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UNIFORMLY CONVEXIFYING OPERATORS IN CLASSICAL BANACH SPACES

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We obtain a new characterisation of finite representability of operators and present new results about uniformly convexifying, Rademacher cotype and Rademacher type operators on some classical Banach spaces, including JB^*-triples and spaces of analytic functions.

1. INTRODUCTION

Different notions of finite representability of operators have been introduced by Beauzamy [**3, 4, 6**], Bellenot [**7**], and Heinrich [**13**]. In this paper, we consider a variant of the finite representability of operators given by Beauzamy in his monograph. In particular, we prove that this notion is equivalent to the one given by Bellenot.

This characterisation allows us to establish new relationships between uniformly convexifying, Rademacher cotype, and Rademacher type operators. These three kinds of operators were introduced by Beauzamy [**2, 3, 4**], and studied by several authors [**8, 13, 16, 17**]. We point out that these ideals are closely related to the ideal of operators which factor through Banach spaces having the Banach-Saks property [**5, 11**].

It is well-known that every uniformly convexifying operator is of Rademacher type and every Rademacher type operator is of Rademacher cotype [**3, 5**]. The essential aim of this paper is to study non-trivial converse implications in this scheme for some classical Banach spaces. Namely, we deduce several new properties of operators defined on JB^*-triples, the disk algebra, the space H^∞ of bounded analytic functions on the disc, the space *VMOA* of analytic functions on the disc with vanishing mean oscillation and the space *BMOA* of analytic functions on the disc with bounded mean oscillation. We also obtain new properties of operators defined from arbitrary Banach spaces into preduals of JBW^*-triples, the Hardy space H^1, the space L^1/H^1_0, and Banach lattices with finite cotype.

In addition to modifying and refining certain of Beauzamy's results (Lemma 3.1), our arguments rely in a new 'commutative-diagram' formulation of finite representability

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of operators (Theorem 2.1) which allows us to benefit from some recent results about JB^*-triples [9] (see also [20]) and spaces of analytic functions [10].

Our terminology and notations are standard and we refer the reader to the monographs of Lindenstrauss and Tzafriri [18] and Wojtaszczyk [22] for Banach space theory, to the survey of Heinrich [14] for finite representability and ultrapowers, and to the survey of Rodríguez-Palacios [21] for JB^*-triples.

2. Finite representability of operators

The notion of finite representability of operators (linear and continuous) that we are interested in is the following (compare with [6, p.241]).

Definition 2.1: Let X, Y, X_0, and Y_0 be Banach spaces and let $T : X \longrightarrow Y$ and $T_0 : X_0 \longrightarrow Y_0$ be operators. We say that T_0 is *finitely representable* in T if for every $\varepsilon > 0$ and every finite-dimensional subspace M_0 of X_0 there exist a finite-dimensional subspace M of X and two ε-isometries $V : M_0 \longrightarrow M$ and $W : T_0(M_0) \longrightarrow T(M)$ such that $WT_0 = TV$.

We point out that an ε-isometry is a continuous linear operator $T : Z_1 \longrightarrow Z_2$ satisfying

$$(1-\varepsilon)\|z\| \leqslant \|Tz\| \leqslant (1+\varepsilon)\|z\| \qquad \text{for all } z \in Z_1.$$

Beauzamy's notion replaces ε-isometry by ε-isomorphism in the above definition. Obviously, our concept is stronger than Beauzamy's, but it can be shown that is indeed different [1]. Moreover, using the ideas given in [14, Proposition 6.1], it can be shown that $(T)_{\mathcal{U}}$ is finitely representable in T for each ultrafilter $\mathcal{U}$.

