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A Duality Theorem for Étale p-Torsion Sheaves on Complete Varieties over a Finite Field

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Abstract. Let X be an arbitrary variety over a finite field k and $p = \operatorname{char} k, n \in \mathbb{N}$. We will construct a complex of étale sheaves on X together with trace isomorphism from the highest étale cohomology group of this complex onto $\mathbb{Z}/p^n\mathbb{Z}$ such that for every constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on X the Yoneda pairing is a nondegenerate pairing of finite groups. If X is smooth, this complex is the Gersten resolution of the logarithmic de Rham–Witt sheaf introduced by Gros and Suwa. The proof is based on the special case proven by Milne when the sheaf is constant and X is smooth, as well as on a purity theorem which in turn follows from a theorem about the cohomological dimension of C_i-fields due to Kato and Kuzumaki. If the existence of the Lichtenbaum complex is proven, the theorem will be the p-part of a general duality theorem for varieties over finite fields.

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Key words: p-torsion duality theory, Lichtenbaum complex, singular varieties.

Introduction

Let X be a regular Noetherian scheme. In [Li] Lichtenbaum has conjectured the existence of complexes of Abelian étale sheaves $\mathbb{Z}(r), r \geqslant 0$ which are subject to certain axioms and which are a coefficient system for a higher dimensional arithmetic duality theory. For example one expects the following theorem to be true.

CONJECTURE ([Mi-1], p. 264). Let U be a connected smooth proper algebraic k-scheme of dimension d over a finite field k. Then there is a canonical isomorphism tr: $H_c^{2d+2}(U, \mathbb{Z}(d)) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$ and for all $i \in \mathbb{Z}$ and all constructible sheaves $\mathcal{F}$ on U the Yoneda pairing

$$H_c^i(U, \mathcal{F}) \times \operatorname{Ext}_U^{2d+2-i}(\mathcal{F}, \mathbb{Z}(d)) \longrightarrow H_c^{2d+2}(U, \mathbb{Z}(d)) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$$

is a nondegenerate pairing of finite groups.

One of the axioms postulated by Lichtenbaum is the Kummer sequence. For every natural number m prime to the characteristic p of k there is a distinguished triangle $\mathbb{Z}(r) \xrightarrow{m} \mathbb{Z}(r) \to \mu_m^{\otimes r} \to \mathbb{Z}(r)\,[1]$ in the derived category $\mathcal{D}(U)$ of étale Abelian sheaves on U. Using this sequence, the conjecture above implies a duality theorem for smooth varieties U over a finite field analogous to Poincaré duality (cf. [De-2]): For all constructible sheaves $\mathcal{F}$ such that $m\mathcal{F} = 0$ the Yoneda pairing

$$H_c^i(U, \mathcal{F}) \times \mathrm{Ext}_{U,m}^{2d+1-i}(\mathcal{F}, \mu_m^{\otimes d}) \longrightarrow H_c^{2d+1}(U, \mu_m^{\otimes d}) \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

is a nondegenerate pairing of finite groups.

The above conjecture also considers p-torsion sheaves. In [Mi-4], Milne has proposed as an additional axiom the Kummer-p-sequence, i.e. the existence of another distinguished triangle

$$\mathbb{Z}(r) \xrightarrow{p^n} \mathbb{Z}(r) \longrightarrow \nu_{n,U}^r[-r] \longrightarrow \mathbb{Z}(r)[1]$$

in $\mathcal{D}(U)$. Here $\nu_{n,U}^r = W_n\Omega_{U,\log}^r$ denotes the logarithmic de Rham–Witt sheaf defined in [Il]. The conjecture then would imply:

THEOREM. *Let U be a connected smooth separated algebraic k-scheme of dimension d over a finite field k. Then there is a canonical trace homomorphism*

$$\mathrm{tr}\colon H_c^{d+1}(U, \nu_{n,X}^d) \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$$

such that for all $i \in \mathbb{Z}$ and all constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves $\mathcal{F}$ on U the Yoneda pairing

$$H_c^i(U, \mathcal{F}) \times \mathrm{Ext}_{U,p^n}^{d+1-i}(\mathcal{F}, \nu_{n,X}^d) \longrightarrow H_c^{d+1}(U, \nu_{n,X}^d) \xrightarrow{\mathrm{tr}} \mathbb{Z}/p^n\mathbb{Z}$$

is a nondegenerate pairing of finite groups.

In this paper we will prove this theorem using an étale version of resolution of singularities obtained recently by de Jong ([dJ]). The trace homomorphism will be an isomorphism, if U can be imbedded in a smooth proper k-scheme X. For the remaining part of the introduction, we assume that $U = X$ is proper.

In [Mi-5], Milne proved the above theorem for the constant sheaf $\mathcal{F} = \mathbb{Z}/p^n\mathbb{Z}$. We prove the general case using induction on the dimension of X and proceed as in [De-1] or [Sp]: Any constructible sheaf $\mathcal{F}$ admits as a subsheaf the extension by zero of a locally constant sheaf on an open subset U such that the quotient is the direct image $\iota_*\mathcal{G}$ of a constructible sheaf $\mathcal{G}$ on the closed complement Y of U. Using a purity theorem one gets the desired assertion for $\iota_*\mathcal{G}$ from the induction hypothesis.

Of course, the closed subscheme Y thus obtained need not be smooth. In order to proceed as described above, we also have to consider singular varieties. In this case the sheaf $\nu_{n,X}^d$ has to be replaced by a complex $\widetilde{\nu}_{n,X}^d$ of étale sheaves. In

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[GrSu-1], Gros and Suwa proved, in the case where X is *smooth*, the existence of an exact sequence

$$0 \longrightarrow \nu^d_{n,X} \longrightarrow \bigoplus_{x\in X^{(0)}} \iota_{x*}\nu^d_{n,\kappa(x)} \longrightarrow \bigoplus_{x\in X^{(1)}} \iota_{x*}\nu^{d-1}_{n,\kappa(x)} \longrightarrow \cdots$$

called the Gersten resolution of $\nu^d_{n,X}$. We construct the complex $\widetilde{\nu}^d_{n,X}$ for arbitrary varieties using the theorem of Bloch and Kato (cf. also [KaCo]). A theorem of Kato and Kuzumaki about the cohomological dimension of C_i-fields then implies the purity theorem. For a closed immersion $\iota: Y \to X$ of pure codimension c one has a canonical isomorphism $\widetilde{\nu}^{d-c}_{n,Y}[-c] \cong R\iota^!\widetilde{\nu}^d_{n,X}$ in $\mathcal{D}(Y)$.

Finally we construct a trace homomorphism tr: $H^{d+1}(X, \widetilde{\nu}^d_{n,X}) \to \mathbb{Z}/p^n\mathbb{Z}$ and prove the following.

THEOREM. *Let X be a connected proper k-scheme of pure dimension d over a finite field k. Then the trace homomorphism* tr: $H^{d+1}(X, \widetilde{\nu}^d_{n,X}) \to \mathbb{Z}/p^n\mathbb{Z}$ *is an isomorphism, and for all $i \in \mathbb{Z}$ and all constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves $\mathcal{F}$ on X the Yoneda pairing*

$$H^i(X, \mathcal{F}) \times \mathrm{Ext}^{d+1-i}_{X,p^n}(\mathcal{F}, \widetilde{\nu}^d_{n,X}) \longrightarrow H^{d+1}(X, \widetilde{\nu}^d_{n,X}) \xrightarrow{\mathrm{tr}} \mathbb{Z}/p^n\mathbb{Z}$$

is a nondegenerate pairing of finite groups.

Definitions and Notations

By an étale sheaf on a scheme X we mean a sheaf on the small étale site on X. If $\mathcal{F}$ is an étale sheaf on X (a complex of étale sheaves on X), we denote by $H^q(X, \mathcal{F})$ the qth étale (hyper-)cohomology group. More generally, for any left exact functor f, we denote the hyperderived functors simply by $R^q f$. We regard an object of an Abelian category $\mathcal{A}$ also as an object of the derived category $\mathcal{D}(\mathcal{A})$. If $f: X \to Y$ is a morphism of schemes, we have the functor f_* from the category of Abelian étale sheaves on X into the category of Abelian étale sheaves on Y whose left adjoint we denote by f^{-1} (instead of f^*).

For a scheme X and $d \geqslant 0$ we denote by $X^{(d)}$ the points of codimension d and by $|X|$ the set of closed points. For any $x \in X$, let $\iota_x: \mathrm{Spec}\,\kappa(x) \to X$ be the canonical inclusion.

A morphism $f: X \to Y$ will be called *pro-étale*, if there exists a filtered projective system $(X_i)_{i\in I}$ of étale Y-schemes and affine étale morphisms such that X is isomorphic to the limit $\varprojlim X_i$ as a Y-scheme.

By a geometric point of a scheme X we mean a morphism $P: \mathrm{Spec}\,\Omega \to X$, where Ω is a separable closure of the residue field of the image point x of P. The strict Henselization of $\mathcal{O}_X$ in P is defined to be the strict Henselization of $\mathcal{O}_{X,x}$ with respect to the imbedding $\kappa(x) \to \Omega$; it will be denoted by $\mathcal{O}_{X,P}$. By $\mathcal{O}^h_{X,x}$ we denote the Henselization of $\mathcal{O}_{X,x}$.

For any $\mathbb{F}_p$-scheme X, let $\Omega^\bullet_X$ denote the absolute de Rham complex, regarded as a complex of Abelian étale sheaves on X. More generally, for any $n \geqslant 1$, let $W_n\Omega^\bullet_X$ denote the *de Rham–Witt complex* as defined in [Il], (I. 1.3). It comes along with the *restriction* $R\colon W_{n+1}\Omega^r_X \to W_n\Omega^r_X$, which commutes with the exterior product and the differential, as well as the *Verschiebung* $V\colon W_n\Omega^r_X \to W_{n+1}\Omega^r_X$, which is just a map of Abelian étale sheaves.

For $r, n \geqslant 1$, let $W_n\Omega^r_{X,\log}$ be the *logarithmic de Rham–Witt sheaf*, i.e. the subsheaf of $W_n\Omega^r_X$ generated étale locally by sections of the form

$$\mathrm{d}\underline{f_1}/\underline{f_1} \wedge \cdots \wedge \mathrm{d}\underline{f_r}/\underline{f_r},$$

$f_1, \ldots, f_r \in \mathcal{O}^*_X$, where $\underline{f} \in W_n\mathcal{O}_X$ denotes the Teichmüller representative of f, for any section f of $\mathcal{O}_X$.

Any morphism $f\colon X \to Y$ of $\mathbb{F}_p$-schemes induces a morphism $W_n\Omega^r_Y \to f_*W_n\Omega^r_X$ of étale $W_n\mathcal{O}_Y$-modules, which commutes with the exterior product, the differential and the maps R and V. If f is pro-étale, then the adjoint morphism $f^{-1}W_n\Omega^r_Y \to W_n\Omega^r_X$ is an isomorphism (cf. (loc. cit.), (I. 1.12.3)).

Let X be a scheme such that there exists a pro-étale morphism $X \to X'$ to some smooth algebraic k-scheme X', where k is a perfect field. In this case we also write $\nu^r_{n,X}$ for the étale sheaf $W_n\Omega^r_{X,\log}$. A morphism $f\colon X \to Y$ of schemes of the above type induces morphisms

$$\nu^r_{n,Y} \longrightarrow f_*\nu^r_{n,X} \quad \text{and} \quad f^{-1}\nu^r_{n,Y} \longrightarrow \nu^r_{n,X},$$

if f is pro-étale, the second one is an isomorphism. For $n = 1$ we write $\nu^r_X := \nu^r_{1,X}$.

1. The Gersten Resolution of $\nu^r_{n,X}$

In what follows, let k be a perfect field of characteristic $p > 0$, and $n \geqslant 1$ an integer.

(1.1) Let us call an extension field K of k *admissible*, if it is separably generated and of finite transcendence degree. In this case the étale sheaves $\nu^r_{n,K}$ are defined. A k-scheme X will be called *admissible*, if all of its residue fields are admissible extensions of k. If $X \to Y$ is a pro-étale morphism of k-schemes and Y is admissible then so is X.

(1.2) Let X be an excellent scheme. In [Ka-5] Kato has defined a complex

$$\bigoplus_{x\in X^{(0)}} \mathrm{K}^M_r(\kappa(x)) \longrightarrow \bigoplus_{x\in X^{(1)}} \mathrm{K}^M_{r-1}(\kappa(x)) \longrightarrow \cdots$$

using the tame symbol and the norm map of Milnor's K-Theory. We use this complex and the following theorem of Bloch–Kato in order to define the Gersten resolution of $\nu^r_{n,X}$. We do this by transferring the norm map and the tame symbol

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from K-theory to the logarithmic de Rham–Witt sheaves, where these maps will be called trace and residue map, respectively.

THEOREM ([BK], Thm. (2.1), Cor. (2.8)). *There is a unique isomorphism of Abelian groups* $d\log\colon \mathrm{K}_r^M(K)/p^n \to \nu_{n,K}^r(K)$, *which maps the symbol* $\{x_1, \ldots, x_r\}$ *to* $\mathrm{d}\underline{x_1}/\underline{x_1} \wedge \cdots \wedge \mathrm{d}\underline{x_r}/\underline{x_r}$.

