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ON REALISING MOD-2 HOMOLOGY CLASSES OF MANIFOLDS BY SUBMANIFOLDS

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1. Statement of results

In his fundamental paper (3), Thom proved, among other things, that a mod-2 homology class of an n-dimensional, closed, compact, C^∞ manifold, which has dimension $\leq n/2$, can be realised by a submanifold, (see (3), Théorème II.1 and Corollaire II.13).

In this note we examine the question of realisability of mod-2 homology classes of the next higher dimension.

Throughout this note "manifold" will mean "closed, compact, C^∞ manifold".

From now on, M will be an $(2n+1)$-dimensional manifold, $z \in H_{n+1}(M; Z_2)$ will be an $(n+1)$-dimensional mod-2 homology class, and $u \in H^n(M; Z_2)$ its dual cohomology class.

Our main results are the following.

Theorem 1. *If n is of the form* $2^r - 1$, *then z is always realisable by a submanifold.*

Theorem 2. *Let n be an odd number not of the form* $(2^r - 1)$. *Then z can be realised by a submanifold, if and only if*

(i) $u \cdot \mathrm{Sq}^1 u + \mathrm{Sq}^n \mathrm{Sq}^1 u = 0$, *or*

(ii) *at least one of the Wu classes* $v_{n-1}, v_{n-3}, \ldots, v_{n-2^r+1}, \ldots$, *of M, is non-zero.*

Remark. The i-dimensional *Wu* class of M, can be defined by the property $\mathrm{Sq}^i x = v_i \cdot x$, where $x \in H^*(M; Z_2)$ $v_i \in H^i(M; Z_2)$ and $\deg(\mathrm{Sq}^i x) = \dim M$, (see (1), p. 39).

Theorem 3. *If n is an odd number not of the form* $2^r - 1$, *then there is an* $(2n+1)$-*dimensional*, $(n-1)$-*connected manifold M, which has an* $(n+1)$-*dimensional mod-2 homology class, which cannot be realised by a submanifold.*

Theorem 4. *If n is an even number and M is orientable, then all mod-2,* $(n+1)$-*dimensional homology classes of M, can be realised by submanifolds.*

2. Proof of Theorems 1, 2, 3, 4

Our results depend heavily on P. J. Ledden's paper (2).

Let $MO(n)$ be the Thom space of the universal n-linear bundle. Then, (following Ledden's notation) there a product of $K(Z_2,)$'s $\bar{K}$ and a map $\bar{F}: MO(n) \to \bar{K}$ which induces an isomorphism in mod-2 cohomology, up to dimension $2n$. Ledden computes the first Postnikov invariant of the map $\bar{F}$, let us call it θ.

Specifically he finds the results stated here as Lemmas 5, 6 and 7.

Lemma 5. *If $n = 2^{r+1} - 1$, then $\theta = 0$*, (see (2), Lemma 2).

Lemma 6. *If n is odd and $2^r < n < 2^{r+1} - 1$, then*

$$\theta = \epsilon_0 \mathrm{Sq}^1 \epsilon_0 + \mathrm{Sq}^n \mathrm{Sq}^1 \epsilon_0 + \mathrm{Sq}^{n-1} \epsilon_1 + \mathrm{Sq}^{n-3} \epsilon_2 + \cdots \mathrm{Sq}^{n-2^r+1} \epsilon_r.$$

The ϵ_i's are fundamental classes of factors of $\bar{K}$, such that $\deg \epsilon_i = 2^r + n$ if $i \geq 1$ and $\epsilon_0 =$ the Thom class of $MO(n)$. For details on the ϵ_i's see (2), and particularly Lemma 2.

Lemma 7. *If n is even then $\theta \in H^{2n+1}(\bar{K}; Z)$ is a class of finite order*, (see (2), Lemma 2').

Lemma 8. *The homology class z is realisable by a submanifold if and only if there is a map $f: M \to \bar{K}$ such that $f^*(\epsilon_0) = u$ and $f^*(\theta) = 0$.*

Proof. Obvious by Théorème II.1 of (3) and the fact that θ is the appropriate Postnikov invariant.