Theorem 2.1. *Let X, Y, X_0, and Y_0 be Banach spaces and let $T : X \longrightarrow Y$ and $T_0 : X_0 \longrightarrow Y_0$ be operators. The following assertions are equivalent.*

1. *The operator T_0 is finitely representable in T.*
2. *Given $\varepsilon > 0$ and a finite-dimensional subspace M_0 of X_0, there exists an operator $V : M_0 \longrightarrow X$ such that*
   - *(a) $\big|\|Vx\| - \|x\|\big| \leqslant \varepsilon\|x\|$ for all $x \in M_0$.*
   - *(b) $\big|\|TVx\| - \|T_0x\|\big| \leqslant \varepsilon\|x\|$ for all $x \in M_0$.*
3. *There exists an ultrafilter $\mathcal{U}$ over a set I and two isometries J and $\widetilde{J}$ such that the following diagram commutes:*

$$\begin{array}{ccc} X_0 & \xrightarrow{T_0} & T_0(X_0) \\ J\downarrow & & \downarrow\widetilde{J} \\ (X)_{\mathcal{U}} & \xrightarrow{(T)_{\mathcal{U}}} & (Y)_{\mathcal{U}}. \end{array}$$

PROOF: ($1 \Rightarrow 2$) Without loss of generality, we take $T_0 \neq 0$. Fix $\varepsilon > 0$ and a finite-dimensional subspace M_0 of X_0. By assumption, there are a finite-dimensional subspace M of X and two ε-isometries $V : M_0 \longrightarrow M$ and $W : T_0(M_0) \longrightarrow T(M)$ such that $WT_0 = TV$. It is clear that V satisfies (a) and given $x \in M_0$ we have that

$$\big|\|TVx\| - \|T_0x\|\big| \leqslant \big|\|WT_0x\| - \|T_0x\|\big| \leqslant \varepsilon \|T_0x\| \leqslant \varepsilon \|T_0\| \|x\|.$$

($2 \Rightarrow 3$) Let us consider the set

$$I := \big\{i = (\varepsilon_i, M_i) : 0 < \varepsilon_i < 1,\ M_i \text{ a finite-dimensional subspace of } X_0\big\}.$$

On this set, we define an order relation by

$$i_1 \leqslant i_2 \Longleftrightarrow \varepsilon_{i_1} \geqslant \varepsilon_{i_2} \quad \text{and} \quad M_{i_1} \subseteq M_{i_2}.$$

Take an ultrafilter $\mathcal{U}$ which dominates the filter induced by this order on I. By hypothesis, for each $i \in I$ we can fix an operator $V_i : M_i \longrightarrow X$ such that

(a) $\big|\|V_ix\| - \|x\|\big| \leqslant \varepsilon_i \|x\|$ for all $x \in M_i$.

(b) $\big|\|TV_ix\| - \|T_0x\|\big| \leqslant \varepsilon_i \|x\|$ for all $x \in M_i$.

Now, for each $i \in I$ and $x \in X_0$, define

$$J_i(x) = \begin{cases} V_ix & \text{if } \ x \in M_i \\ 0 & \text{if } \ x \notin M_i. \end{cases}$$

We have that $\big\|J_i(x)\big\| \leqslant \|V_ix\| \leqslant (1 + \varepsilon_i)\|x\| \leqslant 2\|x\|$. This implies that $\big(J_i(x)\big)_{\mathcal{U}}$ is an element of $(X)_{\mathcal{U}}$ and we consider $J : X_0 \longrightarrow (X)_{\mathcal{U}}$ given by $J(x) := \big(J_i(x)\big)_{\mathcal{U}}$. We show that J is a linear map. Take $x_1, x_2 \in X_0$ and $\alpha, \beta \in \mathbb{K}$. We have to prove that

$$\lim_{i\mathcal{U}} \big\|J_i(\alpha x_1 + \beta x_2) - \alpha J_i(x_1) - \beta J_i(x_2)\big\| = 0.$$

Given $\varepsilon > 0$, consider $i_0 = \big(\varepsilon, \operatorname{span}\{x_1, x_2\}\big)$. If $i = (\varepsilon_i, M_i) \geqslant i_0$, then $\varepsilon_i \leqslant \varepsilon$ and $x_1, x_2, \alpha x_1 + \beta x_2 \in M_i$. Thus, we have that

$$V_i(\alpha x_1 + \beta x_2) - \alpha V_i(x_1) - \beta V_i(x_2) = 0,$$

and $\big\|J_i(\alpha x_1 + \beta x_2) - \alpha J_i(x_1) - \beta J_i(x_2)\big\| = 0$.

