BXC and XYZ lie on the same sphere ; they meet at X, but they cannot meet again, since the circle BXC meets the spherical ball, on which the circle XYZ lies, at one point only; the circle XYZ therefore *touches* BXC; and similarly it touches CYA and AZB.

Instead of a spherical ball a plane, in the form of a flat board, may be used.

Using the flat board, we can even extend the construction to the case when three plane sections of a quadric, say of an ellipsoid, are taken instead of three circles on a sphere. It easily proved that through three ellipses lying in different planes, so that any two of them have two common points, a quadric can be drawn.[1]

If the board is brought down on three such ellipses, realised in pieces of wire properly fastened together, the common section of the plane and quadric will touch the three ellipses; in other words, a conic can be drawn through the three points where the plane meets the three ellipses so as to touch them at these points; the proof is practically the same as before.

An ellipsoidal ball would not serve in this case. Its section by the plane XYZ would certainly meet each of the three ellipses at one point only, but there is no guarantee that that section is identical with the section of the given quadric by the plane XYZ, so that *meeting* in one point only does not imply *contact*.

Dr JOHN DOUGALL.

A Note on Conjugate Permutations.

§ 1. Two permutations of the natural order $(123 \ldots n)$ are said to be conjugate when each number and the number of the place in the one permutation are interchanged in the case of the other permutation.

For example (32541) and (52143) are conjugate permutations of (12345).

Conjugate permutations seem to have been first considered by H. A. Rothe[2] in 1800. They arise spontaneously, of course, when we

[1] *Cf.* H. F. Baker, *Principles of Geometry*, Vol. III., p. 10.

[2] See Muir, *History*, Part I., pp. 59-60. Netto (*Combinatorik*, p. 118) seems unaware of Rothe's work.

order the elements appearing as factors in any term of a determinant by columns instead of by rows. Thus the term $a_{13} a_{22} a_{35} a_{44} a_{51}$ when ordered by columns becomes $a_{51} a_{22} a_{13} a_{44} a_{35}$, and the second suffixes in the first ordering, (32541), and the first suffixes in the second ordering, (52143), are conjugate permutations.

Rothe gave the theorem that conjugate permutations have the same number of inversions of natural order. This he proved by a simple but ingenious argument from a chessboard of squares, reproduced by Muir[1] in his *History*. He also found but did not prove the recurrence-relation of self-conjugate permutations, viz.:—

If U_n be the number of self-conjugate permutations of the first n integers, then

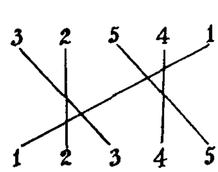
$$U_{n+1} = U_n + nU_{n-1},$$

with initial values $U_1 = 1$, $U_2 = 2$.

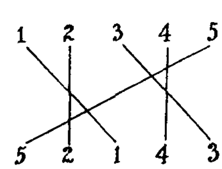
In 1899[2] Muir gave a simple proof of this relation, besides expressing U_n in various interesting forms.

In the following we give simple diagrammatic proofs of these theorems. The first is almost intuitive, the second gives visual form to Muir's proof.

§ 2. Consider then (32541) and (52143). We may compare (32541) with (12345) in respect of inversions of natural order by diagram I., in which corresponding numbers are joined by lines. Since each intersection of two lines implies an inversion of order, the number of such inversions is the number of intersections.



I.



II.

Now, leaving the lines unchanged, let us put the order-numbers (12345) *above instead of below*, and complete the lower numbers as in

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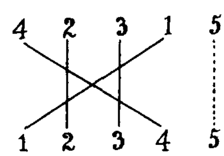
[1] *Ibid.*

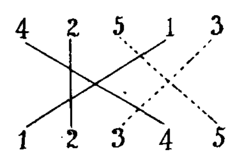
[2] *Proc. R. S. Edin.* **17** (1889), pp. 7-13.

diagram II. By definition the latter must be the conjugate permutation. But the number of intersections, that is of inversions, remains the same as before, which proves the result.

§ 3. Next, the diagram of a self-conjugate permutation must evidently remain *the same when inverted*, that is, must be symmetrical about its horizontal bisector. Consider such a diagram for any self-conjugate permutation of n numbers, and let us add points representing an additional number on the right. Then the diagram for $n + 1$ numbers may arise in two distinct ways. The two added points may represent the same number, in which case we join them by a vertical line (dotted in diagram III.), or they may represent different numbers, in which case the upper added point may be joined to one of the n lower points on the left by an oblique line, which will have an image from the lower added point, as in diagram IV.



III.



IV.

In the first case we are adding one vertical line to an n-line diagram, in each of the other n possible cases we are adding two symmetrical oblique lines to an $(n - 1)$-line diagram. Thus we have at once Rothe's relation.

$$U_{n+1} = U_n + nU_{n-1}.$$

Incidentally these criss-cross diagrams have a use in finding the sign of a term in the expansion of a determinant, or, what is the same thing, the relative class of two permutations. The elements *need not be ordered* first according to rows or columns; we merely write row-suffixes above, column-suffixes below and join corresponding numbers by lines as before. If the number of intersections is even, the term is positive, if odd, the term is negative. This rule is slightly easier to apply than a similar one given by Lloyd Tanner.[1]

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[1] Muir, *History*, Vol. III., p. 64.