Canad. Math. Bull. Vol. **14** (1), 1971

COLORING A DODECAHEDRON WITH FOUR COLORS(1)

BY

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The purpose of this note is to show that there is one and only one coloring of the sides of a dodecahedron with four colors, under the condition that no two adjacent sides share the same color.

In [1] Busacker and Saaty give a graphical solution to the game known as "Instant Insanity". The game is played as follows: we have four cubes to be stacked vertically, and each cube has each of its faces colored with one of four colors. The object is to stack the cubes in such a way that each vertical face of the stack contains one and only one cube face of each color. The solution (if one exists) uses a finite graph whose vertices represent the colors and whose lines represent opposition of colors across an axis of symmetry of each cube. The stacking problem has been generalized by Joan VanDeventer [2] and others.

The same technique will be applied here to show that if a dodecahedron is to be colored in such a way that each face is one of four colors, and no two faces with the same color share a side, then there is only one way to color it (up to an exchange of colors or a rotation of the figure).

To explain the proof it is convenient to adopt a chart of a dodecahedron. Imagine it unfolded on a plane. Using dots instead of pentagons for convenience and numbering the faces, the unfolded dodecahedron takes the appearance indicated in Figure 1.

The lines in the diagram represent adjacency, i.e. two pentagons share a common side if they are connected by a line in the diagram.

Using the same arrangement, the six axes of symmetry of the dodecahedron will be found by connecting opposing pairs of sides thus:

$$1\text{–}12,\ 2\text{–}9,\ 3\text{–}10,\ 4\text{–}11,\ 5\text{–}7,\ 6\text{–}8.$$

Let the four colors be white (W), red (R), green (G), and blue (B). Construct a graph having four points R, W, B, G. Let the graph have a line connecting point x with point y if the colors x and y oppose each other along an axis of symmetry of the dodecahedron. If a color x opposes itself, draw a loop at the point x. Call this the *opposition graph* of the dodecahedron.

Notice that there are exactly six lines because the dodecahedron has six axes of symmetry.

(1) Presented at the annual meeting of the Michigan section of the Mathematical Association of America, March 31, 1969.

https://doi.org/10.4153/CMB-1971-018-1 Published online by Cambridge University Press

LEMMA 1. *If the dodecahedron is colored with four colors, with no two adjacent faces having the same color, then the opposition graph can have no loops, i.e. no two opposing faces can have the same color.*

Proof. In Fig. 1, color faces 1 and 12 red (1R, 12R). Then the remaining faces must be colored with the remaining three colors. That this is impossible is easily seen by coloring (without loss of generality) 2W, 3B, which forces 7G, 8W, etc.

LEMMA 2. *With the same hypotheses as Lemma 1, the opposition graph has no doubled lines, i.e. two colors can be opposite at most once.*

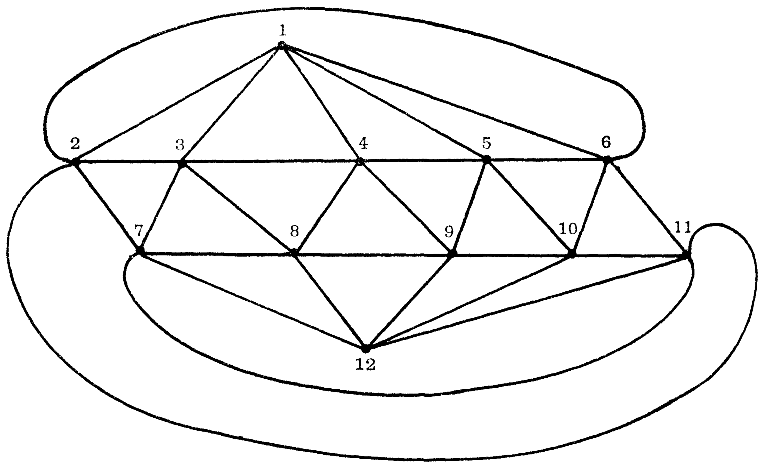

Figure 1.

Proof. Suppose for example, that in Fig. 1 we have 1R, 12W, 2W, 9R. Without loss of generality we may take 3B. This forces 4W, 8G and thus 7R. This implies that the four faces 5, 6, 10, and 11 must be colored with the two colors B and G, which is impossible.

Lemmas 1 and 2 show that our four-point six-line opposition graph can have only one form:

THEOREM 1. *The opposition graph of a dodecahedron under the hypotheses above is a complete 4-graph, i.e. each pair of vertices is on exactly one common line.*

This is a graph in which each point has degree three (i.e. is incident with exactly three lines), so each of the four colors appears on exactly three faces of the dodecahedron. This is not yet enough to see directly that there is only one coloring. If, in Fig. 1, we arbitrarily choose 1R, 12W, 2W, 4W, we may choose 9B and 11G without loss of generality (neither 9 nor 11 can be R because R already opposes W along the 1–12 axis). Since 10 has neighbors 9B, 11G, and 12W, we must have

10R. Now we have two choices for face 3. If we choose 3B then Theorem 1 forces the coloring C_1:

1	2	3	4	5	6	7	8	9	10	11	12
R	W	B	W	G	B	R	G	B	R	G	W

If, on the other hand, we set 3G we force the coloring C_2:

1	2	3	4	5	6	7	8	9	10	11	12
R	W	G	W	G	B	B	R	B	R	G	W

These two colorings are equivalent under exchange of colors and rotation, as we can see from Fig. 1, by taking the coloring C_1, exchanging R for W and inverting the figure. This gives the coloring C_2.

THEOREM 2. *If a dodecahedron is colored with four colors so that no two adjacent faces have the same color, there is only one coloring, up to a permutation of the colors or a rotation.*

REFERENCES

1. R. G. Busacker and T. L. Saaty, *Finite graphs and networks*, McGraw-Hill, New York, 1965, 153–155.

2. J. VanDeventer, *Instant insanity: On a solution by methods of graph theory*, Proc. of the Graph Theory Conference at Western Michigan Univ., Nov. 1968. Springer-Verlag lecture notes, Vol. 110.

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