We show that J is an isometry. We have to check that $\lim_{i\mathcal{U}} \big\|J_i(x)\big\| = \|x\|$. This is clear when $x = 0$. Otherwise, fix $\varepsilon > 0$ and take $i_0 = \big(\varepsilon/\|x\|, \operatorname{span}\{x\}\big)$. Now, if $i = (\varepsilon_i, M_i) \geqslant i_0$, we have that

$$\big|\|J_i(x)\| - \|x\|\big| = \big|\|V_i(x)\| - \|x\|\big| \leqslant \varepsilon_i \|x\| \leqslant \varepsilon.$$

To define the map $\widetilde{J}$ let us note that if $T_0x_1 = T_0x_2$ with $x_1 \neq x_2$ then $(T)_{\mathcal{U}}(Jx_1) = (T)_{\mathcal{U}}(Jx_2)$, that is,

$$\lim_{i,\mathcal{U}} \|T(J_i(x_1)) - T(J_i(x_2))\| = 0.$$

Given $\varepsilon > 0$, we take $i_0 = (\varepsilon / \|x_1 - x_2\|, \operatorname{span}\{x_1, x_2\})$ so that if $i = (\varepsilon_i, M_i) \geqslant i_0$ then

$$\left\|T(J_i(x_1)) - T(J_i(x_2))\right\| = \left\|T(V_i(x_1)) - T(V_i(x_2))\right\| = \|T \circ V_i(x_1 - x_2)\|.$$

By appealing to condition (b) and since $T_0x_1 = T_0x_2$, we finally deduce

$$\left\|T(J_i(x_1)) - T(J_i(x_2))\right\| \leqslant \varepsilon_i \|x_1 - x_2\| \leqslant \varepsilon.$$

Therefore, the map $\widetilde{J} : T_0(X_0) \longrightarrow (Y)_{\mathcal{U}}$ given by $\widetilde{J}(y) = (T)_{\mathcal{U}}(Jx)$, for some $x \in X_0$ with $y = T_0(x)$, is well-defined. It is not difficult to see that it is linear. Clearly, to finish this implication, it is enough to prove that $\widetilde{J}$ is an isometry. Indeed, given $y = T_0(x) \in T_0(X_0)$, $y \neq 0$, and $\varepsilon > 0$ we take $i_0 = (\varepsilon/\|x\|, \operatorname{span}\{x\})$. If $i = (\varepsilon_i, M_i) \geqslant i_0$, we have that $x \in M_i$ and

$$\left|\|T(J_i(x))\| - \|T_0(x)\|\right| = \left|\|T(V_i(x))\| - \|T_0(x)\|\right| \leqslant \varepsilon_i \|x\| \leqslant \varepsilon.$$

Therefore, we have $\lim_{i,\mathcal{U}} \|T \circ J_i(x)\| = \|T_0x\| = \|y\|$, that is, $\|\widetilde{J}(y)\| = \|y\|$.

($3 \Rightarrow 1$) Statement (3) clearly shows that the operator T_0 is finitely representable in $(T)_{\mathcal{U}}$. On the other hand, $(T)_{\mathcal{U}}$ is finitely representable in T. The implication follows by noting that finite representability of operators is a transitive property. ∎

Remark 2.1. Condition 2 of the above theorem is the notion of finite representability of operators introduced by Bellenot [**7**]. Actually, his definition has a third condition but, as he observed, it is implied by the second one. Bellenot also notes that his definition is different from the ones given by Beauzamy in his papers [**3**] and [**4**].

