bd(x). \tag{5.9}$$

The constant c in (5.9) is determined by estimates (5.6), and its value is not essential. For a fixed a we now choose b big enough to satisfy the inequality $\gamma = (ca/b) \leqslant \frac{1}{2}$. This means that in the case (5.1) the following inclusion holds:

$$\{(1+x^2)^2, \mathrm{e}^{2x}\} \in \mathcal{K}(\gamma), \quad \gamma \leqslant \tfrac{1}{2}. \tag{5.10}$$

Thus, having (5.10) at our disposal, we can apply Theorem 3.7. By (5.8), for $x \geqslant x_2$ we have

$$\omega(x,b) = \sup_{\xi,\eta \in \mathcal{D}_b(x)} \left| \int_\xi^\eta \Delta q(t)\,\mathrm{d}t \right| \leqslant \sup_{\xi,\eta \in \tilde{\mathcal{D}}_b(x)} \left| \int_\xi^\eta \Delta q(\xi)\,\mathrm{d}\xi \right|. \tag{5.11}$$

Now we use the second mean theorem (see [**13**, Chapter XII, § 12.3]). In our case this theorem can be applied for $x \gg 1$ and the inequalities

$$c^{-1} \leqslant \frac{1+t^2}{1+x^2} \leqslant c, \quad c^{-1} \leqslant \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2x}} \leqslant c \quad \text{for } |t-x| \leqslant 30b\hat{d}(x), \quad x \geqslant x_2, \tag{5.12}$$

which follows from (5.2), we obtain for $x \geqslant x_2$

$$\begin{aligned} \tilde{\omega}_b(x) &\overset{\text{def}}{=} \sup_{\xi,\eta \in \tilde{\mathcal{D}}_b(x)} \left| \int_\xi^\eta (1+t^2)\mathrm{e}^{(\theta-\gamma)t}[\mathrm{e}^{\gamma t} \cos \mathrm{e}^{\gamma t}]\mathrm{d}t \right| \\ &\leqslant c(1+x^2)\mathrm{e}^{(\theta-\gamma)x} \sup_{\xi,\eta \in \tilde{\mathcal{D}}_b(x)} \left| \int_\xi^\eta \mathrm{e}^{\gamma t} \cos \mathrm{e}^{\gamma t}\,\mathrm{d}t \right| \\ &\leqslant c(1+x^2)\mathrm{e}^{(\theta-\gamma)x}. \end{aligned} \tag{5.13}$$

Below we use estimate (5.13) (see (3.4)):

$$\begin{aligned} I(x) &= \int_x^\infty \frac{\omega(t,b)\,\mathrm{d}t}{r(t)} \leqslant c \int_x^\infty \frac{\tilde{\omega}(t,b)}{r(t)}\,\mathrm{d}t \leqslant c \int_x^\infty \frac{(1+t^2)\mathrm{e}^{(\theta-\gamma)t}}{(1+t^2)^2}\,\mathrm{d}t \\ &= c \int_x^\infty \frac{\mathrm{e}^{(\theta-\gamma)t}\,\mathrm{d}t}{1+t^2} \leqslant c\mathrm{e}^{(\theta-\gamma)x}. \end{aligned} \tag{5.14}$$

From (5.14) and (3.4) it follows that $J(x)$ converges (at least conditionally) for $\gamma \geqslant \theta$. According to Theorem 3.7, condition (3.6) together with (5.14) and (5.5) give (for $x \geqslant x_2$)

$$\int_x^\infty |\Delta q(t)| h(t) I(t)\,\mathrm{d}t \leqslant c \int_x^\infty \frac{(1+t^2)\mathrm{e}^{\theta t}|\cos(\mathrm{e}^{\gamma t})|}{(1+t^2)\mathrm{e}^t} \mathrm{e}^{(\theta-\gamma)t}\,\mathrm{d}t$$
$$= c\int_x^\infty \mathrm{e}^{(2\theta-\gamma-1)t}\,\mathrm{d}t < \infty \;\Rightarrow\; \theta < \tfrac{1}{2}(\gamma+1). \tag{5.15}$$

By Theorem 3.7 (see (3.6)), problem (1.6)–(1.8) is solvable provided condition $\theta < \min\{\gamma, \frac{1}{2}(\gamma+1)\}$ holds. Similarly, by condition (3.7),

$$\int_x^\infty \frac{I^2(t)\,\mathrm{d}t}{r(t)h(t)} \leqslant c\int_x^\infty \frac{\mathrm{e}^{2(\theta-\gamma)t}}{(1+t^2)^2}(1+t^2)\mathrm{e}^t\,\mathrm{d}t \leqslant c\mathrm{e}^{(2\theta-2\gamma+1)x}\int_x^\infty \frac{\mathrm{d}t}{1+t^2}$$
$$= c\mathrm{e}^{(2\theta-2\gamma+1)x} < \infty \;\Rightarrow\; \theta \leqslant \gamma - \tfrac{1}{2}. \tag{5.16}$$

Hence problem (1.6)–(1.8) is solvable under condition (5.16).

Thus the Hartman–Wintner problem for Equations (1.1), (1.2) in the case (5.1) is solvable provided any of the following three conditions holds:

$$(1)\ \theta < 1,\ \gamma > 0; \qquad (2)\ \theta < \min\{\gamma, \tfrac{1}{2}(\gamma+1)\}; \qquad (3)\ \theta \leqslant \gamma - \tfrac{1}{2}. \tag{5.17}$$

Moreover, by Theorem 3.9 and inequalities (5.4) we conclude that under either of conditions (5.17) all FSSs of these equations are asymptotically equivalent as $x \to \infty$. Note that in the case (5.1) one can find an asymptotic of an FSS of Equation (1.2) using the standard JWKB method, and thus obtain an asymptotic of an FSS of Equation (1.1) under either of conditions (5.17). Thus under conditions (5.17) one can say that asymptotic analysis of (1.1) is completed. This last stage of asymptotic integration of (1.1) is not presented here because it contains no new details.

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