(1.3) Let $K'|K$ be a finite extension of admissible extension fields of k and $\pi\colon \operatorname{Spec} K' \to \operatorname{Spec} K$ the corresponding morphism. Then there is a unique morphism of étale Abelian sheaves on $\operatorname{Spec} K$, called the *trace map*, $\mathrm{tr}\colon \pi_*\nu_{n,K'}^r \to \nu_{n,K}^r$ such that for every finite separable field extension $L|K$ and every point of the scheme $\operatorname{Spec} L \otimes_K K'$ with residue field L' the diagram

$$\begin{array}{ccc}
\mathrm{K}_r^M(L') & \xrightarrow{N_{L'|L}} & \mathrm{K}_r^M(L) \\
\downarrow{\scriptstyle \mathrm{d}\log} & & \downarrow{\scriptstyle \mathrm{d}\log} \\
\nu_{n,K'}^r(L \otimes_K K') & \xrightarrow{\mathrm{tr}} & \nu_{n,K}^r(L)
\end{array}$$

commutes.

(1.4) Let X be a locally Noetherian admissible integral normal k-scheme. Let η be the generic point of X and $x \in X^{(1)}$. Then there exists a unique morphism of Abelian étale sheaves on X, called the *residue map* $\mathrm{res}\colon \iota_{\eta*}\nu_{n,\kappa(\eta)}^r \to \iota_{x*}\nu_{n,\kappa(x)}^{r-1}$ such that for every étale morphism $f\colon U \to X$, every generic point η' of U and every point $y \in f^{-1}(x) \cap \overline{\{\eta'\}}$ the diagram

$$\begin{array}{ccc}
\mathrm{K}_r^M(\kappa(\eta')) & \xrightarrow{\partial_y} & \mathrm{K}_{r-1}^M(\kappa(y)) \\
\downarrow{\scriptstyle \mathrm{d}\log} & & \downarrow{\scriptstyle \mathrm{d}\log} \\
\nu_{n,\kappa(\eta)}^r(U_\eta) & \xrightarrow{\mathrm{res}} & \nu_{n,\kappa(x)}^{r-1}(U_x),
\end{array}$$

commutes. Here ∂_y denotes the tame symbol corresponding to the valuation of $\kappa(\eta')$ defined by y.

(1.5) Now let X be an excellent admissible k-scheme. For $d \geqslant 0$, $\eta \in X^{(d)}$ and $x \in X^{(d+1)} \cap \overline{\{\eta\}}$ we will define a morphism of étale sheaves $\mathrm{res}_x^\eta\colon \iota_{\eta*}\nu_{n,\kappa(\eta)}^r \to \iota_{x*}\nu_{n,\kappa(x)}^{r-1}$. After replacing X by the reduced subscheme $\overline{\{\eta\}}$ we can assume that X is integral with generic point η. Let $\pi\colon X' \to X$ be the normalization of X. For $x \in X^{(1)}$ let the morphism res_x^η be defined by the diagram

$$\begin{array}{ccc}
\pi_*\iota_{\eta*}\nu_{n,\kappa(\eta)}^r & \xrightarrow{\mathrm{res}} & \bigoplus_{y|x} \pi_*\iota_{y*}\nu_{n,\kappa(y)}^{r-1} \\
\| & & \downarrow{\scriptstyle \mathrm{tr}} \\
\iota_{\eta*}\nu_{n,\kappa(\eta)}^r & \xrightarrow{(-1)^d \mathrm{res}_x^\eta} & \iota_{x*}\nu_{n,\kappa(x)}^{r-1},
\end{array}$$

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where the upper morphism arises from applying the functor π_* to the residue maps corresponding to the points $y \in \pi^{-1}(x)$ as defined in (1.4). It is easy to see that one gets a map from $\iota_{\eta *}\nu^r_{n,\kappa(\eta)}$ into the *sum* of the sheaves $\iota_{x*}\nu^{r-1}_{n,\kappa(x)}$, $x \in X^{(d+1)} \cap \overline{\{\eta\}}$. Taking the sum over all $\eta \in X^{(d)}$, one gets the differentials of the complex

$$\widetilde{\nu}^r_{n,X} := \bigoplus_{x\in X^{(0)}} \iota_{x*}\nu^r_{n,\kappa(x)} \longrightarrow \bigoplus_{x\in X^{(1)}} \iota_{x*}\nu^{r-1}_{n,\kappa(x)} \longrightarrow \cdots,$$

where the first sheaf is meant to be placed in degree zero.

If $f: U \to X$ is a pro-étale morphism of excellent admissible k-schemes, then the restriction of $\widetilde{\nu}^r_{n,X}$ to U is equal to $\widetilde{\nu}^r_{n,U}$.

(1.6) If X is k-scheme such that there exists a pro-étale map into a smooth algebraic k-scheme, then the morphism $\nu^r_{n,X} \to \oplus_{\eta\in X^{(0)}}\iota_{\eta*}\nu^r_{n,\kappa(\eta)}$ obtained by functoriality gives rise to a morphism $\nu^r_{n,X} \to \widetilde{\nu}^r_{n,X}$ of complexes.

THEOREM ([GrSu-1], Cor. (1.6)). *Let X be a smooth algebraic k-scheme. Then the sequence of étale Abelian sheaves on X*

$$0 \longrightarrow \nu^r_{n,X} \longrightarrow \bigoplus_{x\in X^{(0)}} \iota_{x*}\nu^r_{n,\kappa(x)} \longrightarrow \bigoplus_{x\in X^{(1)}} \iota_{x*}\nu^{r-1}_{n,\kappa(x)} \longrightarrow \cdots$$

is exact. In other words, the morphism $\nu^r_{n,X} \to \widetilde{\nu}^r_{n,X}$ is an isomorphism in $\mathcal{D}(X)$.

Proof. Since, for any étale morphism $U \to X$, the restriction of $\widetilde{\nu}^r_{n,X}$ to U is $\widetilde{\nu}^r_{n,U}$, it is enough to prove the stronger assertion that the restriction of the above sequence to the Zariski site is exact. However, the commutativity of the diagram (4.21) in [GrSu-1] shows that this sequence agrees with the one in (loc. cit.) Cor. (1.6). □

2. The Purity Theorem for $\widetilde{\nu}^r_{n,X}$

As before, let k be a perfect field of characteristic $p > 0$.

LEMMA (2.1). *Let X be an affine k-scheme such that there exists a pro-étale morphism into a smooth algebraic k-scheme. Then $H^q(X, \nu^r_{n,X}) = 0$ for every $q \geqslant 2$ and all $r \geqslant 0, n \geqslant 1$.*

Proof. If $n = 1$, we have an exact sequence of étale Abelian sheaves on X

$$0 \longrightarrow \nu^r_X \longrightarrow \Omega^r_X \xrightarrow{C^{-1}-1} \Omega^r_X/B^r_X \longrightarrow 0,$$

where C^{-1} is the inverse of the Cartier operator and B^r_X denotes the sheaf of boundaries of the de Rham complex (cf. [Mi-2], La. (1.5)). Note that Milne's definition of ν^r_X agrees with the above by [Il], (2.4.1). Since Ω^r_X and Ω^r_X/B^r_X are quasicoherent, the long exact cohomology sequence proves the assertion. We deduce the result for $n > 1$ by induction using the exact sequence

$$0 \longrightarrow \nu^r_{n,X} \xrightarrow{\times p^m} \nu^r_{n+m,X} \xrightarrow{R^n} \nu^r_{m,X} \longrightarrow 0$$

for $n, m \geqslant 1$, which is shown to exist in [CoSS], Lemme 3. $\square$

(2.2) On the other hand, the group $H^1(X, \nu^r_{n,X})$ is an interesting invariant, even if X is the spectrum of a field. Recall that a field K is called C_i if every homogenous polynomial $f \in K[X_1, \dots, X_n]$ of degree d has a nontrivial zero whenever $n > d^i$.

PROPOSITION. *Let $i \geqslant 0$. If K is an admissible extension field of k which is C_i, then $H^1(K, \nu^i_{n,K}) = 0$ for every $n \geqslant 1$.*

Proof. The group $H^1(K, \nu^i_K)$ is the cokernel of the map on global sections induced by $C^{-1} - 1 \colon \Omega^i_K \to \Omega^i_K / B^i_K$. Since Ω^i_K and B^i_K are quasicoherent, this map agrees with the map $\mathfrak{p}$ defined in [KaKu], Definition 1. Therefore the assertion for $n = 1$ is part of (loc. cit.) Proposition 2 (2). For $n > 1$ we use the same sequence as in the proof of (2.1). $\square$

The crucial step in the proof of the purity theorem will be the following

PROPOSITION 2.3. *Let X be an algebraic k-scheme, $x \in X$ a point and $d := \dim\overline{\{x\}}$. We then have $R^q \iota_{x*} \nu^d_{n,\kappa(x)} = 0$, for all $n \geqslant 1$ and $q \geqslant 1$.*

Proof. Replacing X by the normalization of the reduced subscheme $\overline{\{x\}}$ and using the Leray spectral sequence, we can assume that X is integral and normal. For every geometric point P of X the stalk of the sheaf in question is

$$(R^q \iota_{x*} \nu^d_{n,\kappa(x)})_P = H^q(K_P, \nu^d_{n,K_P}),$$

where K_P is the quotient field of the strict henselization $\mathcal{O}_{X,P}$. K_P contains the algebraic closure $\bar{k}$ of k and is algebraic over the function field $K(X)$ of X. Therefore $K_P|\bar{k}$ is like $K(X)|k$ an extension of transcendence degree d. Hence K_P is a C_d-field from which the assertion follows, using (2.1) and (2.2).

THEOREM 2.4. *Let X be an algebraic k-scheme of dimension d and $\iota \colon Y \to X$ a closed immersion. Then the canonical morphism $\iota^! \widetilde{\nu}^d_{n,X} \to R\iota^! \widetilde{\nu}^d_{n,X}$ is an isomorphism in $\mathcal{D}(Y)$ for all $n \geqslant 1$.*

Proof. It is enough to show that the complex $\widetilde{\nu}^d_{n,X}$ consists of $\iota^!$-acyclic objects. But from (2.3) one gets $R^q \iota^! \iota_{x*} \nu^{d-i}_{n,\kappa(x)} = R^q(\iota^! \iota_{x*}) \nu^{d-i}_{n,\kappa(x)}$ for every $i \geqslant 0$, $x \in X^{(i)}$ and $q \geqslant 1$, using the Leray spectral sequence. This group vanishes trivially, if $x \notin Y$. If $x \in Y$, then $\iota^! \iota_{x*}$ is equal to the direct image functor corresponding to the inclusion $\operatorname{Spec} \kappa(x) \to Y$. Another application of (2.3) completes the proof. $\square$

Let X be an equidimensional algebraic k-scheme and $\iota\colon Y \to X$ a closed immersion of pure codimension c. Then we have canonically $\widetilde{\nu}^{d-c}_{n,Y}[-c] = \iota^! \widetilde{\nu}^d_{n,X}$. Thus the above theorem gives.

COROLLARY. *Let X be an algebraic k-scheme of pure dimension d and $\iota\colon Y \to X$ a closed immersion of pure codimension c. Then one has a canonical isomorphism $\widetilde{\nu}^{d-c}_{n,Y}[-c] \to R\iota^! \widetilde{\nu}^d_{n,X}$ in $\mathcal{D}(Y)$, for all $n \geqslant 1$.*

The following proposition is another important consequence of the theorem of Kato and Kuzumaki.

PROPOSITION 2.5. *Let X be a smooth algebraic k-scheme of dimension d and f its structure morphism. Then $\nu^d_{n,X} \to \widetilde{\nu}^d_{n,X}$ is an f_*-acyclic resolution of $\nu^d_{n,X}$ for all $n \geqslant 1$.*

Proof. Let $n \geqslant 1$, $i \geqslant 0$ and $q \geqslant 1$. We have to show that $R^q f_*(\iota_{x*}\nu^{d-i}_{n,\kappa(x)}) = 0$ for all $x \in X^{(i)}$. Let $\bar{k}$ be an algebraic closure of k. The above sheaf vanishes if and only if its stalk at the geometric point given by $\bar{k}$ does. Using (2.3) we therefore have to show that $H^q(\kappa(x), \nu^{d-i}_{n,\kappa(x)}) = 0$, for all points x of $X \times_k \bar{k}$ of codimension i. This group vanishes for $q \geqslant 2$, since the cohomological p-dimension of a field of characteristic p is at most 1. Now let $q = 1$. Since $\dim\overline{\{x\}} = d - i$, the field $\kappa(x)$ is an extension of $\bar{k}$ of transcendence degree $d - i$ and therefore a C_{d-i}-field. Now the assertion follows from (2.2).

3. The Cohomology of $\nu^r_{n,X}$ with Compact Support

We first recall some elementary facts about sheaves with supports (cf. for example [Ha-1], Chapter IV).