3. Some ideals of operators on function spaces

First of all, we recall the definitions of the three kinds of operators introduced by Beauzamy.

Definition 3.1: Let X, Y be Banach spaces. An operator T from X into Y is said to be *uniformly convexifying* if there exists an equivalent norm $|\cdot|$ on X such that, given $\varepsilon > 0$, there is $\delta > 0$ such that, for all $x_1, x_2 \in X$ with $|x_1| = |x_2| = 1$, we have $\|Tx_1 - Tx_2\| < \varepsilon$ whenever $|(x_1 + x_2)/2| > 1 - \delta$.

Using the ideas given by Beauzamy in [**3**], it is not difficult to prove that T is uniformly convexifying if and only if every finitely representable operator in T is weakly compact. This fact will be used throughout the paper without further comment.

Beauzamy also proved that every uniformly convexifying operator factors through a Banach space with the Banach-Saks property [5, Théorème 1 and 3]. It is worth mentioning that there is an uniformly convexifying operator defined on a certain $C(K)$ which does not factor through a super-reflexive Banach space [19] (see also [3]).

DEFINITION 3.2: Let X, Y be Banach spaces. An operator T from X into Y is said to be of *Rademacher type* if

$$\lim_{n\to\infty} \sup_{x_1,\ldots,x_n\in B_X} \frac{1}{n}\int_0^1 \left\| \sum_{i=1}^n \varepsilon_i(t) T x_i \right\| dt = 0,$$

where the ε_i's denote Rademacher functions defined on the interval $[0,1]$. Likewise, T is said to be of *Rademacher cotype* if

$$\lim_{n\to\infty} \inf_{x_1,\ldots,x_n\in X,\, \|Tx_i\|\geqslant 1} \frac{1}{n}\int_0^1 \left\| \sum_{i=1}^n \varepsilon_i(t) T x_i \right\| dt = \infty.$$

Beauzamy proved that an operator is of Rademacher type if and only if it does not fix a uniform copy of ℓ_n^1 and an operator is of Rademacher cotype if and only if it does not fix a uniform copy of ℓ_n^∞ [2]. We recall that an operator T fixes a uniform copy of ℓ_n^r $(r = 1,\infty)$ if there is $\theta > 0$ such that, for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$ there are points $x_1, \ldots, x_n \in X$ satisfying

$$(1-\varepsilon)\left\|(\alpha_1,\ldots,\alpha_n)\right\|_r \leqslant \left\| \sum_{p=1}^n \alpha_p x_p \right\| \leqslant \left\|(\alpha_1,\ldots,\alpha_n)\right\|_r$$

$$(1-\varepsilon)\,\theta\left\|(\alpha_1,\ldots,\alpha_n)\right\|_r \leqslant \left\| \sum_{p=1}^n \alpha_p T x_p \right\| \leqslant \theta\left\|(\alpha_1,\ldots,\alpha_n)\right\|_r$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

LEMMA 3.1. *Let X, Y be Banach spaces and let T be an operator from X into Y. Then,*

1. *The operator T is not of Rademacher cotype if and only if there is an ultrafilter $\mathcal{U}$ such that $(T)_{\mathcal{U}}$ fixes a copy of c_0.*
2. *The operator T is not of Rademacher type if and only if there is an ultrafilter $\mathcal{U}$ such that $(T)_{\mathcal{U}}$ fixes a copy of ℓ_1.*

PROOF: Suppose that T fixes a uniform copy of ℓ_n^∞. Then, there is $\theta \geqslant 0$ such that, for each $n \in \mathbb{N}$, we can find a finite sequence $x_1^n, \ldots, x_n^n \in X$ satisfying

$$\frac{1}{2}\max_{p=1,\ldots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^n \alpha_p x_p^n \right\| \leqslant \max_{p=1,\ldots,n} |\alpha_p|$$

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$$\frac{1}{2}\theta \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p T x_p^n \right\| \leqslant \theta \max_{p=1,\dots,n} |\alpha_p|$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Take an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ dominating the Fréchet filter and consider the elements of $(X)_{\mathcal{U}}$ given by $x_p = \left(x_p^i\right)_{\mathcal{U}}$, where $x_p^i = 0$ if $p > i$. From the above inequalities, it is not difficult to see that the sequences (x_p) and $\left((T)_{\mathcal{U}}\, x_p\right)$ are isomorphic to the unit basis of c_0.

