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DUALITY MAP CHARACTERISATIONS FOR OPIAL CONDITIONS

TIM DALBY AND BRAILEY SIMS

We characterise Opial's condition, the non-strict Opial condition, and the uniform Opial condition for a Banach space X in terms of properties of the duality mapping from X into X^*.

In 1967, Opial [4] introduced the following condition on a Banach space X. If (x_n) converges weakly to x_∞ then

$$\liminf_{n\to\infty} \|x_n - x_\infty\| < \liminf_n \|x_n - x\|$$

for all $x \neq x_\infty$.

This condition has been used in the study of the existence of fixed points for non–expansive maps. For example, Gossez and Lami Dozo [2] have shown that Opial's condition implies weak normal structure and hence the weak fixed point property. A weaker condition, non–strict Opial, is that (x_n) converging weakly to x_∞ implies

$$\liminf_{n\to\infty} \|x_n - x_\infty\| \leqslant \liminf_n \|x_n - x\|$$

for all x. Again, this condition is associated with the weak fixed point property. See, for example, Sims [7].

In the opposite direction Prus [5] in 1992 introduced the uniform Opial condition. For $c > 0$ define the Opial modulus of X to be

$$r(c) = \inf\left\{\liminf_n \|x_n + x\| - 1 : \|x\| \geqslant c,\ x_n \xrightarrow{w} 0, \text{ and } \liminf_n \|x_n\| \geqslant 1\right\}.$$

Then $r(c)$ is an increasing function of c, and we say X has the uniform Opial property if $r(c) > 0$, for $c > 0$, in which case we have

$$1 + r(c) \leqslant \liminf_n \|x_n + x\|$$

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whenever $x_n \xrightarrow{w} 0$, $\liminf_n \|x_n\| \geqslant 1$, and $\|x\| \geqslant c$. For $1 < p < \infty$, the space ℓ_p satisfies the uniform Opial condition whilst $L_p[0,1]$, $p \neq 2$, fails even the non-strict Opial condition.

A gauge, μ, is a continuous strictly increasing real-valued function on $[0, \infty)$ satisfying $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = \infty$. A mapping $J_\mu : X \to X^*$ is called a duality mapping with gauge function μ if for every $x \in X$

$$J_\mu(x) := \{x^* \in X^* : x^*(x) = \|x\|\,\mu(\|x\|) \text{ and } \|x^*\| = \mu(\|x\|)\}.$$

If $\mu(t) = t$ we write J instead of J_μ. X is said to have a weakly continuous duality map if there exists a gauge μ such that the duality map J_μ is single-valued and sequentially continuous from X with the weak topology to X^* with the weak * topology. Gossez and Lami Dozo [**2**], in 1972, showed that a Banach space with a weakly continuous duality map satisfies Opial's condition. Recently, Lin, Tan and Xu [**3**] improved on this by showing that such a space has the uniform Opial condition.

More recently still, Benavides, Acedo and Xu [**1**] have produced an example, $\ell_{p,1}$, that satisfies the uniform Opial condition but fails to have a weakly continuous duality map. This naturally raises the question of a duality map characterisation of the uniform Opial condition.

Sims [**6**] in 1985 characterised Opial's condition in terms of the asymptotic nature of $J(x_n)$ where (x_n) is a non-null weakly convergent sequence. More precisely we have the following.

Theorem 1. *A Banach space satisfies Opial's condition if and only if whenever (x_n) converges weakly to a non-zero limit x_∞, for $x_n^* \in J(x_n)$ we have*

$$\liminf_n \; x_n^*(x_\infty) > 0.$$

An examination of the proof shows that the following is also true.

Theorem 2. *A Banach space satisfies the non-strict Opial condition if and only if whenever (x_n) converges weakly to a non-zero limit x_∞, for $x_n^* \in J(x_n)$ we have*

$$\liminf_n \; x_n^*(x_\infty) \geqslant 0.$$

Here we complete the cycle by extending the techniques of [**6**] to obtain a characterisation of the uniform Opial condition.

We begin by showing that the uniform Opial condition is determined in the following way. Note: the subsequential form of this characterisation is not needed for our later proofs, but is included for its potential utility.

Lemma 3. *For a Banach space X the following are equivalent:*

(i) *X has the uniform Opial condition.*

(ii) *There exists a strictly positive function ρ such that whenever $x_n \xrightarrow{w} 0$, $\lim_n \|x_n\| = 1$ and $\|x\| \geqslant c$, there exists a subsequence (x_{n_k}) with*

$$\liminf_k \|x_{n_k} + x\| \geqslant 1 + \rho(c).$$

Proof: Clearly (i) implies (ii) and (ii) implies Opial's condition.

Now suppose X has (ii) but fails to have the uniform Opial condition. Then there exists a $c > 0$ and, for each $m \in \mathbf{N}$, a sequence $x_n^m \xrightarrow{w} 0$, as $n \to \infty$, with

$$r_m := \liminf_n \|x_n^m\| \geqslant 1$$

and an x^m with $\|x^m\| \geqslant c$ so that

$$\liminf_n \|x_n^m + x^m\| < 1 + \frac{1}{m}.$$

Note. By passing to a subsequence we can, and shall, assume that both of the above lim inf's are in fact limits.