(3.1) For any scheme X, an Abelian étale sheaf $\mathcal{F}$ on X and a closed subset $Y \subset X$ we denote by $\Gamma_Y(X, \mathcal{F}) = \ker(\mathcal{F}(X) \to \mathcal{F}(X \setminus Y))$ the group of all sections s with support $\operatorname{supp} s \subset Y$. This gives a functor from the category of Abelian étale sheaves into the category of Abelian groups, whose qth right derived functor is denoted by $H^q_Y(X, \mathcal{F})$. For every $p \geqslant 0$ we set

$$\Gamma_{[p]}(X, \mathcal{F}) := \{s \in \mathcal{F}(X) | \operatorname{codim}_X(\operatorname{supp} s) \geqslant p\}.$$

If $x \in X$ is a point, we define the group $\Gamma_x(\mathcal{F}) := \lim_{\substack{\to \\ U}} \Gamma_{\overline{\{x\}}\cap U}(U, \mathcal{F})$, where U runs through all *Zariski*-open neighbourhoods of x. It consists of the elements of the Zariski stalk of $\mathcal{F}$ at x, which can be represented by a local section with support in $\overline{\{x\}}$. Again we have a functor $\mathcal{F} \longmapsto \Gamma_x(\mathcal{F})$ from the category of Abelian étale sheaves into the category of Abelian group. Its right derived functors are $H^q_x(\mathcal{F}) = \lim_{\substack{\to \\ U}} H^q_{\overline{\{x\}}\cap U}(U, \mathcal{F})$.

(3.2) Let X be a locally Noetherian scheme, $\mathcal{F}$ an Abelian étale sheaf on X, $p \geqslant 0$ and $x \in X^{(p)}$. Then the image of the canonical map $\Gamma_{[p]}(X, \mathcal{F}) \to \mathcal{F}_x$ is contained in $\Gamma_x(\mathcal{F})$ and it is easy to prove the following

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LEMMA. *Let X be a Noetherian scheme and $\mathcal{F}$ a flabby Abelian étale sheaf on X. Then, for any $p \geqslant 0$, the stalk maps induce an isomorphism*

$$\Gamma_{[p]}(X, \mathcal{F})/\Gamma_{[p+1]}(X, \mathcal{F}) \longrightarrow \bigoplus_{x \in X^{(p)}} \Gamma_x(\mathcal{F})$$

of groups.

PROPOSITION 3.3. *Let X be a Noetherian scheme and $\mathcal{F}$ be an Abelian étale sheaf on X. Then there is a spectral sequence*

$$E_1^{pq} = \bigoplus_{x \in X^{(p)}} H_x^{p+q}(\mathcal{O}_{X,x}^h, \mathcal{F}) \Longrightarrow E^n = H^n(X, \mathcal{F}),$$

which is functorial in $\mathcal{F}$.

Proof. If $\mathcal{F} \to I^\bullet$ is an injective resolution, then the complex $K^\bullet := \Gamma(X, I^\bullet)$ of global sections is filtered by the subcomplexes

$$F^p K^\bullet := \Gamma_{[p]}(X, I^\bullet) \subset \Gamma(X, I^\bullet).$$

By (3.2) we have an isomorphism of complexes of Abelian groups

$$\Gamma_{[p]}(X, I^\bullet)/\Gamma_{[p+1]}(X, I^\bullet) \cong \bigoplus_{x \in X^{(p)}} \Gamma_x(I^\bullet).$$

The cohomology groups of this complex can be identified with the $E_1^{p,q}$-terms above. Therefore the spectral sequence associated to the filtered complex $K^\bullet$ is the desired one. □

(3.4) If X is a Noetherian $\mathbb{F}_p$-scheme of dimension $d = \dim X$ and $\mathcal{F}$ an étale p-torsion sheaf, then $H^q(X, \mathcal{F}) = 0$ for all $q > d+1$ ([SGA IV], exp. 10, Theorem (5.1)). As an immediate consequence, if A is a d-dimensional Noetherian local ring of characteristic p, x the closed point of Spec A and $\mathcal{F}$ an étale p-torsion sheaf, we have

$$H_x^q(A, \mathcal{F}) = 0 \quad \text{for} \quad q > d+1.$$

COROLLARY (3.5). *Let X be a Noetherian $\mathbb{F}_p$-scheme of dimension d and $\mathcal{F}$ an étale p-torsion sheaf on X. Then there is a canonical surjective homomorphism*

$$\bigoplus_{x \in X^{(d)}} H_x^{d+1}(X, \mathcal{F}) \longrightarrow H^{d+1}(X, \mathcal{F}).$$

Proof. For the spectral sequence (3.3) we have $E_1^{pq} = 0$ for $p > d$, for dimension reasons. Therefore there is an edge morphism $E_1^{d1} \to E^{d+1}$. From

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(3.4) it follows that $E_1^{pq} = 0$ for $q \geqslant 2$, and therefore this edge morphism is surjective. Since every point of codimension d is closed, the excision theorem gives $H_x^{d+1}(X, \mathcal{F}) \cong H_x^{d+1}(\mathcal{F})$. □

(3.6) A morphism $A \to B$ of local rings will be called an *iterated Henselization*, if it is the composition of a sequence of morphisms

$$A = A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_n = B,$$

where one has for every i: Either $A_i \to A_{i+1} = A_{i\,\mathfrak{p}}$ is the canonical map into the localization at a prime ideal $\mathfrak{p}$ of A_i or it is the canonical map $A_i \to A_{i+1} = A_i^h$ into the Henselization.

Let X be a scheme. By an *iterated Henselization of* X we understand a Henselian local ring R together with a morphism Spec $R \to X$ which factorizes over Spec $\mathcal{O}_{X,x}$ for some $x \in X$, such that the induced morphism $\mathcal{O}_{X,x} \to R$ is an iterated Henselization as defined above.

If R is an iterated Henselization of X, then the corresponding morphism Spec $R \to X$ is pro-étale.

LEMMA (3.7). *Let X be a Noetherian scheme of dimension d. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of étale p-torsion sheaves. Suppose there is a dense open subset on which φ induces an isomorphism. Further suppose that, for every iterated Henselization R of X with* dim $R = 1$ *the morphism $H^1(R, \mathcal{F}) \to H^1(R, \mathcal{G})$ induced by φ is an isomorphism. Then φ induces an isomorphism $H^{d+1}(X, \mathcal{F}) \to H^{d+1}(X, \mathcal{G})$.*

Proof. We proceed by induction on $d = \dim X$. If $d = 0$, φ is an isomorphism by assumption. Now let $d \geqslant 1$. We first assert that φ induces an isomorphism on the relative cohomology groups

$$H_x^{n+1}(R, \mathcal{F}) \longrightarrow H_x^{n+1}(R, \mathcal{G}),$$

for every iterated Henselization R of X; here x is the closed point of Spec R and $n = \dim R$. If $n = 0$, then Spec $R \to X$ factorizes over Spec $\mathcal{O}_{X,\eta}$, for some generic point η of X. Hence in this case, φ even induces an isomorphism of the restrictions $\mathcal{F}|_R \to \mathcal{G}|_R$. Now let $n > 0$ and $V := \operatorname{Spec} R \setminus \{x\}$. The relative cohomology sequence gives rise to a diagram

$$\begin{array}{ccccccc}
H^n(R, \mathcal{F}) & \longrightarrow & H^n(V, \mathcal{F}) & \longrightarrow & H_x^{n+1}(R, \mathcal{F}) & \longrightarrow & H^{n+1}(R, \mathcal{F}) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow \\
H^n(R, \mathcal{G}) & \longrightarrow & H^n(V, \mathcal{G}) & \longrightarrow & H_x^{n+1}(R, \mathcal{G}) & \longrightarrow & H^{n+1}(R, \mathcal{G})
\end{array}$$

with exact rows. Since the local ring R is Henselian and $\mathcal{F}$ is a p-torsion sheaf, we have $H^{n+1}(R, \mathcal{F}) = H^{n+1}(\kappa, \mathcal{F}|_\kappa) = 0$, where κ is the residue field of R.

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Therefore the objects on the very right of the diagram vanish, as well as the objects on the very left, if $n > 1$. If $n = 1, \alpha$ is an isomorphism by assumption. Therefore it is enough to show that β is an isomorphism. But V is a Noetherian $(n-1)$-dimensional scheme and $n-1 < d$. Now the restriction $\varphi\colon \mathcal{F}|_V \to \mathcal{G}|_V$ also satisfies the conditions of the lemma and therefore the induction hypothesis concludes the proof of the above assertion.

Now regard the spectral sequences (3.3) for the scheme X and the sheaves $\mathcal{F}$ and $\mathcal{G}$ and the morphism of spectral sequences induced by φ. For both spectral sequences one has $E_1^{pq} = 0$ for $p > d$ and $q \geq 2$. This gives $E^{d+1} = E_2^{d,1}$. The above assertion implies that φ induces an isomorphism on the $E_1^{p,1}$-terms

$$\bigoplus_{x\in X^{(p)}} H_x^{p+1}(\mathcal{O}_{X,x}^h, \mathcal{F}) \longrightarrow \bigoplus_{x\in X^{(p)}} H_x^{p+1}(\mathcal{O}_{X,x}^h, \mathcal{G})$$

for $p = d-1, d$. Hence φ also induces an isomorphism on the $E_2^{d,1}$-terms which concludes the proof. □

PROPOSITION (3.8). *Let $R|k$ be an excellent Henselian discrete valuation ring such that there exists a pro-étale morphism from Spec R into some smooth algebraic k-scheme. Suppose the residue field κ of R is an extension of k of transcendence degree $\leqslant d$. Then $H^q(R, \widetilde{\nu}_{n,R}^{d+1}) = 0$ for all $q \geqslant 1, n \geqslant 1$.*

Proof. It is enough to show the assertion for $n = 1$ since the general case can be deduced from that by induction using the second exact sequence of the proof of (2.1). Let F be the quotient field of R and $j\colon \mathrm{Spec}\, F \to \mathrm{Spec}\, R$ and $\iota\colon \mathrm{Spec}\,\kappa \to \mathrm{Spec}\, R$ the canonical morphisms. Looking at the hypercohomology spectral sequence we see that we have to show that the residue map res: $j_*\nu_F^{d+1} \to \iota_*\nu_\kappa^d$ induces a surjective map on global sections $H^0(F, \nu_F^{d+1}) \to H^0(\kappa, \nu_\kappa^d)$ and an isomorphism on the first cohomology groups

$$H^1(R, j_*\nu_F^{d+1}) \longrightarrow H^1(R, \iota_*\nu_\kappa^d). \tag{*}$$

The first assertion is clear. In order to proof the second assertion, we first show that the higher direct images $R^q j_*\nu_F^{d+1}, q \geqslant 1$ are zero. The stalk of this sheaf at the generic point vanishes trivially, and the stalk at the closed point x is $(R^q j_*\nu_F^{d+1})_{\bar{x}} = H^q(F^{sh}, \nu_{F^{sh}}^{d+1})$, where F^{sh} denotes the quotient field of the strict Henselization R^{sh} of R at the point $\bar{x}$. Now R^{sh} is an excellent strictly Henselian discrete valuation ring with residue field κ^{sep}. By hypothesis this field is a C_d-field. By [KaKu], Proposition 2 (2) and Theorem 1 (4) it follows from this that $H^1(F^{sh}, \nu_{F^{sh}}^{d+1}) = 0$, arguing as in the proof of (2.2). Thus we have shown the above assertion and we can view $(*)$ as a homomorphism

$$H^1(F, \nu_F^{d+1}) \longrightarrow H^1(\kappa, \nu_\kappa^d),$$

which is equal to the homomorphism defined in [KaCo], (1.3) and, hence, an isomorphism by (loc. cit), La. (1.4), (3). □

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PROPOSITION (3.9). *Let X be a normal algebraic k-scheme of dimension d and $j: U \to X$ the inclusion of a dense open subset. Then the adjunction map $j_!\nu^d_{n,U} \to \nu^d_{n,X}$ induces an isomorphism $H^{d+1}(X, j_!\nu^d_{n,U}) \to H^{d+1}(X, \nu^d_{n,X})$.*

Proof. By Lemma (3.7) it is enough to show that, for every iterated Henselization R of X with $\dim R = 1$, the adjunction induces an isomorphism

$$H^1(R, j_!\nu^d_{n,U}) \longrightarrow H^1(R, \nu^d_{n,X}).$$

Since X is normal, R is an excellent Henselian discrete valuation ring. Let κ its residue field. If the morphism $\operatorname{Spec} R \to X$ factorizes over U, the assertion is trivial. Otherwise we have $(j_!\nu^d_{n,U})|_\kappa = 0$, hence $H^1(R, j_!\nu^d_{n,U}) = H^1(\kappa, (j_!\nu^d_{n,U})|_\kappa) = 0$. Thus in this case we have to show that $H^1(R, \nu^d_{n,X}) = 0$ as well. But the pro-étale morphism $\operatorname{Spec} R \to X$ leads to the open subset $V := X_{\mathrm{reg}}$ of all regular points of X. Therefore the canonical map $\nu^d_{n,R} \to \widetilde{\nu}^d_{n,R}$ is the restriction of $\nu^d_{n,V} \to \widetilde{\nu}^d_{n,V}$ and, hence, an isomorphism in $\mathcal{D}(\operatorname{Spec} R)$ by (1.6). This implies that $H^1(R, \nu^d_{n,R}) = H^1(R, \widetilde{\nu}^d_{n,R})$. Let $x \in X^{(1)}$ be the image of the closed point of $\operatorname{Spec} R$. Then the transcendence degree of the extension $x|\mathbb{F}_p$ is at most $d-1$. Since $\kappa|\kappa(x)$ is an algebraic extension, we can apply (3.8) to obtain the desired result. □

4. The Trace Homomorphism

PROPOSITION (4.1). *Let k be a perfect field of characteristic $p > 0$ and X be a proper k-scheme of pure dimension d with structure morphism $f: X \to \operatorname{Spec} k$. Then there is a unique morphism*

$$\mathrm{tr}\colon Rf_*\widetilde{\nu}^d_{n,X} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}[-d]$$

in $\mathcal{D}(\operatorname{Spec} k)$ whose composition with the canonical morphism $f_*\widetilde{\nu}^d_{n,X} \to Rf_*\widetilde{\nu}^d_{n,X}$ is given in degree d by the morphism of étale sheaves on $\operatorname{Spec} k$

$$f_* \bigoplus_{x\in|X|} \iota_{x*}\mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$$

defined by the trace maps corresponding to the finite extensions $\kappa(x)|k$, $x \in |X|$ (1.3).