Now, suppose that $\mathcal{U}$ is an ultrafilter over a set I and $(T)_{\mathcal{U}}$ fixes a copy of c_0, that is, there are two constants m and M and a sequence (x_p) in $(X)_{\mathcal{U}}$ such that

$$m \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p x_p \right\| \leqslant M \max_{p=1,\dots,n} |\alpha_p|$$

$$m \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p \,(T)_{\mathcal{U}}\, x_p \right\| \leqslant M \max_{p=1,\dots,n} |\alpha_p|$$

for all $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Let

$$K := \lim_{n\to\infty} \sup\left\{ \left\| \sum_{k=n}^{p} \alpha_k x_k \right\| : \sup |\alpha_k| = 1,\ p > n \right\}$$

and take $\theta := m/2K$. Fix $\varepsilon > 0$. By [**15**, Proof of Lemma 2.2], there are a constant $1 < \varphi < 2$ and a block sequence of (x_p) given by $y_n := 1/(\varphi K) \sum_{i=p_n}^{p_{n+1}-1} a_i^n x_i$ with $\sup_{i=p_n,\dots,p_{n+1}-1} |a_i^n| = 1$ such that

$$(1-\varepsilon) \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p y_p \right\| \leqslant \max_{p=1,\dots,n} |\alpha_p|$$

for all $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. It is clear that

$$\frac{m}{\varphi K} \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p \,(T)_{\mathcal{U}}\, y_p \right\|$$

for all $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. In particular, $\left\|(T)_{\mathcal{U}}\, y_n\right\| \geqslant \theta$. By [**2**, Lemme 2], this implies that $(T)_{\mathcal{U}}$ fixes a uniform copy of ℓ_n^∞. That is, there is $\rho > 0$ such that for all $\varepsilon > 0$ and for all n there are points $z_1, \dots, z_n \in (X)_{\mathcal{U}}$ satisfying that

$$(1-\varepsilon) \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p z_p \right\| \leqslant \max_{p=1,\dots,n} |\alpha_p|$$

$$(1-\varepsilon)\,\rho \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p \,(T)_{\mathcal{U}}\, z_p \right\| \leqslant \rho \max_{p=1,\dots,n} |\alpha_p|$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. To finish the proof we have to show that T has the same property.

Fix $\varepsilon > 0$ and take $z_1, \dots, z_n$ as above. Since $(T)_{\mathcal{U}}$ is finitely representable in T, by Theorem 2.1, given the subspace $M_n = \mathit{span}\{z_1, \dots, z_n\}$, there are a subspace $\widetilde{M_n}$ of X and two ε-isometries, $V : M_n \longrightarrow \widetilde{M_n}$ and $W : (T)_{\mathcal{U}} M_n \longrightarrow TV\widetilde{M_n}$ such that $W(T)_{\mathcal{U}} = TV$. We may assume that $\|V\| \leqslant 1$ and $\|W\| \leqslant 1$. Take $\widetilde{z}_p = Vz_p$ for $p = 1, \dots, n$. For all $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ we have that

$$(1-\varepsilon)^2 \max_{p=1,\dots,n} |\alpha_p| \leqslant \left\| \sum_{p=1}^{n} \alpha_p \widetilde{z}_p \right\| \leqslant \max_{p=1,\dots,n} |\alpha_p|,$$