Proof of Theorem 1. Since ϵ_0 is the fundamental class of a factor of $\bar{K}$, there is a map $f: M \to \bar{K}$ such that $f^*(\epsilon_0) = u$. Because $\theta = 0$ (by Lemma 5) the conditions of Lemma 8 are satisfied, and the result follows.

Proof of Theorem 2. First we prove necessity. Let us assume that the class z can be realised by a submanifold. Then from Lemmas 8 and 6, there is a map $f: M \to \bar{K}$ such that $u \mathrm{Sq}^1 u + \mathrm{Sq}^n \mathrm{Sq}^1 u + \mathrm{Sq}^{n-1} f^*(\epsilon_1) + \mathrm{Sq}^{n-3} f^*(\epsilon_2) + \cdots + \mathrm{Sq}^{n-2^r+1} f^*(\epsilon_r) = 0$. But this means that either $u \mathrm{Sq}^1 u + \mathrm{Sq}^n \mathrm{Sq}^1 u = 0$, or that at least one of the terms $\mathrm{Sq}^{n-2^i+1} f^*(\epsilon_i)$ is non zero, for $i = 1, 2, \ldots, r$. But this last condition implies that one of the v_{n-2^i+1}'s of M is non zero.

Proof of sufficiency. The key remark in order to prove sufficiency, is that a map $f: M \to \bar{K}$ can be defined, such that $f^*(\epsilon_0) = u$ and the $f^*(\epsilon_1), f^*(\epsilon_2), \ldots, f^*(\epsilon_r)$ have any preassigned values. That ends the proof.

Proof of Theorem 4. It is exactly the same as Theorem 1. By Lemma 7, for any map $f: M \to \bar{K}$, $f^*(\theta) = 0$ because $H^{2n+1}(M; Z) = Z$ and θ has finite order.

Lemma 9. *If n is odd, then there is an $(2n+1)$-dimensional, $(n-1)$-connected manifold M, such that $H_n(M; Z) = Z_2$.*

Proof. We consider the Stiefel manifold, $V_{n+2,2} = O(n+2)/O(n)$, which is the space of all orthonormal 2-frames in $\mathbb{R}^{n+2}$. It is well-known, that $V_{n+2,2}$ is $(n-1)$-connected, and that if n is odd then $\pi_n(V_{n+2,2})=Z_2$. That ends the proof, because $V_{n+2,2}$ is $(2n+1)$-dimensional.

Lemma 10. *Let $n>1$ and let M be an $(n-1)$-connected, $(2n+1)$-dimensional manifold, such that $H_n(M;Z)=Z_2$. Then $H^n(M;Z_2)=Z_2$, and if x is the generator of $H^n(M;Z_2)$ then $x\cdot \mathrm{Sq}^1x\neq O$.*

Proof. By the Universal coefficient theorem and Poincare duality we have:

$$H^n(M;Z)=H^{n+1}(M;Z)=O \text{ so } H^n(M;Z_2)=Z_2 \text{ and } H^{n+1}(M;Z_2)=Z_2.$$

Next we consider the long exact sequence in cohomology of M, induced by the short exact sequence $O\to Z\xrightarrow{2\cdot}Z\to Z_2\to O$. From this we get easily that the Bockstein operator $b_2: H^n(M;Z_2)\to H^{n+1}(M;Z)$ is an isomorphism and so, by the previous calculations $\mathrm{Sq}^1: H^n(M;Z_2)\to H^{n+1}(M;Z_2)=Z_2$ is an isomorphism. Because of Poincare duality $x\cdot \mathrm{Sq}^1x\neq O$.

Proof of Theorem 3. Let M be a manifold with the specifications of Lemma 9. Then since it is $(n-1)$-connected we must have $v_{n-1}, v_{n-3},\dots, v_{n-2^r+1},\dots=O$. For the same reason $\mathrm{Sq}^n\mathrm{Sq}^1x=O$, because Sq^n decomposes. So by the previous Lemma and Theorem 2, the result follows.

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