Proof. If Y is a connected smooth proper curve over k and g is an element of the function field $K(Y)$ of Y we have the equation $\sum_{y\in|Y|}[\kappa(y): k] \cdot v_y(g) = 0$, where we have denoted by v_y the normalized valuation of $K(Y)$ corresponding to y. Therefore the above morphism is actually a morphism of complexes $f_*\widetilde{\nu}^d_{n,X} \to \mathbb{Z}/p^n\mathbb{Z}[-d]$ of étale sheaves on $\operatorname{Spec} k$. Since by (2.5) the canonical morphism $f_*\widetilde{\nu}^d_{n,X} \to Rf_*\widetilde{\nu}^d_{n,X}$ is an isomorphism in $\mathcal{D}(\operatorname{Spec} k)$, the proof is complete. □

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In case k is a finite field we want to characterize the trace map by its values on certain fundamental classes which are now going to be defined.

(4.2) Let X and Y be algebraic k-schemes of pure dimension d and d_0, respectively, and let $\iota: Y \to X$ be a closed immersion. By $\varphi(Y, X)$ we denote the composition

$$H^{d_0+1}(Y, \tilde{\nu}_{n,Y}^{d_0}) \xrightarrow{\cong} H_Y^{d+1}(X, \tilde{\nu}_{n,X}^{d}) \longrightarrow H^{d+1}(X, \tilde{\nu}_{n,X}^{d}),$$

where the first morphism is induced by the isomorphism $\tilde{\nu}_{n,Y}^{d_0}[d_0 - d] \to R\iota^! \tilde{\nu}_{n,X}^{d}$ in $\mathcal{D}(Y)$ obtained from the purity theorem (2.4) and the second one is the canonical morphism. The morphism $\varphi(Y, X)$ has the following functorial property: If $Z \to Y \to X$ is a composition of closed immersions and X, Y and Z are of pure dimension d, d_0 and d', respectively, then the diagram

$$\begin{array}{ccc} H^{d'+1}(Z, \tilde{\nu}_{n,Z}^{d'}) & \xrightarrow{\varphi(Z,X)} & H^{d+1}(X, \tilde{\nu}_{n,X}^{d}). \\ \Big\downarrow{\scriptstyle \varphi(Z,Y)} & & \Big\| \\ H^{d_0+1}(Y, \tilde{\nu}_{n,Y}^{d_0}) & \xrightarrow{\varphi(Y,X)} & H^{d+1}(X, \tilde{\nu}_{n,X}^{d}). \end{array}$$

commutes.

(4.3) Now let k be a finite field and p its characteristic. Let X be an algebraic k-scheme of pure dimension d. If $x \in X$ is a closed point, we have just defined a morphism

$$\varphi(x, X): \mathbb{Z}/p^n\mathbb{Z} = H^1(\kappa(x), \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow H^{d+1}(X, \tilde{\nu}_{n,X}^{d}),$$

which maps $1 \in \mathbb{Z}/p^n\mathbb{Z}$ to an element denoted by ϵ_x and called the *fundamental class* at the point x. We will show that, if X is connected, the fundamental class does not depend on the closed point x chosen. If X happens to be smooth, then by (1.6) we have an isomorphism

$$H^{d+1}(X, \nu_{n,X}^{d}) \xrightarrow{\cong} H^{d+1}(X, \tilde{\nu}_{n,X}^{d}),$$

in this case we view the fundamental class ϵ_x also as an element of $H^{d+1}(X, \nu_{n,X}^{d})$, for every closed point x of X. Corollary (3.5) tells us, that $H^{d+1}(X, \nu_{n,X}^{d})$ is generated as an Abelian group by the fundamental classes of all the closed points of X.

PROPOSITION (4.4). *Let k be a finite field and p its characteristic. Let X be a proper k-scheme of pure dimension d. Then there is a unique homomorphism*

$$\mathrm{tr}: H^{d+1}(X, \tilde{\nu}_{n,X}^{d}) \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$$

such that $\mathrm{tr}(\epsilon_x) = 1$ *for every closed point* $x \in X$.

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Proof. We first show the uniqueness of tr. Using (2.3) and the fact that the cohomological p-dimension of a field of characteristic p is at most 1, the hypercohomology spectral sequence gives rise to an exact sequence

$$\bigoplus_{x\in X^{(d-1)}} H^1(\kappa(x), \nu^1_{n,\kappa(x)}) \longrightarrow \bigoplus_{x\in|X|} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow H^{d+1}(X, \tilde{\nu}^d_{n,X}) \longrightarrow 0,$$

where the second map is given by the sum of the homomorphisms $\varphi(x, X)$ defined in (4.3). This means that $H^{d+1}(X, \tilde{\nu}^d_{n,X})$ is generated as an Abelian group by the fundamental classes ϵ_x of the closed points x of X.

To show existence, define tr to be the map induced on the $(d+1)$-st cohomology groups by the trace map defined in (4.1), using the identification $H^1(k, \mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}/p^n\mathbb{Z}$. Now let x be a closed point of X. We will show that $\mathrm{tr}(\epsilon_x) = 1$: We have a morphism of complexes of étale sheaves $\alpha\colon \iota_{x*}\mathbb{Z}/p^n\mathbb{Z}[-d] \to \tilde{\nu}^d_{n,X}$, which is just given by the inclusion of the x-component of the direct sum on the right-hand side. On cohomology it induces the morphism

$$\varphi(x, X)\colon \mathbb{Z}/p^n\mathbb{Z} = H^{d+1}(\kappa(x), \mathbb{Z}/p^n\mathbb{Z}[-d]) \longrightarrow H^{d+1}(X, \tilde{\nu}^d_{n,X})$$

defined in (4.3). Since the trace map defined in (1.3) corresponding to the finite extension $\kappa(x)|k$ coincides with the usual trace map given by the étale morphism $\mathrm{Spec}\,\kappa(x) \to \mathrm{Spec}\,k$, the composition of $f_*(\alpha)$ with the trace map defined in (4.1) induces the corestriction map on cohomology groups

$$\mathrm{cor}\colon H^1(\kappa(x), \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow H^1(k, \mathbb{Z}/p^n\mathbb{Z});$$

If both groups are identified with $\mathbb{Z}/p^n\mathbb{Z}$, this map becomes the identity. This shows $\mathrm{tr}(\epsilon_x) = 1$, as required. □

The main assertion of this chapter says that the trace map is an isomorphism, if X is connected. If X is a smooth curve, we prove this fact by comparing it with the trace homomorphism defined by Milne in [Mi-2]. In the general case we proceed by induction on the dimension of X using the purity theorem.

From now on, k denotes a finite field and p its characteristic.

(4.5) Let X be a connected smooth proper k-scheme of dimension d. In [Mi-2], Milne has defined an isomorphism $\mathrm{tr}_M\colon H^{d+1}(X, \nu^d_X) \to \mathbb{Z}/p\mathbb{Z}$ which is uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc}
H^d(X, \Omega^d_X) & \xrightarrow{\ t\ } & k \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\mathrm{tr}} \\
H^{d+1}(X, \nu^d_X) & \xrightarrow{\ \mathrm{tr}_M\ } & \mathbb{Z}/p\mathbb{Z}.
\end{array}$$

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Here t is the trace map of Serre duality (cf. [Mi-2], definition of $\eta = \mathrm{tr}$ preceding (1.9)), δ the connecting homomorphism arising from the first exact sequence of the proof of (2.1), and the vertical arrow on the right is the trace of the finite field extension $k|\mathbb{F}_p$.

For the proof of the next proposition we need the following elementary result from homological algebra:

LEMMA (4.6). *Let X be a scheme, $j: U \to X$ the inclusion of an open subset of X and Y the complement of U. Let $\mathcal{F}$ be an Abelian étale sheaf on X such that the adjunction map $\mathcal{F} \to j_* j^{-1}\mathcal{F}$ is injective. Then the diagram*

$$\begin{array}{ccc} H^0(X, j_* j^{-1}\mathcal{F}) & \longrightarrow & H^0(X, \mathcal{C}) = H^0_Y(X, \mathcal{C}) \\ \| & & \downarrow \delta \\ H^0(U, j^{-1}\mathcal{F}) & \longrightarrow & H^1_Y(X, \mathcal{F}) \end{array}$$

is anticommutative, where $\mathcal{C}$ denotes the cokernel of the adjunction map, the lower horizontal map is taken from the relative cohomology sequence and δ is the connecting homomorphism induced by the short exact sequence which defines $\mathcal{C}$.

PROPOSITION (4.7). *Let X be a connected smooth proper curve over k. Then* $\mathrm{tr}_M(\epsilon_x) = 1$ *for every closed point $x \in X$, i.e. for $n = 1$ the trace map* tr *defined in* (4.4) *coincides with the trace map* tr_M *defined by Milne.*

Proof. Let $x \in X$ be a closed point, $A = \mathcal{O}_{X,x}$ and $K = \mathrm{Quot}A$. Using excision and the relative cohomology sequence for the closed subset $\{x\}$ of Spec A we get an isomorphism $H^1_x(X, \Omega^1_X) = \Omega^1_K/\Omega^1_A$. The composition

$$\mathrm{Res}_x\colon \Omega^1_K \longrightarrow \Omega^1_K/\Omega^1_A \longrightarrow H^1(X, \Omega^1_X) \xrightarrow{\ t\ } k$$

is just the Tate residue map (cf. [AK], Chapter VIII, Theorem (4.4)). On the other hand there is an A-linear map

$$\begin{array}{ccc} \kappa(x) & \longrightarrow & \Omega^1_K/\Omega^1_A \\ \bar{a} & \longmapsto & a\, d\pi/\pi, \end{array}$$

independent of the choice of a prime element π of A. Now consider the diagram

$$\begin{array}{ccccccc} \kappa(x) & \xrightarrow{z_x} & H^1_x(X, \Omega^1_X) & \longrightarrow & H^1(X, \Omega^1_X) & \xrightarrow{\ t\ } & k \\ \downarrow \mathrm{tr} & & \downarrow \delta & & \downarrow \delta & & \downarrow \mathrm{tr} \\ \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\varphi_x} & H^2_x(X, \nu^1_X) & \longrightarrow & H^2(X, \nu^1_X) & \xrightarrow{\ \mathrm{tr}\ } & \mathbb{Z}/p\mathbb{Z}. \end{array} \qquad (1)$$

By construction of the Tate residue map the composition of the morphisms of the first row is just the trace of the finite extension $\kappa(x)|k$. Now the squares in the

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middle and on the right commute. Once we have shown that the square on the left commutes as well, we are done. We can assume that X is the spectrum of the local ring $A = \mathcal{O}_{X,x}$. Let $j\colon \operatorname{Spec} K \to X$ and $\iota\colon \operatorname{Spec} \kappa(x) \to X$ be the canonical inclusions. Consider the following commutative diagram of étale sheaves

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nu_X^1 & \longrightarrow & \Omega_X^1 & \xrightarrow{1-C} & \Omega_X^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_*\nu_K^1 & \longrightarrow & j_*\Omega_K^1 & \xrightarrow{1-C} & j_*\Omega_K^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \iota_*\mathbb{Z}/p\mathbb{Z} & \longrightarrow & j_*\Omega_K^1/\Omega_X^1 & \longrightarrow & j_*\Omega_K^1/\Omega_X^1 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where C is the Cartier operator (which exists since X is smooth). The first row is exact as shown in [Mi] Lemma (1.3). From this follows the exactness of the second row since $R^1 j_*\nu_K^1 = 0$. The first column is exact by (1.6). The two remaining columns are exact, since Ω_A^1 is a free A-module. Finally, the last row is exact by the snake lemma. From this we get an anticommutative diagram of connecting homomorphisms

$$
\begin{array}{ccc}
H_x^0(X, j_*\Omega_K^1/\Omega_X^1) & \longrightarrow & H_x^1(X, \Omega_X^1) \\
\downarrow & & \downarrow \\
H_x^1(X, \iota_*\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & H_x^2(X, \nu_X^1).
\end{array}
\qquad (2)
$$