and

$$\left\| \sum_{p=1}^{n} \alpha_p T\widetilde{z}_p \right\| = \left\| \sum_{p=1}^{n} \alpha_p TVz_p \right\| = \left\| \sum_{p=1}^{n} \alpha_p W(T)_{\mathcal{U}} z_p \right\| \leqslant \rho \max_{p=1,\dots,n} |\alpha_p|,$$

$$\left\| \sum_{p=1}^{n} \alpha_p T\widetilde{z}_p \right\| = \left\| \sum_{p=1}^{n} \alpha_p TVz_p \right\| = \left\| \sum_{p=1}^{n} \alpha_p W(T)_{\mathcal{U}} z_p \right\| \geqslant \rho(1-\varepsilon)^2 \max_{p=1,\dots,n} |\alpha_p|.$$

So, T fixes a uniform copy of ℓ_n^∞.

The second part of this lemma can be shown in a similar way but by using [**15**, Proof of Lemma 2.1] and [**2**, Lemme 1]. ∎

In our next result, we relate the above ideals with property (V). We recall that a Banach space X is said to have property (V) if for every Banach space Y, each operator $T : X \longrightarrow Y$ either fixes a copy of c_0 or is weakly compact. A Banach space X is said to have local property (V) if every ultrapower of X has property (V).

Theorem 3.2. *Let X be a Banach space such that X^{**} has local property (V). Then, an operator T from X into a Banach space Y is of Rademacher cotype if and only if it is uniformly convexifying. In particular, if Y has finite cotype, then T is uniformly convexifying and factors through a Banach space with the Banach-Saks property.*

Proof: Suppose that T is a Rademacher cotype operator. First of all, we prove that T^{**} is also a Rademacher cotype operator. If this were not the case then, by Lemma 3.1, there is an ultrafilter $\mathcal{U}$ such that $(T^{**})_{\mathcal{U}}$ fixes a copy of c_0. Since $(T^{**})_{\mathcal{U}}$ is finitely representable in T^{**} and T^{**} is finitely representable in T [**7**, Corollary 7] (see also [**1**]), we can deduce from Theorem 2.1 that there is an ultrafilter $\mathcal{V}$ such that the following diagram commutes

$$\begin{array}{ccc} (X^{**})_{\mathcal{U}} & \xrightarrow{(T^{**})_{\mathcal{U}}} & (T^{**})_{\mathcal{U}}\left((X^{**})_{\mathcal{U}}\right) \\ J\downarrow & & \downarrow \widetilde{J} \\ (X)_{\mathcal{V}} & \xrightarrow{(T)_{\mathcal{V}}} & (Y)_{\mathcal{V}}, \end{array}$$

where J and $\widetilde{J}$ are isometries. This implies that $(T)_{\mathcal{V}}$ fixes a copy of c_0 and, again by Lemma 3.1, T is not a Rademacher cotype operator. Therefore, T^{**} is a Rademacher cotype operator.

Now, take an operator $T_0 : X_0 \longrightarrow Y_0$, finitely representable in T. We have to prove that T_0 is weakly compact. It is clear that T_0 is finitely representable in T^{**}. By Theorem 2.1, there is an ultrafilter $\mathcal{U}$ over a set I and two isometries J and $\widetilde{J}$ such that the following diagram commutes

$$\begin{array}{ccc} X_0 & \xrightarrow{T_0} & T_0\,(X_0) \\ J\downarrow & & \downarrow\widetilde{J} \\ (X^{**})_{\mathcal{U}} & \xrightarrow{(T^{**})_{\mathcal{U}}} & (Y^{**})_{\mathcal{U}}. \end{array}$$

By Lemma 3.1, $(T^{**})_{\mathcal{U}}$ does not fix a copy c_0. Since $(X^{**})_{\mathcal{U}}$ has property (V), we have that $(T^{**})_{\mathcal{U}}$ is a weakly compact operator and so T_0 is weakly compact. ∎

We begin applying the above theorem to JB^*-triples. The ideal of uniformly convexifying operators defined on a JB^*-triple has been widely studied. In particular, Diestel and Seifert [**11**] proved that a weakly compact operator defined on a space of continuous functions over a compact space is a Banach-Saks operator and Jarchow, Niculescu, and Pełczyński proved that, indeed, it is uniformly convexifying [**16**]. This was generalised to C^*-algebras by Jarchow [**16, 17**], and extended to JB^*-triples by Chu and Iochum [**8**].

