NOTES ON NUMERICAL ANALYSIS III

Further Remarks on Sectionally Linear Functions

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This note is to complement the earlier paper on the same subject (Canad. Math. Bull. 3 (1960), 41-57) in two points. The first part presents a simpler proof of the minimum property (cf. l. c. section 3) of the orthogonal functions $\psi_\nu(x)$ (cf. l. c. p. 46). In the second part we introduce another orthogonal system of sectionally linear functions $\chi_o(x), \ldots, \chi_n(x)$ which leads to a particularly simple interpolation formula. These functions appeared, mutatis mutandis, in the author's study on sectionally linear functions over an infinite range about which a report will be given elsewhere.

1. <u>Minimum property of the functions</u> $\psi_\nu(x)$. Since the functions $\psi_o(x), \ldots, \psi_n(x)$ are linearly independent, it will be possible to express the $\phi_\nu(x)$ as linear combinations of the $\psi_\nu(x)$, viz.

$$\phi_o(x) = \psi_o(x),\ \phi_1(x) = \psi_1(x) - \alpha_o^{(1)}, \ldots,$$

$$\phi_m(x) = \psi_m(x) + \beta_m^{(1)}\psi_{m-1}(x) + \beta_m^{(2)}\psi_{m-2}(x) + \ldots + \beta_m^{(m)}\psi_o(x), \ \ldots$$

with certain numerical coefficients $\beta_m^{(\mu)}$. Hence

$$f_m(x) = \eta_o + \eta_1\psi_1(x) + \ldots + \eta_{m-1}\psi_{m-1}(x) + \psi_m(x)$$

with coefficients η_μ depending linearly on the ξ_μ. Thus with regard to the orthogonality relations

$$(2) \qquad (\psi_\mu, \psi_\nu) = 0,$$

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and

(4): $$(\psi_\mu, \psi_\mu) = \sigma_\mu :$$

$$(f_m, f_m) = \sigma_o \eta_o^2 + \sigma_1 \eta_1^2 + \ldots + \sigma_{m-1} \eta_{m-1}^2 + (\psi_m, \psi_m).$$

This will have its least possible value if all η_μ vanish, that is if $f_m(x) = \psi_m(x)$, q. e. d.

It need hardly be mentioned that this method of proof is well known.

2. An orthonormal system of sectionally linear functions. We consider the following system of $n + 1$ sectionally linear functions:

$$\chi_o(x) = \phi_o(x) - \frac{1}{x_1 - x_o} \phi_1(x) + \frac{1}{x_1 - x_o} \phi_2(x),$$

$$\chi_\nu(x) = \frac{1}{x_\nu - x_{\nu-1}} \phi_\nu(x) - \frac{x_{\nu+1} - x_{\nu-1}}{(x_{\nu+1} - x_\nu)(x_\nu - x_{\nu-1})} \phi_{\nu+1}(x)$$

$$+ \frac{1}{x_{\nu+1} - x_\nu} \phi_{\nu+2}(x) \qquad (\nu = 1, 2, \ldots, n-2),$$

$$\chi_{n-1}(x) = \frac{1}{x_{n-1} - x_{n-2}} \phi_{n-1}(x) - \frac{x_n - x_{n-2}}{(x_n - x_{n-1})(x_{n-1} - x_{n-2})} \phi_n(x),$$

$$\chi_n(x) = \frac{1}{x_n - x_{n-1}} \phi_n(x).$$

It is readily established that

$$\chi_\nu(x) = \begin{cases} 0 \text{ for } x \leq x_{\nu-1} \\ \text{linear increasing for } x_{\nu-1} \leq x \leq x_\nu \\ 1 \text{ for } x = x_\nu \\ \text{linear decreasing for } x_\nu \leq x \leq x_{\nu+1} \\ 0 \text{ for } x \geq x_{\nu+1} \end{cases}$$

where for $\nu = 0$ the first two, for $\nu = n$ the last two entries are to be neglected. These functions represent an orthonormal system:

$$(\chi_\mu, \chi_\nu) = \chi_\mu(x_\nu) = \begin{cases} 0 \text{ if } \mu \neq \nu \\ 1 \text{ if } \mu = \nu \end{cases}$$

and therefore a basis of the space of all sectionally linear functions over the partition $\mathcal{P}_n$.

Any such function can thus be written in the form

$$f(x) = \Sigma_{\nu=0}^{n} b_\nu \, \chi_\nu(x)$$

with the coefficients

$$b_\nu = (f, \chi_\nu) = f(x_\nu).$$

The coefficients, being the "vertex values" of the function $f(x)$, therefore require no computation at all. In particular one has

$$\phi_\nu(x) = (x_\nu - x_{\nu-1}) \, \chi_\nu(x) + (x_{\nu+1} - x_{\nu-1}) \, \chi_{\nu+1}(x) + \ldots$$

$$+ (x_n - x_{\nu-1}) \, \chi_n(x), \qquad (\nu = 1, 2, \ldots, n),$$

$$\phi_0(x) = 1 = \chi_0(x) + \chi_1(x) + \ldots + \chi_n(x).$$

It may be pointed out that in the sum

$$f(x) = \Sigma_{\nu=0}^{n} f(x_\nu) \, \chi_\nu(x)$$

for every fixed value of x in the interval $[a,b]$ at most two, consecutive, terms are different from zero: If $x_m \le x \le x_{m+1}$,

$$f(x) = f(x_m) \, \chi_m(x) + f(x_{m+1}) \, \chi_{m+1}(x).$$

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