If we identify both groups of the upper row with Ω_K^1/Ω_A^1, the map between them becomes *multiplication by* -1, by the preceding lemma. The A-linear map z_x defines a morphism of étale $\mathcal{O}_X$-modules $\iota_*\mathcal{O}_{\kappa(x)} \to j_*\Omega_K^1/\Omega_X^1$ which gives rise to a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \iota_*\mathbb{Z}/p\mathbb{Z} & \longrightarrow & \iota_*\mathcal{O}_{\kappa(x)} & \xrightarrow{1-F^{-1}} & \iota_*\mathcal{O}_{\kappa(x)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \iota_*\mathbb{Z}/p\mathbb{Z} & \longrightarrow & j_*\Omega_K^1/\Omega_X^1 & \xrightarrow{1-F^{-1}} & j_*\Omega_K^1/\Omega_X^1 & \longrightarrow & 0
\end{array}
$$

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with exact rows. Here $F\colon \mathcal{O}_{\kappa(x)} \to \mathcal{O}_{\kappa(x)}$, $s \mapsto s^p$ denotes the Frobenius map. From that we get another diagram of relative cohomology groups

$$\begin{array}{ccc} H^0_x(X, \iota_*\mathcal{O}_{\kappa(x)}) & \longrightarrow & H^0_x(X, j_*\Omega^1_K/\Omega^1_X) \\ \downarrow & & \downarrow \\ H^1_x(X, \iota_*\mathbb{Z}/p\mathbb{Z}) & = & H^1_x(X, \iota_*\mathbb{Z}/p\mathbb{Z}), \end{array} \tag{3}$$

where we can identify the morphism on the left with the trace map $\mathrm{tr}\colon \kappa(x) \to \mathbb{Z}/p\mathbb{Z}$. Matching the diagrams (2) and (3) together gives the square on the left in (1). This completes the proof. □

COROLLARY (4.8). *Let X be a smooth connected proper curve over k. Then the trace map* $\mathrm{tr}\colon H^2(X, \nu^1_{n,X}) \to \mathbb{Z}/p^n\mathbb{Z}$ *is an isomorphism, for every* $n \geqslant 1$.

Proof. For $n = 1$ this is an immediate consequence of (4.7). The trace map is obviously surjective for all $n \geqslant 1$. For $n > 1$ the second exact sequence of the proof of (2.1) gives an exact sequence

$$H^2(X, \nu^1_{n-1,X}) \longrightarrow H^2(X, \nu^1_{n,X}) \longrightarrow H^2(X, \nu^1_X)$$

Inductively, we conclude that the group in the middle has at most p^n elements. Therefore tr is an isomorphism. □

The next theorem states that the trace map is an isomorphism in the general case. For its proof, which is based on induction on the dimension, we need some assertions about the existence of appropriate closed subschemes.

LEMMA (4.9). *Let K be an arbitrary field. Let X be a connected (resp. irreducible quasiprojective) algebraic K-scheme of dimension* $\dim X \geqslant 1$. *If x and y are closed points of X, then there exists a connected (resp. irreducible) curve $C \subset X$, which contains x and y.*

Proof. It is enough to prove the statement in brackets. We can assume that X is integral and projective. Further it is easy to see that we can assume that K is algebraically closed, in which case the lemma is well known. □

LEMMA (4.10). *Let K be a perfect field and X a reduced connected proper K-scheme with the property that every irreducible component of X is of dimension $\geqslant 2$. Suppose Z is a closed subset of X which does not contain any irreducible component of X. Then there is a connected closed subset Y of X of pure codimension* 1 *which contains Z and the complement of which is* smooth *and* affine.

Proof. If X is a Noetherian reduced affine scheme and Z a closed subset of pure codimension 1, there is always a closed subset Y of pure codimension 1 which contains Z and the complement of which is affine. In order to prove this assertion we can assume that Z is irreducible. Let A be the ring of global sections of X

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and $\mathfrak{p}$ the prime ideal of A defining Z. Since A is reduced, there is an element $f \in \mathfrak{p}$ which is not a zero divisor of A. Then $Y := \mathrm{Spec}(A/fA)$ has the desired properties.

Now let X and Z be as in the hypothesis of the lemma. In every irreducible component of X we choose a nonempty smooth affine open subset, which does not meet Z. Now if Z' is the complement of the union of the open subsets thus obtained, Z' does not contain any irreducible component of X and we have $Z \subset Z'$. Replacing Z by Z' we can thus assume that the complement V of Z is smooth and affine. By (4.9) Z is contained in a connected closed subset Y_0 of X of pure codimension 1. By the first paragraph $Y_0 \cap V$ is contained in a closed subset Y' of V of pure codimension 1 the complement of which is affine. The closure of every irreducible component of Y' in X intersects Z, since it is a proper K-scheme of dimension $\geqslant 1$ which cannot be contained in the affine scheme V. If we define Y to be the union of the closure of Y' in X and the irreducible components of Y_0 which do not meet V, then Y has the desired properties.

(4.11) The following easy counting lemma can be stated in terms of $\mathbb{Z}/p^n\mathbb{Z}$-divisors. We quickly give their definition: If X is an algebraic curve over some field K, by a $\mathbb{Z}/p^n\mathbb{Z}$-divisor D we mean an element of the free $\mathbb{Z}/p^n\mathbb{Z}$-module generated by the closed points of X. By the degree $\deg D$ of D we mean the sum of all of its coefficients. If $f: X \to Y$ is morphism of algebraic curves and D is a $\mathbb{Z}/p^n\mathbb{Z}$-divisor on Y, then the $\mathbb{Z}/p^n\mathbb{Z}$-divisor $f^{-1}D$ or $f|_X$ is defined by the rule that its coefficient at a closed point x of X is the coefficent of D at the point $f(x)$. Conversely, if D is a $\mathbb{Z}/p^n\mathbb{Z}$-divisor on X, then f_*D is defined to be the $\mathbb{Z}/p^n\mathbb{Z}$-divisor on Y whose coefficient at a closed point $y \in Y$ is the sum of the coefficients of D at all closed points of the fibre $f^{-1}(y)$.

LEMMA. *Let $X = X_1 \cup \cdots \cup X_s$ be the decomposition of a connected algebraic curve over some field K into its irreducible components, and suppose that $s > 1$. Let $Y = X_1 \cup \cdots \cup X_{s-1}$, $Z = X_s$ and $p: Y \amalg Z \to X$ the unique morphism whose restrictions to Y and Z are the inclusions. If D is a $\mathbb{Z}/p^n\mathbb{Z}$-divisor on X with $\deg D = 0$, then there is a $\mathbb{Z}/p^n\mathbb{Z}$-divisor D' on $Y \amalg Z$ with $p_*D' = D$ and $\deg D|_Y = \deg D|_Z = 0$.* □

THEOREM (4.12). *Let X be a proper k-scheme of pure dimension d. If X is connected, then the trace map $\mathrm{tr}: H^{d+1}(X, \widetilde{\nu}^d_{n,X}) \to \mathbb{Z}/p^n\mathbb{Z}$ is an isomorphism.*

Proof. We can assume that X is reduced. From the proof of (4.4) we know that we have

$$H^{d+1}(X, \widetilde{\nu}^d_{n,X}) = \ker\left(\bigoplus_{x \in X^{(d-1)}} H^1(\kappa(x), \nu^1_{n,\kappa(x)}) \longrightarrow \bigoplus_{x \in |X|} \mathbb{Z}/p^n\mathbb{Z}\right)$$

and that the trace is induced by the summation map $\oplus_{x\in|X|}\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. For $d = 0$ the theorem is trivial. Now let $d = 1$. First assume that X is irreducible. Let $\pi\colon X' \to X$ its normalization. The diagram in (1.5) gives a morphism of complexes of étale sheaves $\pi_*\tilde{\nu}^1_{n,X'} \to \tilde{\nu}^1_{n,X}$. It induces the square on the left in the diagram

$$
\begin{array}{ccccccc}
H^1(K(X'), \nu^1_{n,K(X')}) & \longrightarrow & \bigoplus_{x\in|X'|} \mathbb{Z}/p^n\mathbb{Z} & \longrightarrow & \mathbb{Z}/p^n\mathbb{Z} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
H^1(K(X), \nu^1_{n,K(X)}) & \longrightarrow & \bigoplus_{x\in|X|} \mathbb{Z}/p^n\mathbb{Z} & \longrightarrow & \mathbb{Z}/p^n\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

where the right horizontal arrows are given by the trace maps of X' and X. It is easy to see that the vertical arrow in the middle is given by summation over the fibres of π, i.e. equal to π_* in the notation of (4.11). This implies that the square on the right is also commutative. Now the upper sequence is exact by (4.8). It follows that the lower sequence is exact as well which concludes the proof if X is an irreducible curve.

In the general case of a curve we conclude the proof by induction on the number r of irreducible components of X. Let $r > 1$. Let Y be the union of the first $r - 1$ irreducible components of X, Z the rth component and $p\colon Y \amalg Z \to X$ be the map induced by the inclusions. Since X and $Y \amalg Z$ have the same normalization, there is a commutative diagram

$$
\begin{array}{ccccc}
\bigoplus_{\eta\in X^{(0)}} H^1(\kappa(\eta), \nu^1_{n,\kappa(\eta)}) & \longrightarrow & \bigoplus_{y\in|Y\coprod Z|} \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{\beta} & \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z} \\
\| & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle s} \\
\bigoplus_{\eta\in X^{(0)}} H^1(\kappa(\eta), \nu^1_{n,\kappa(\eta)}) & \longrightarrow & \bigoplus_{x\in|X|} \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{\deg} & \mathbb{Z}/p^n\mathbb{Z},
\end{array}
$$

where β maps a $\mathbb{Z}/p^n\mathbb{Z}$-divisor D to the pair with components $\deg D|_Y$ and $\deg D|_Z$, and s denotes the summation map. By induction hypothesis, the upper sequence is exact. Hence, using Lemma (4.11), we get that the lower sequence is exact as well. This concludes the proof of the theorem for $d = 1$.

For the general case we use induction on d. Let $d > 1$ and let X be a reduced connected proper algebraic k-scheme of pure dimension d. By (4.10) there is a connected closed subscheme $\iota\colon Y \to X$ of pure codimension 1, the complement U of which is smooth and affine. By (1.6) and (2.1) these properties of U imply that the morphism $\varphi(Y, X)\colon H^d(Y, \tilde{\nu}^{d-1}_{n,Y}) \xrightarrow{\cong} H^{d+1}(X, \tilde{\nu}^d_{n,X})$ defined in (4.2) is an isomorphism. For any closed point $y \in Y$ it maps the fundamental class ϵ_y to

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the fundamental class of y viewed as a closed point of X. This implies that the diagram

$$\begin{array}{ccc} H^d(Y, \widetilde{\nu}^{d-1}_{n,Y}) & \xrightarrow{\text{tr}} & \mathbb{Z}/p^n\mathbb{Z} \\ \downarrow{\scriptstyle \varphi(Y,X)} & & \| \\ H^{d+1}(X, \widetilde{\nu}^{d}_{n,X}) & \xrightarrow{\text{tr}} & \mathbb{Z}/p^n\mathbb{Z} \end{array}$$

commutes. The theorem follows by the induction hypothesis. □

(4.13) We finally extend the definition of the trace map to cohomology with compact supports.

If U is a separated algebraic k-scheme, by a *compactification* of U we understand a dominant open immersion $U \to X$ where X is a proper k-scheme. Such a compactification always exists by [Na]. We call U *smoothly compactifiable*, if there exists a compactification into a smooth k-scheme X.

If $\mathcal{F}$ is an étale Abelian torsion sheaf on U and $q \geqslant 0$, the qth cohomology group with compact support is defined by $H^q_c(U, \mathcal{F}) := H^q(X, j_!\mathcal{F})$, where $j\colon U \to X$ is a compactification. By the proper base change theorem this is independent of the chosen compactification.