Corollary 3.3. *Let T be an operator from a JB^*-triple into an arbitrary Banach space. Then T is a Rademacher cotype operator if and only if it is uniformly convexifying if and only if T does not fix a copy of c_0.*

Proof: Let X be a JB^*-triple. Initially, we obtain the first equivalence. Bearing in mind the above theorem, it is enough to show that X has local property (V), that is, if $\mathcal{U}$ is an ultrafilter, we have to prove that $(X)_{\mathcal{U}}$ has property (V). But, this quickly follows from [**12**] and [**9**, Theorem 3].

In order to obtain the second equivalence, bearing in mind Lemma 3.1, we have to check that if the operator T does not fix a copy of c_0 then it is of Rademacher cotype. So, suppose that T does not fix a copy of c_0. Since X has property (V) [**9**, Theorem 3], T is a weakly compact operator and, by [**8**, p.457], it is uniformly convexifying. Hence, T is a Rademacher cotype operator. ∎

Now, we study uniformly convexifying operators on spaces of analytic functions.

Corollary 3.4. *Let T be an operator from BMOA, VMOA, the disc algebra A, or H^∞ into an arbitrary Banach space. Then T is a Rademacher cotype operator if and only if it is uniformly convexifying.*

Proof: First of all, note that $(VMOA)^{**}$ is isomorphic to $BMOA$ [**23**, Theorems 8.4.9 and 8.3.8]. Therefore, if we check that $BMOA$ has local property (V), the corollary, for the space $VMOA$, follows from Theorem 3.2. Bearing in mind [**14**, Proposition 6.7], it is clear that the $BMOA$ case can be deduced in a similar way. So, take an ultrafilter $\mathcal{U}$ over a set I. Since there is a projection P from $L^\infty(\partial D)$ onto $BMOA$ [**23**, Theorem 8.3.10], we have that $(P)_\mathcal{U}$ is a projection from $\left(L^\infty(\partial D)\right)_\mathcal{U}$ onto $(BMOA)_\mathcal{U}$. Using [**14**, Theorem 3.3] and because of property (V) is inherited by quotients, we have that $(BMOA)_\mathcal{U}$ has property (V).

The cases of the disc algebra and H^∞ are also consequences of Theorem 3.2, using that these spaces have local property (V) [**10**, Theorem 3.6 and Corollary 3.8]. ∎

A weakly compact operator from the disc algebra is uniformly convexifying [**16**]. Therefore, following a similar argument to the one given in the proof of Corollary 3.3, we have the following.

PROPOSITION 3.5. *An operator from the disc algebra into an arbitrary Banach space does not fix a copy of* c_0 *if and only if it is a Rademacher cotype operator.*

We study now the relationship between Rademacher type and uniformly convexifying operators.

COROLLARY 3.6. *Let* Y *be a Banach space such that* Y^* *has local property (V) and let* T *be an operator from an arbitrary Banach space into* Y. *Then,* T *is a Rademacher type operator if and only if it is uniformly convexifying. In particular, if* X *has non-trivial type, then* T *is uniformly convexifying and factors through a Banach space with the Banach-Saks property.*

PROOF: Suppose that T is a Rademacher type operator. Then, by [**2**, Proposition 1], T^* is a Rademacher cotype operator and, by Theorem 3.2, T^* is uniformly convexifying. Finally, using [**3**, Proposition II.4], we have that T is also uniformly convexifying. ∎

Looking at the proofs of Corollary 3.3 and above Corollary, we quickly have the following.