Also, since X has Opial's condition,

$$r_m < \lim_n \|x_n^m + x^m\| < 1 + \frac{1}{m} \leqslant 2.$$

Now, let $y_n^m = x_n^m / r_m$ and $y^m = x^m / r_m$, then $y_n^m \xrightarrow{w} 0$, as $n \to \infty$, $\lim_n \|y_n^m\| = 1$, $\|y^m\| \geqslant c/r_m \geqslant c/2$, while

$$\begin{aligned} \lim_n \|y_n^m + y^m\| &\leqslant \left(1 + \frac{1}{m}\right) / r_m \\ &\leqslant 1 + \frac{1}{m}. \end{aligned}$$

The sequence (y_n^m) for $m > 1/\rho(c/2)$ contradicts (ii), so (ii) implies (i). ∎

We shall say that a Banach space X has *Property (D)* if there exists an increasing strictly positive function α on $(0, \infty)$ such that whenever $x_n \xrightarrow{w} x_\infty \neq 0$, $\lim_n \|x_n - x_\infty\| = 1$, and $x_n^* \in J(x_n)$, we have

$$\liminf_n x_n^*(x_\infty) \geqslant \alpha(\|x_\infty\|).$$

Observation. From Theorem 1 it is clear that (D) implies Opial's condition.

We now show that (D) is necessary for the uniform Opial condition.

Lemma 4. *If X has uniform Opial condition then X has property (D) with $\alpha(t) = tr(t)$*

Proof: Let $x_n \xrightarrow{w} x_\infty \neq 0$ with $\lim_n \|x_n - x_\infty\| = 1$ and suppose there exists $x_n^* \in J(x_n)$ such that

$$\liminf_n x_n^*(x_\infty) < \|x_\infty\| \, r(\|x_\infty\|).$$

Then there exists a subsequence $\left(x_{n_k}^*\right)$ with $\lim_k x_{n_k}^*(x_\infty) < \|x_\infty\| \, r(\|x_\infty\|)$. By the uniform Opial condition,

$$\begin{aligned}
\liminf_k \|x_{n_k}\| &= \liminf_k \left\|\left(x_{n_k} - x_\infty\right) + x_\infty\right\| \\
&\geqslant 1 + r(\|x_\infty\|) \\
&= \lim_k \left\|x_{n_k} - x_\infty\right\| + r(\|x_\infty\|) \\
&\geqslant \liminf_k \frac{x_{n_k}^*}{\left\|x_{n_k}\right\|}\left(x_{n_k} - x_\infty\right) + r(\|x_\infty\|) \\
&\geqslant \liminf_k \left\|x_{n_k}\right\| - \limsup_k \frac{x_{n_k}^*}{\left\|x_{n_k}\right\|}(x_\infty) + r(\|x_\infty\|).
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_k x_{n_k}^*(x_\infty) &\geqslant r(\|x_\infty\|) \liminf_k \left\|x_{n_k}\right\| \\
&\geqslant r(\|x_\infty\|) \, \|x_\infty\|,
\end{aligned}$$

contradicting the choice of $\left(x_{n_k}^*\right)$. ∎

We now use a modification of an argument suggested in [**2**] and developed in [**6**] to establish a converse to Lemma 4.

Theorem 5. *A Banach space X has the uniform Opial condition if and only if its duality map satisfies property (D).*

Proof: $(\Rightarrow)$ has been established in lemma 4.

$(\Leftarrow)$ We use the characterisation of the uniform Opial condition given in Lemma 3. Thus, let (x_n) be a weak null sequence with $\|x_n\| \to 1$. Then, for $x \neq 0$

$$\frac{1}{2}\|x_n + x\|^2 = \frac{1}{2}\|x_n\|^2 + \int_0^1 g_n^+(t)\,dt$$

where

$$g_n^+(t) := \lim_{h \to t+} \frac{\frac{1}{2}\|x_n + hx\|^2 - \frac{1}{2}\|x_n + tx\|^2}{h - t}$$

is the upper Gateaux derivative at t of the convex function $t \mapsto 1/2\left\|x_n + tx\right\|^2$, and so is an increasing function of t, equal to $\max\{x_n^*(x) : x_n^* \in J(x_n + tx)\}$.

Now, for any $\varepsilon > 0$, $x_n + \varepsilon x \xrightarrow{w} \varepsilon x \neq 0$ and so, since (D) implies Opial's condition, we see from Theorem 1 that for n sufficiently large, $g_n^+(\varepsilon) > 0$. Thus for n sufficiently large

$$\int_\varepsilon^1 g_n^+(t)dt \geqslant \frac{1}{2}\, g_n^+\left(\frac{1}{2}\right).$$

Since the $g_n^+(t)$ are uniformly bounded it follows that

$$\begin{aligned}\liminf_n \left\|x_n + x\right\|^2 &\geqslant \liminf_n \left\|x_n\right\|^2 + 2\liminf_n \int_0^1 g_n^+(t)dt \\ &\geqslant 1 + \liminf_n g_n^+\left(\frac{1}{2}\right) \\ &\geqslant 1 + \alpha\left(\frac{1}{2}\left\|x\right\|\right).\end{aligned}$$

Thus, X satisfies (ii) of lemma 3 with

$$\rho(c) = \sqrt{1 + \alpha(c/2)} - 1. \qquad \square$$

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Department of Mathematics
The University of Newcastle
New South Wales 2308
Australia
e-mail: bsims@frey.newcastle.edu.au