(4.14) Let U a smooth separated algebraic k-scheme of pure dimension d. Let $j\colon U \to X$ be a compactification. If $x \in U$ is a closed point, the excision theorem as well as (2.4) and (1.6) give isomorphisms

$$\mathbb{Z}/p^n\mathbb{Z} \cong H^{d+1}_x(U, \widetilde{\nu}^d_{n,U}) \cong H^{d+1}_x(U, \nu^d_{n,U}) \cong H^{d+1}_x(X, j_!\nu^d_{n,U}).$$

Therefore we have a morphism $\mathbb{Z}/p^n\mathbb{Z} \to H^{d+1}_c(U, \nu^d_{n,U})$, which does not depend on the compactification j. The image of 1 will again be called the *fundamental class* of x and denoted by ϵ_x. The morphism

$$H^{d+1}_c(U, \nu^d_{n,U}) \longrightarrow H^{d+1}(X, \widetilde{\nu}^d_{n,X})$$

induced by the composition of the adjunction map $j_!\nu^d_{n,U} \to \nu^d_{n,X}$ and the canonical morphism $\nu^d_{n,X} \to \widetilde{\nu}^d_{n,X}$ maps this fundamental class to the fundamental class already defined in (4.1). From (3.5) we get that $H^{d+1}_c(U, \nu^d_{n,U})$ is generated as an Abelian group by the fundamental classes of the closed points of U. (3.9) and (4.12) imply

COROLLARY. *Let U be a smooth separated algebraic k-scheme of pure dimension d. Then there is a unique homomorphism* $\text{tr}\colon H^{d+1}_c(U, \nu^d_{n,U}) \to \mathbb{Z}/p^n\mathbb{Z}$, *which maps the fundamental class ϵ_x of every closed point $x \in U$ to* 1. *If U is connected and smoothly compactifiable, then* tr *is an isomorphism.*

(4.15) Let $f: U' \to U$ be a finite étale morphism of separated algebraic k-schemes and $\mathcal{F}$ an étale Abelian torsion sheaf on U. There is a cartesian diagram

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ U & \xrightarrow{j} & X \end{array}$$

where j and j' are compactifications and $\bar{f}$ is finite. Applying the functor $j_!$ to the trace morphism tr: $f_* f^{-1}\mathcal{F} \to \mathcal{F}$ and using $\bar{f}_* j'_! = j_! f_*$, one gets a homomorphism

$$\mathrm{tr}: H_c^q(U', f^{-1}\mathcal{F}) \longrightarrow H_c^q(U, \mathcal{F}),$$

which does not depend on the diagram chosen. It is easy to prove the following

LEMMA. *Let $f: U' \to U$ be finite étale morphism of smooth separated algebraic k-schemes. Then the diagram*

$$\begin{array}{ccc} H_c^{d+1}(U', \nu_{n,U'}^d) & \xrightarrow{\mathrm{tr}} & \mathbb{Z}/p^n\mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{tr}} & & \| \\ H_c^{d+1}(U, \nu_{n,U}^d) & \xrightarrow{\mathrm{tr}} & \mathbb{Z}/p^n\mathbb{Z} \end{array}$$

commutes.

5. The Duality Theorem

We will need the following theorem concerning resolution of singularities.

THEOREM (5.1) [dJ], Theorem (4.1). *Let k be a perfect field and X an integral separated algebraic k-scheme. Then there is a smoothly compactifiable integral algebraic k-scheme V and an étale morphism $V \to X$.*

From that we get the following statement: Let k be a perfect field, X a reduced separated algebraic k-scheme and $\mathcal{F}$ a constructible étale Abelian sheaf on X. Then there exists a dense open subset $U \subset X$ and a surjective finite étale morphism $V \to U$ such that V is smoothly compactifiable and $\mathcal{F}|_V$ is a constant sheaf.

Now let k again be a finite field and $p = \mathrm{char}\, k$.

(5.2) Let X be a proper k-scheme of pure dimension d and $\mathcal{F}$ an étale Abelian $\mathbb{Z}/p^n\mathbb{Z}$-sheaf. Then the composition of the Yoneda pairing with the trace map (4.12) is a $\mathbb{Z}$-bilinear map

$$H^i(X, \mathcal{F}) \times \mathrm{Ext}_{X,p^n}^{d+1-i}(\mathcal{F}, \tilde{\nu}_{n,X}^d) \longrightarrow H^{d+1}(X, \tilde{\nu}_{n,X}^d) \xrightarrow{\mathrm{tr}} \mathbb{Z}/p^n\mathbb{Z}.$$

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It induces a homomorphism of groups

$$\alpha^i(X,\mathcal{F})\colon H^i(X,\mathcal{F}) \longrightarrow \mathrm{Ext}^{d+1-i}_{X,p^n}(\mathcal{F},\widetilde{\nu}^d_{n,X})^*,$$

where $A^* = \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$ denotes the dual group of any Abelian group A; by $\mathrm{Ext}^q_{X,p^n}(\mathcal{F},\_\_)$ we denote the qth derived functor of the functor $\mathrm{Hom}(\mathcal{F},\_\_)$ from the category of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X into the category of groups.

If U is a separated algebraic k-scheme of pure dimension d and $\mathcal{F}$ an étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaf, there is a unique pairing

$$H^i_c(U,\mathcal{F}) \times \mathrm{Ext}^{d+1-i}_{U,p^n}(\mathcal{F},\nu^d_{n,U}) \longrightarrow H^{d+1}_c(U,\nu^d_{n,U})$$

such that for every compactification $j\colon U \to X$ the diagram

$$\begin{array}{ccccc} H^i_c(U,\mathcal{F}) & \times & \mathrm{Ext}^{d+1-i}_{U,p^n}(\mathcal{F},\nu^d_{n,U}) & \longrightarrow & H^{d+1}_c(U,\nu^d_{n,U}) \\ \| & & \downarrow\cong & & \| \\ H^i(X,j_!\mathcal{F}) & \times & \mathrm{Ext}^{d+1-i}_{X,p^n}(j_!\mathcal{F},j_!\nu^d_{n,U}) & \longrightarrow & H^{d+1}(X,j_!\nu^d_{n,U}) \end{array}$$

commutes.

We first prove the duality theorem for constant sheaves. Milne has shown (cf. [Mi-1], Chapter II, (7.12)) that if X is a smooth proper connected algebraic k-scheme of dimension d and $n \geqslant 1$, the Yoneda pairing

$$H^i(X,\mathcal{F}) \times \mathrm{Ext}^{d+1-i}_{X,p^n}(\mathcal{F},\nu^d_{n,X}) \longrightarrow H^{d+1}(X,\nu^d_{n,X}) \cong \mathbb{Z}/p^n\mathbb{Z}$$

is a nondegenerate pairing of finite groups for the sheaf $\mathcal{F} = \mathbb{Z}/p^n\mathbb{Z}$. In order to get the assertion for the constant sheaves $\mathcal{F} = \mathbb{Z}/p^m\mathbb{Z}$, $m \leqslant n$ we prove some easy facts from homological algebra:

(5.3) Let X be a scheme. For every $n \geqslant 1$ the category $S(X,\mathbb{Z}/p^n\mathbb{Z})$ of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X is an Abelian category with enough injectives. For $m \leqslant n$ the inclusion functor $S(X,\mathbb{Z}/p^m\mathbb{Z}) \subset S(X,\mathbb{Z}/p^n\mathbb{Z})$ is exact but does not preserve injectives. It is left adjoint to the functor

$$t_{n,m}\colon S(X,\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow S(X,\mathbb{Z}/p^m\mathbb{Z})$$
$$\mathcal{F} \longmapsto {}_{p^m}\mathcal{F},$$

where ${}_{p^m}\mathcal{F}$ is the kernel of the multiplication with p^m. The functor $t_{n,m}$ is left exact and preserves injectives. We now calculate its derived functors: Let $\mathcal{F}$ be an étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaf and $\mathcal{F} \to I^\bullet$ an injective resolution in $S(X,\mathbb{Z}/p^n\mathbb{Z})$. By applying the functor $\mathrm{Hom}(\_\_, I^q)$, $q \geqslant 0$, to the exact sequence

$$0 \longrightarrow \mathbb{Z}/p^{n-m}\mathbb{Z} \xrightarrow{p^m} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z} \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow {}_{p^m}I^\bullet \longrightarrow I^\bullet \xrightarrow{p^m} {}_{p^{n-m}}I^\bullet \longrightarrow 0$$

of complexes of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X. The corresponding long exact cohomology sequence yields an exact sequence

$$0 \longrightarrow {}_{p^m}\mathcal{F} \longrightarrow \mathcal{F} \xrightarrow{p^m} {}_{p^{n-m}}\mathcal{F} \longrightarrow R^1 t_{n,m}\mathcal{F} \longrightarrow 0$$

and isomorphisms $R^q t_{n,n-m}\mathcal{F} \cong R^{q+1}t_{n,m}\mathcal{F}$ of Abelian étale sheaves, for $q \geqslant 1$.

LEMMA (5.4). *Let X be a smooth algebraic k-scheme. Let $r \geqslant 0$, $n \geqslant 1$ and $m \leqslant n$. Then the morphism $\times p^{n-m}$ defined in* [CoSS], p. 778, *Equation* (20) *induces an isomorphism $\times p^{n-m}\colon \nu^r_{m,X} \xrightarrow{\cong} t_{n,m}\nu^r_{n,X} = {}_{p^m}\nu^r_{n,X}$, and for all $q \geqslant 1$ we have $R^q t_{n,m}\nu^r_{n,X} = 0$.*

Proof. The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \nu^r_{m,X} & \xrightarrow{\times p^{n-m}} & \nu^r_{n,X} & \xrightarrow{R^m} & \nu^r_{n-m,X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle p^m} & & \downarrow{\scriptstyle \times p^m} & & \\
 & & 0 & \longrightarrow & \nu^r_{n,X} & \xrightarrow{\mathrm{id}} & \nu^r_{n,X} & \longrightarrow & 0
\end{array}$$

has exact lines (cf. proof of (2.1)). Regarding the induced morphism on kernels proves the first assertion. It follows that the composition

$$\nu^r_{n,X} \xrightarrow{R^m} \nu^r_{n-m,X} \xrightarrow{\times p^m} {}_{p^{n-m}}\nu^r_{n,X}$$

is surjective. On the other hand, by definition of $\times p^m$ this composition is the multiplication by p^m. By (5.3) we therefore get $R^1 t_{n,m}\nu^r_{n,X} = 0$, for all $m < n$. The second assertion follows by induction on q. □

PROPOSITION (5.5). *Let X be a connected smooth proper k-scheme of dimension d and $n \geqslant 1$. Then for every $m \leqslant n$ the composition of the Yoneda pairing with the trace map*

$$H^i(X, \mathbb{Z}/p^m\mathbb{Z}) \times \mathrm{Ext}^{d+1-i}_{X,p^n}(\mathbb{Z}/p^m\mathbb{Z}, \nu^d_{n,X}) \longrightarrow H^{d+1}(X, \nu^d_{n,X}) \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$$

is a nondegenerate pairing of finite groups.

Proof. If $m = n$, this is just the assertion of [Mi-1], Chapter II, (7.12). Now let $m < n$. By the preceding lemma we have an isomorphism $\nu^d_{m,X} \cong Rt_{n,m}\nu^d_{n,X}$ in $\mathcal{D}(X, \mathbb{Z}/p^m\mathbb{Z})$; since $t_{n,m}$ is right adjoint to the inclusion functor $S(X, \mathbb{Z}/p^m\mathbb{Z}) \subset S(X, \mathbb{Z}/p^n\mathbb{Z})$, one has isomorphisms

$$\mathrm{Ext}^q_{X,p^n}(\mathbb{Z}/p^m\mathbb{Z}, \nu^d_{n,X}) = \mathrm{Ext}^q_{X,p^m}(\mathbb{Z}/p^m\mathbb{Z}, Rt_{n,m}\nu^d_{n,X}) \cong \mathrm{Ext}^q_{X,p^m}(\mathbb{Z}/p^m\mathbb{Z}, \nu^d_{m,X}),$$

for all $q \geqslant 0$. In the commutative diagram

$$\begin{array}{ccc} H^i(X,\mathbb{Z}/p^m\mathbb{Z})\times \mathrm{Ext}^{d+1-i}_{X,p^n}(\mathbb{Z}/p^m\mathbb{Z},\nu^d_{n,X}) & \longrightarrow & H^{d+1}(X,\nu^d_{n,X}) \\ \| \qquad\qquad \uparrow\cong & & \uparrow\alpha \\ H^i(X,\mathbb{Z}/p^m\mathbb{Z})\times \mathrm{Ext}^{d+1-i}_{X,p^m}(\mathbb{Z}/p^m\mathbb{Z},\nu^d_{m,X}) & \longrightarrow & H^{d+1}(X,\nu^d_{m,X}) \end{array}$$

α is induced by the morphism $\times p^{n-m}$. The lower line is a nondegenerate pairing, hence also the upper line, if we can show that α is injective. The short exact sequence used in the proof of (2.1) induces an exact sequence

$$H^{d+1}(X,\nu^d_{m,X}) \xrightarrow{\alpha} H^{d+1}(X,\nu^d_{n,X}) \longrightarrow H^{d+1}(X,\nu^d_{n-m,X}) \longrightarrow 0$$

where we have used (3.4). Since by (4.12) the group $H^{d+1}(X,\nu^d_{s,X}) \cong \mathbb{Z}/p^s\mathbb{Z}$ has p^s elements, this implies the injectivity of α. □

THEOREM (5.6). *Let k be a finite field and $p = \mathrm{char}\, k$. Let X be a proper k-scheme of pure dimension d. Then for every $n \geqslant 1$ and every constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf $\mathcal{F}$ the composition of the Yoneda pairing and the trace homomorphism*

$$H^i(X,\mathcal{F}) \times \mathrm{Ext}^{d+1-i}_{X,p^n}(\mathcal{F},\tilde{\nu}^d_{n,X}) \longrightarrow H^{d+1}(X,\tilde{\nu}^d_{n,X}) \xrightarrow{\mathrm{tr}} \mathbb{Z}/p^n\mathbb{Z}$$

is a nondegenerate pairing of finite groups.