COROLLARY 3.7. *Let* T *be an operator from a Banach space into a predual of a* JBW^**-triple. Then,* T *is a Rademacher type operator if and only if it is uniformly convexifying.*

COROLLARY 3.8. *Let* T *be an operator from a Banach space into* H^1 *or any odd dual of the disc algebra or* H^∞. *Then* T *is a Rademacher type operator if and only if it is uniformly convexifying.*

PROOF: Since $(H^1)^*$ is isomorphic to $BMOA$ [**23**, Theorem 8.3.8] and, bearing in mind the proof of Corollary 3.4, the result, in the case of H^1, follows from Corollary 3.6.

The cases of the disc algebra and H^∞ can be shown in a similar way, but by using this time [**10**] and [**14**, Proposition 6.7]. ∎

Our next result gives a new characterisation of Rademacher type operators (compare with [**6**, p.241]).

THEOREM 3.9. *An operator T is of Rademacher type if and only if every finitely representable operator in T is weakly conditionally compact.*

PROOF: Suppose that $T : X \longrightarrow Y$ is a Rademacher type operator and take $T_0 : X_0 \longrightarrow Y_0$ a finitely representable operator in T. By Theorem 2.1, there are an ultrafilter $\mathcal{U}$ and two isometries J and $\widetilde{J}$ such that the following diagram commutes:

$$\begin{array}{ccc} X_0 & \xrightarrow{T_0} & T_0(X_0) \\ J\downarrow & & \downarrow \widetilde{J} \\ (X)_{\mathcal{U}} & \xrightarrow{(T)_{\mathcal{U}}} & (Y)_{\mathcal{U}}. \end{array}$$

By Lemma 3.1, we see that $(T)_{\mathcal{U}}$ does not fix a copy of ℓ_1, so $(T)_{\mathcal{U}}$ is weakly conditionally compact. Hence, T_0 is also weakly conditionally compact.

On the other hand, if T is not a Rademacher type operator then, using Lemma 3.1, we can find an ultrafilter $\mathcal{U}$ such that $(T)_{\mathcal{U}}$ fixes a copy of ℓ_1. The implication ends by noting that $(T)_{\mathcal{U}}$ is finitely representable in T. ∎

In the framework of Banach lattices, we also have the following.

COROLLARY 3.10. *Let T be an operator from a Banach space into a Banach lattice with finite cotype. Then, T is Rademacher type if and only if it is uniformly convexifying.*

PROOF: Suppose that $T : X \longrightarrow Y$ is a Rademacher type operator and take $T_0 : X_0 \longrightarrow Y_0$ a finitely representable operator in T. By Theorem 2.1, there are an ultrafilter $\mathcal{U}$ and two isometries J and $\widetilde{J}$ such that the following diagram commutes:

$$\begin{array}{ccc} X_0 & \xrightarrow{T_0} & T_0(X_0) \\ J\downarrow & & \downarrow \widetilde{J} \\ (X)_{\mathcal{U}} & \xrightarrow{(T)_{\mathcal{U}}} & (Y)_{\mathcal{U}}. \end{array}$$

Moreover, by Theorem 3.9, we see that $(T)_{\mathcal{U}}$ is weakly conditionally compact. Now, since Y is a Banach lattice with finite cotype, we have that $(Y)_{\mathcal{U}}$ is a Banach lattice without a copy of c_0, so by [**18**, Theorem 1.c.4], we obtain that $(Y)_{\mathcal{U}}$ is weakly sequentially complete. Therefore, $(T)_{\mathcal{U}}$ is weakly compact and, since $\widetilde{J}$ is an isometry, we deduce that T_0 is weakly compact.

Therefore, we have shown that every finitely representable operator in T is weakly compact, thus T, is uniformly convexifying. ∎

Different classical function spaces of Banach lattices with finite cotype can be found in [**18**].

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