Proof. Let $n \geqslant 1$. We have to show that $\alpha^i(X,\mathcal{F})$ is an isomorphism for all proper equidimensional k-schemes X, all constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves $\mathcal{F}$ on X and all $i \in \mathbb{Z}$. We will do this in several steps.

(a) *Let X be a proper k-scheme of pure dimension d. Let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X. If $\alpha^i(X,\mathcal{G})$ is an isomorphism for two of the sheaves $\mathcal{G} \in \{\mathcal{F},\mathcal{F}',\mathcal{F}''\}$ so it is for the third.

(b) *Let X be as in (a) and $\mathcal{F}$ an étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on X. Then we have for every $i \in \mathbb{Z}$: The map $\alpha^i(X,\mathcal{F})$ is an isomorphism if and only if $\alpha^i(X_{\mathrm{red}},\mathcal{F})$ is.*

By [EGA IV], (18.1.2) the functor $U \mapsto U \times_X X_{\mathrm{red}}$ is an equivalence of topologies. If $\iota\colon X_{\mathrm{red}} \to X$ denotes the inclusion, then ι^{-1} and ι_* are mutually quasi-inverse equivalences of the categories of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X and X_{red}.

(c) *Let X and $\mathcal{F}$ be as in (b). If $X_1,\ldots,X_s$ are the connected components of X, then we have for every $i \in \mathbb{Z}$: The map $\alpha^i(X,\mathcal{F})$ is an isomorphism if and only if all the $\alpha^i(X_\nu,\mathcal{F})$ are isomorphisms, $\nu = 1,\ldots,s$.*

The intervening cohomology groups as well as the Ext-group are direct sums of the corresponding groups for the components X_ν, and the trace map $H^{d+1}(X,\tilde{\nu}^d_{n,X}) \to \mathbb{Z}/p^n\mathbb{Z}$ is the sum of the trace maps of the components.

(d) *Let X be a smooth proper k-scheme of pure dimension d. Then $\alpha^i(X, \mathcal{F})$ is an isomorphism for all constant constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves $\mathcal{F}$ on X and all $i \in \mathbb{Z}$.*

This follows from (5.5) using (a) and (c).

(e) *Let $K|k$ be a finite field extension, $X = \operatorname{Spec} K$ and $\mathcal{F}$ a constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on X. Then $\alpha^i(X, \mathcal{F})$ is an isomorphism of finite groups, for all $i \in \mathbb{Z}$.*

The assertion follows from (d), if $\mathcal{F}$ is constant. Let $\mathcal{F}$ be arbitrary and $L|K$ a finite extension such that $f^{-1}\mathcal{F}$ is constant, where $f\colon \operatorname{Spec} L \to \operatorname{Spec} K$ denotes the induced morphism. Consider the commutative diagram

$$\begin{array}{ccccccc}
H^i(L, f^{-1}\mathcal{F}) \times \operatorname{Ext}^{1-i}_{L,p^n}(f^{-1}\mathcal{F}, \mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & H^1(L, \mathbb{Z}/p^n\mathbb{Z}) & \xrightarrow{\text{tr}} & \mathbb{Z}/p^n\mathbb{Z} \\
\uparrow\cong \qquad\qquad \downarrow\cong & & \downarrow \text{cor} & & \| \\
H^i(K, f_*f^{-1}\mathcal{F}) \times \operatorname{Ext}^{1-i}_{K,p^n}(f_*f^{-1}\mathcal{F}, \mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & H^1(K, \mathbb{Z}/p^n\mathbb{Z}) & \xrightarrow{\text{tr}} & \mathbb{Z}/p^n\mathbb{Z},
\end{array}$$

for $i = 0, 1$. Since the upper line is a nondegenerate pairing of finite groups so is the lower. It follows that $\alpha^i(\operatorname{Spec} K, f_*f^{-1}\mathcal{F})$ is an isomorphism for all $i \in \mathbb{Z}$. We now prove the following assertion by descending induction on i: For every constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on $\operatorname{Spec} K$ the map $\alpha^i(\operatorname{Spec} K, \mathcal{F})$ is surjective. This is trivial for $i > 1$. Now let $i \leqslant 1$. For a given sheaf $\mathcal{F}$ we choose a morphism $f\colon \operatorname{Spec} L \to \operatorname{Spec} K$ such that $f^{-1}\mathcal{F}$ is constant. The trace map $f_*f^{-1}\mathcal{F} \to \mathcal{F}$ is surjective and therefore gives rise to an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow f_*f^{-1}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow 0;$$

using the abbreviations $H^q(\mathcal{G}) := H^q(K, \mathcal{G})$ and $E^q(\mathcal{G}) := \operatorname{Ext}^{1-q}_{K,p^n}(\mathcal{G}, \mathbb{Z}/p^n\mathbb{Z})^*$ we get a commutative diagram

$$\begin{array}{ccccccccc}
H^i(f_*f^{-1}\mathcal{F}) & \longrightarrow & H^i(\mathcal{F}) & \longrightarrow & H^{i+1}(\mathcal{K}) & \longrightarrow & H^{i+1}(f_*f^{-1}\mathcal{F}) & \longrightarrow & H^{i+1}(\mathcal{F}) \\
\downarrow\cong & & \downarrow & & \downarrow & & \cong\downarrow & & \downarrow \\
E^i(f_*f^{-1}\mathcal{F}) & \longrightarrow & E^i(\mathcal{F}) & \longrightarrow & E^{i+1}(\mathcal{K}) & \longrightarrow & E^{i+1}(f_*f^{-1}\mathcal{F}) & \longrightarrow & E^{i+1}(\mathcal{F}).
\end{array}$$

Applying the induction hypothesis to the sheaf $\mathcal{K}$ the diagram shows that the map $\alpha^i(\operatorname{Spec} K, \mathcal{F})$ is surjective. Similarly, using descending induction on i, one shows that $\alpha^i(\operatorname{Spec} K, \mathcal{F})$ is injective, which concludes the proof of (e).

Now we prove the theorem by induction on d. For $d = 0$ the assertion follows from (e) using (b) and (c). Now let $d > 0$ and assume that $\alpha^i(X, \mathcal{F})$ is an isomorphism for all proper k-schemes X of pure dimension $\dim X < d$, all constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves $\mathcal{F}$ on X and all $i \in \mathbb{Z}$.

(f) *Let X be a proper k-scheme of pure dimension d, $\iota\colon Y \to X$ a closed immersion of pure codimension 1 and $\mathcal{G}$ a constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on Y. Then $\alpha^i(X, \iota_*\mathcal{G})$ is an isomorphism for all $i \in \mathbb{Z}$.*

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This follows from the induction hypothesis using the commutative diagram

$$\begin{array}{ccccccc}
H^i(Y, \mathcal{G}) & \times & \mathrm{Ext}^{d-i}_{Y,p^n}(\mathcal{G}, \widetilde{\nu}^{d-1}_{n,Y}) & \longrightarrow & H^d(Y, \widetilde{\nu}^{d-1}_{n,Y}) & \xrightarrow{\mathrm{tr}} & \mathbb{Z}/p^n\mathbb{Z} \\
\uparrow \cong & & \downarrow \cong & & \downarrow \varphi(Y,X) & & \| \\
H^i(X, \iota_*\mathcal{G}) & \times & \mathrm{Ext}^{d+1-i}_{X,p^n}(\iota_*\mathcal{G}, \widetilde{\nu}^{d}_{n,X}) & \longrightarrow & H^{d+1}(X, \widetilde{\nu}^{d}_{n,X}) & \xrightarrow{\mathrm{tr}} & \mathbb{Z}/p^n\mathbb{Z}.
\end{array}$$

The isomorphism of the Ext-groups is obtained from the purity theorem (2.4).

(g) *Let X be a smooth proper k-scheme of pure dimension d. Let $j: U \to X$ be the inclusion of an open subset and $\mathcal{F}$ a constant constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on U. Then $\alpha^i(X, j_!\mathcal{F})$ is an isomorphism for all $i \in \mathbb{Z}$.*

We can assume that X is connected. The constant étale sheaf on X corresponding to the Abelian group which defines $\mathcal{F}$ will be also denoted by $\mathcal{F}$. The cokernel $\mathcal{C}$ of the injective adjunction map $j_!\mathcal{F} \to \mathcal{F}$ is the direct image of the constant étale sheaf $\mathcal{F}$ on the complement of U. Enlarging this complement if necessary we see that $\mathcal{C}$ is also the direct image of a constructible sheaf on a closed subscheme of pure codimension 1. Now the assertion follows from (d), (f) and (a).

(h) *Let V be a smoothly compactifiable k-scheme of pure dimension d and $\mathcal{G}$ a constant constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on V. Then the composition*

$$H^i_c(V, \mathcal{G}) \times \mathrm{Ext}^{d+1-i}_{V,p^n}(\mathcal{G}, \nu^d_{n,V}) \longrightarrow H^{d+1}_c(V, \nu^d_{n,V}) \xrightarrow{\mathrm{tr}} \mathbb{Z}/p^n\mathbb{Z}$$

is a nondegenerate pairing.

In view of the definition of this pairing in (5.2) this is an immediate consequence of (g).

(i) *Let X be a proper k-scheme of pure dimension d, $j: U \to X$ an open immersion and $f: V \to U$ a finite étale morphism such that V is smoothly compactifiable. Let $\mathcal{F}$ a constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on U, whose restriction to V is constant. Then $\alpha^i(X, j_!f_*f^{-1}\mathcal{F})$ is an isomorphism for all $i \in \mathbb{Z}$.*

One has a commutative diagram (cf. [Mi-3], p. 171)

$$\begin{array}{ccccccc}
H^i_c(V, f^{-1}\mathcal{F}) & \times & \mathrm{Ext}^{d+1-i}_{V,p^n}(f^{-1}\mathcal{F}, \nu^d_{n,V}) & \longrightarrow & H^{d+1}_c(V, \nu^d_{n,V}) & \xrightarrow{\mathrm{tr}} & \mathbb{Z}/p^n\mathbb{Z} \\
\uparrow \cong & & \uparrow \cong & & \downarrow \mathrm{tr} & & \| \\
H^i_c(U, f_*f^{-1}\mathcal{F}) & \times & \mathrm{Ext}^{d+1-i}_{U,p^n}(f_*f^{-1}\mathcal{F}, \nu^d_{n,U}) & \longrightarrow & H^{d+1}_c(U, \nu^d_{n,U}) & \xrightarrow{\mathrm{tr}} & \mathbb{Z}/p^n\mathbb{Z}.
\end{array}$$

Hence the assertion follows from (h).

(j) *Let X be a proper k-scheme of pure dimension d. Then $\alpha^i(X, \mathcal{F})$ is injective for all $i \in \mathbb{Z}$ and all constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves $\mathcal{F}$ on X.*

We will prove this assertion by induction on i. If $i < 0$, there is nothing to show. Now let $i \geqslant 0$. We can assume that X is reduced. By (5.1) there is a dominant open immersion $j: U \to X$ and a surjective finite étale morphism $f: V \to U$ such

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that V is smoothly compactifiable and $\mathcal{F}|_V$ is constant. Shrinking U if necessary we can assume that the complement Y of U in X is of pure codimension 1. Let $\mathcal{F}' := \mathcal{F}|_U$ and $\mathcal{C}$ the cokernel of the injective map $j_!\mathcal{F}' \to j_! f_* f^{-1}\mathcal{F}'$. Thus we have two exact sequences

$$0 \longrightarrow j_!\mathcal{F}' \longrightarrow j_! f_* f^{-1}\mathcal{F}' \longrightarrow \mathcal{C} \longrightarrow 0$$

and

$$0 \longrightarrow j_!\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow i_*\mathcal{F}|_Y \longrightarrow 0.$$

They induce commutative diagrams with exact lines

$$\begin{array}{ccccccccc}
H^{i-1}(j_! f_* f^{-1}\mathcal{F}') & \longrightarrow & H^{i-1}(\mathcal{C}) & \longrightarrow & H^i(j_!\mathcal{F}') & \longrightarrow & H^i(j_! f_* f^{-1}\mathcal{F}') & \longrightarrow & H^i(\mathcal{C}) \\
\downarrow {\scriptstyle (i)\,\cong} & & \downarrow & & \downarrow & & \downarrow {\scriptstyle (i)\,\cong} & & \downarrow \\
E^{i-1}(j_* f^* f^{-1}\mathcal{F}') & \longrightarrow & E^{i-1}(\mathcal{C}) & \longrightarrow & E^i(j_!\mathcal{F}') & \longrightarrow & E^i(j_! f_* f^{-1}\mathcal{F}') & \longrightarrow & E^i(\mathcal{C})
\end{array}$$

and

$$\begin{array}{ccccccccc}
H^{i-1}(i_*i^*\mathcal{F}) & \longrightarrow & H^i(j_!\mathcal{F}') & \longrightarrow & H^i(\mathcal{F}) & \longrightarrow & H^i(i_*i^*\mathcal{F}) & \longrightarrow & H^{i+1}(j_!\mathcal{F}') \\
\downarrow {\scriptstyle (f)\,\cong} & & \downarrow & & \downarrow & & \downarrow {\scriptstyle (f)\,\cong} & & \downarrow \\
E^{i-1}(i_*i^*\mathcal{F}) & \longrightarrow & E^i(j_!\mathcal{F}') & \longrightarrow & E^i(\mathcal{F}) & \longrightarrow & E^i(i_*i^*\mathcal{F}) & \longrightarrow & E^{i+1}(j_!\mathcal{F}'),
\end{array}$$

where we have used the abbreviations

$$H^q(\mathcal{G}) := H^q(X, \mathcal{G}) \quad \text{and} \quad E^q(\mathcal{G}) := \operatorname{Ext}^{d+1-q}_{X,p^n}(\mathcal{G}, \tilde{\nu}^d_{n,X})^*.$$

If we apply the induction hypothesis to the sheaf $\mathcal{C}$ and use (i) as well as the first diagram we see that $\alpha^i(X, j_!\mathcal{F}')$ is injective. From that we conclude using (f) and the second diagram that $\alpha^i(X, \mathcal{F})$ is injective as claimed.

(k) *Let X be proper k-scheme of pure dimension d. Then $\alpha^i(X, \mathcal{F})$ is an isomorphism for all $i \in \mathbb{Z}$ and all constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaves $\mathcal{F}$ on X.*

We prove this last assertion by descending induction on i. For $i > d+1$ there is nothing to show. Now let $i \leqslant d+1$. We define $j: U \to X$ and $f: V \to U$ as in (j) and set again $\mathcal{F}' := \mathcal{F}|_U$. If $\mathcal{K}$ is the kernel of the surjective morphism $j_! f_* f^{-1}\mathcal{F}' \to j_!\mathcal{F}'$, we again get two exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow j_! f_* f^{-1}\mathcal{F}' \longrightarrow j_!\mathcal{F}' \longrightarrow 0$$

and

$$0 \longrightarrow j_!\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow i_*\mathcal{F}|_Y \longrightarrow 0,$$

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which induce commutative diagrams with exact lines

$$\begin{array}{ccccccccc}
H^i(j_! f_* f^{-1}\mathcal{F}') & \longrightarrow & H^i(j_!\mathcal{F}') & \longrightarrow & H^{i+1}(\mathcal{K}) & \longrightarrow & H^{i+1}(j_! f_* f^{-1}\mathcal{F}') & \longrightarrow & H^{i+1}(j_!\mathcal{F}') \\
\Big\downarrow{\scriptstyle (i)\,\cong} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle (i)\,\cong} & & \Big\downarrow \\
E^i(j_! f_* f^{-1}\mathcal{F}') & \longrightarrow & E^i(j_!\mathcal{F}') & \longrightarrow & E^{i+1}(\mathcal{K}) & \longrightarrow & E^{i+1}(j_! f_* f^{-1}\mathcal{F}') & \longrightarrow & E^{i+1}(j_!\mathcal{F}')
\end{array}$$

and

$$\begin{array}{ccccccccc}
H^{i-1}(i_* i^*\mathcal{F}) & \longrightarrow & H^i(j_!\mathcal{F}') & \longrightarrow & H^i(\mathcal{F}) & \longrightarrow & H^i(i_* i^*\mathcal{F}) & \longrightarrow & H^{i+1}(j_!\mathcal{F}') \\
\Big\downarrow{\scriptstyle (f)\,\cong} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle (f)\,\cong} & & \Big\downarrow \\
E^{i-1}(i_* i^*\mathcal{F}) & \longrightarrow & E^i(j_!\mathcal{F}') & \longrightarrow & E^i(\mathcal{F}) & \longrightarrow & E^i(i_* i^*\mathcal{F}) & \longrightarrow & E^{i+1}(j_!\mathcal{F}'),
\end{array}$$

where we have used the same abbreviations as in (j). If we apply the induction hypothesis to the sheaf $\mathcal{K}$ and use (i), we get from the first diagram that $\alpha^i(X, j_!\mathcal{F}')$ is surjective, hence an isomorphism. From that we deduce, using the second diagram and (f) that $\alpha^i(X, \mathcal{F})$ is an isomorphism. This concludes the proof of (k) and of the theorem. □

COROLLARY (5.7). *Let k be a finite field and $p = \operatorname{char} k$. Let U be a smooth separated algebraic k-scheme of pure dimension d. Then for every $n \geqslant 1$ and every constructible $\mathbb{Z}/p^n\mathbb{Z}$-sheaf $\mathcal{F}$ the composition of the Yoneda pairing and the trace map*

$$H^i_c(U, \mathcal{F}) \times \operatorname{Ext}^{d+1-i}_{U,p^n}(\mathcal{F}, \nu^d_{n,U}) \longrightarrow H^{d+1}_c(U, \nu^d_{n,U}) \xrightarrow{\text{tr}} \mathbb{Z}/p^n\mathbb{Z}$$

is a nondegenerate pairing of finite groups. If U is smoothly compactifiable, then the trace map is an isomorphism. □

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References

[AK] Altman, A. and Kleiman, S.: *Introduction to Grothendieck Duality Theory*, Lecture Notes in Math. 146, Springer, New York, 1970.

[Ar] Artin, M.: Grothendieck topologies, *Notes on a seminar* (1962).

[Be] Berthelot, B.: Le théorème de dualité plate pour les surfaces (d'apres J. S. Milne). Exp VI in *Surfaces algébriques*, Lecture Notes in Math. 868, Springer, New York, 1981.

[BK] Bloch, S. and Kato, K.: p-adic étale cohomology, *Publ. Math. IHES* **63** (1986), 107–152.

[BT] Bass, H. and Tate, J.: The Milnor ring of a global field, In: *Algebraic K-Theory II*, Lecture Notes in Math. 342, Springer, New York, 1972, pp. 349–428.

[CoSS] Colliot-Thélène, J.L., Sansuc, J.J. and Soule, C.: Torsion dans le groupe de chow de codimension deux, *Duke. Math. J.* **50**(3) (1983), 763–801.

[De-1] Deninger, Ch.: Duality in the étale cohomology of one-dimensional proper schemes and generalizations, *Math. Ann.* **277** (1987), 529–541.

[De-2] Deninger, Ch.: On Artin-Verdier duality for function fields, *Math. Z.* **188** (1984), 91–100.

[dJ] de Jong, A.J.: Smoothness, semi-stability and alterations, *Publ. Math. IHES* **83** (1996), 51–93.

[EGA I] Grothendieck, A. and Dieudonné, J. A.: *Éléments de géométrie algébrique I*, Springer, Berlin, 1971.

[EGA II] Grothendieck, A.: Éléments de géométrie algébrique II, *Publ. Math. IHES* **8** (1961).

[EGA III] Grothendieck, A.: Éléments de géométrie algébrique III, *Publ. Math. IHES* **11, 17** (1961/63).

[EGA IV] Grothendieck, A.: Éléments de géométrie algébrique IV, *Publ. Math. IHES* **20, 24, 28, 32** (1964–1967).

[Gr] Gros, M.: Classes de Chern et classes de cycles en cohomologie de Hodge–Witt logarithmique, *Bull. Soc. Math. France* **113** (Memoires) (1985), Fasc. 4.

[GrSu-1] Gros, M. and Suwa, N.: La conjecture de Gersten pour les faisceaux de Hodge–Witt logarithmique, *Duke Math. J.* **57** (1988), 615–628.

[GrSu-2] Gros, M. and Suwa, N.: Application d'Abel–Jacobi p-adique et cycles algébriques, *Duke Math. J.* **57** (1988), 579–613.

[Gre] Greenberg, M.: *Lectures on Forms in Many Variables*, Benjamin, New York, 1969.

[Gri] Grivel, P. P.: Catégories dérivées et foncteurs dérivés, In: *Algebraic D-Modules*, Perspect. Math., Academic Press, Boston, 1987.

[Gro] Grothendieck, A.: *Théorèmes de dualité pour les faisceaux algébriques cohérents*, Seminaire Bourbaki, exp. 149, Benjamin, New York, 1956–57.

[Ha-1] Hartshorne, R.: *Residues and Duality*, Lecture Notes in Math. 20, Springer, New York, 1966.

[Ha-2] Hartshorne, R.: *Ample Subvarieties of Algebraic Varieties*, Lecture Notes in Math. 156, Springer, New York, 1970.

[Il] Illusie, L.: Complexe de de Rham–Witt et cohomologie cristalline, *Ann. Sci. Ecole Norm. Sup. (4)* **12** (1979), 501–661.

[Ka-1] Kato, K.: A generalization of local class field theory by using K-groups II, *J. Fac. Sci. Univ. Tokyo Sec. IA* **27** (1980), 602–683.

[Ka-2] Kato, K.: Duality theories for the p-primary étale cohomology I, In: *Algebraic and Topological Theories* – to the memory of Dr. Takehiko Miyata (1985), 127–148.

[Ka-3] Kato, K.: Duality theories for the p-primary étale cohomology II, *Compositio Math.* **63** (1987), 259–270.

https://doi.org/10.1023/A:1000892524712 Published online by Cambridge University Press

[Ka-4] Kato, K.: *Galois Cohomology of Complete Discrete Valuation Fields*, Lecture Notes in Math. 967, Springer, New York, 1982, pp. 215–238.

[Ka-5] Kato, K.: K-Theory and the Chow-group of zero cycles, In: *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory*, Contemp. Math. 55, Part I, Amer. Math. Soc., Providence, 1986, pp. 241–253.

[KaCo] Kato, K. and Colliot-Thélène, J. L.: A Hasse principle for two-dimensional global fields, *J. Reine Angew. Math.* **366** (1986), 142–183.

[KaKu] Kato, K. and Kuzumaki, T.: The dimension of fields and algebraic K-theory, *J. Number Theory* **24** (1986), 229–244.

[Kat] Katz, N. M.: Nilpotent connections and the monodromy theorem: Applications of a result of turritin, *Publ. Math. IHES* **39** (1970), 175–232.

[Li] Lichtenbaum, S.: Values of zeta-functions at nonnegative integers, In: *Number Theory, (Noordwijkerhout, 1983)*, Lecture Notes in Math. 1068, Springer, New York, pp. 127–138.

[Lip] Lipman, J.: Derived categories and derived functors, Preprint, 1992.

[Mi-1] Milne, J. S.: *Arithmetic Duality Theorems*, Perspect. Math., Academic Press, Boston, 1986.

[Mi-2] Milne, J. S.: Duality in the flat cohomology of a surface, *Ann. Sci. Ecole Norm. Sup. (4)* **9** (1976), 171–202.

[Mi-3] Milne, J. S.: *Étale Cohomology*, Princeton Univ. Press, 1980.

[Mi-4] Milne, J. S.: Motivic cohomology and values of zeta functions, *Compositio Math.* **68** (1988), 59–102.

[Mi-5] Milne, J. S.: Values of zeta functions of varieties over finite fields, *Amer. J. Math.* **108** (1986), 297–360.

[Na] Nagata, M.: A generalization of the imbedding problem of an abstract variety in a complete variety, *J. Math. Kyoto Univ*. **3** (1963), 89–102.

[Ra] Ramanujam, C. P.: *A Tribute. Hrsg.: K. G. Ramanathan*, Springer, Berlin, 1978.

[Se] Serre, J. P.: *Corps locaux*, Hermann, Paris, 1968.

[SGA IV] Artin, M., Grothendieck, A. and Verdier, J. L.: *Seminaire de géométrie algébrique IV*, Lecture Notes in Math. 269, 270, 305, Springer, New York, 1972/73.

[SGA V] Grothendieck, A.: *Seminaire de géométrie algébrique V*, Lecture Notes in Math. 589, Springer, New York, 1977.

[Sp] Spieß, M.: Artin-Verdier duality for arithmetic surfaces, *Math. Ann.* **305** (1996), 705–792.

[Su] Suwa, N.: A note on Gersten's conjecture for logarithmic Hodge–Witt sheaves, *K-Theory* **9** (1995), 245–271.

[Ta] Tamme, G.: *Introduction to Étale Cohomology*, Springer, Berlin, 1994.

[Tat] Tate, J.: Residues of differentials on curves, *Ann. Sci. Ecole Norm. Sup. (4)* **1** (1968), 149–159.

[Za] Zariski, O.: *Introduction to the Problem of Minimal Models in the Theory of Algebraic Surfaces*. Publ. Math. Soc. Japan, 4, Princeton Univ. Press, 1958; or: Collected Papers, Vol. II, MIT Press, Cambridge, Mass., 1973, pp. 277–369.

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