SEMI-GROUPS OF MAPS IN A LOCALLY COMPACT SPACE

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Suppose that S is a locally compact Hausdorff space. A *one-parameter semi-group of maps in* S is a family $\{\phi_t; t \geqslant 0\}$ of continuous functions from S into S satisfying

(i) $\phi_t \circ \phi_u = \phi_{t+u}$ for $t, u \geqslant 0$, where the circle denotes composition, and

(ii) $\phi_0 = e$, the identity map on S.

A semi-group $\{\phi_t\}$ of maps in S is said to be

(iii) *of class* (C_0) if $\phi_t(x) \to x$ as $t \to 0$ for each x in S,

(iv) *separately continuous* if the function $t \to \phi_t(x)$ is continuous on $[0, \infty)$ for each x in S, and

(v) *doubly continuous* if the function $(t, x) \to \phi_t(x)$ is continuous on $[0, \infty) \times S$.

We show that separate continuity implies double continuity and that if S is σ-compact (the union of countably many compact sets), then every class (C_0) semi-group of maps in S is separately continuous.

We establish a 1–1 correspondence between the class of all separately continuous semi-groups of maps in S and a certain easily describable class of linear operators in $C_b(S)$, the linear space of all bounded, real-valued, continuous functions on S. The correspondence would seem to justify calling the linear operator corresponding to a given semi-group of maps in S the infinitesimal generator of that semi-group. A topology, called the bounded strict topology, is introduced on the space $C_b(S)$, and it is shown that if S is paracompact, then the bounded strict topology coincides with the more familiar strict topology; see **(1)** or **(3)**. It is then shown that if $\{\phi_t; t \geqslant 0\}$ is a separately continuous semi-group of maps in S, $\alpha > 0$, and $T_t f = f \circ \phi_t$ for f in $C_b(S)$ and $t \geqslant 0$, then $\{e^{-\alpha t} T_t; t \geqslant 0\}$ is an equi-continuous semi-group of class (C_0) in $C_b(S)$ with the bounded strict topology; see **(9**, p. 234**)**. This is the major step in establishing the correspondence between semi-groups of maps in S and their generators. The generator of $\{\phi_t\}$ is given by

$$Af = \lim (f \circ \phi_t - f)/t \qquad (t \to 0).$$

The class of generators A of separately continuous semi-groups $\{\phi_t\}$ of maps is the class of derivation operators A in $C_b(S)$ such that the domain of A is

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dense in $C_b(S)$ with the bounded strict topology, and for some $\alpha > 0$ (equivalently, for each $\alpha > 0$), the collection

$$\{[I - n^{-1}(A - \alpha)]^{-m}\}_{m,n=1}^{\infty}$$

is an equi-continuous collection of operators on $C_b(S)$ with the bounded strict topology. Hufford **(6)** has carried out a similar program for S compact.

1. Topologies on function spaces. Let $C(S)$ denote the linear space of all continuous real-valued functions on S. Then $C_b(S)$ is the space of all bounded functions in $C(S)$. Let $C_0(S)$ denote the linear space of all functions in $C(S)$ which vanish at infinity, and let $C_c(S)$ denote the linear space of all functions in $C(S)$, which have compact support.

The norm, $||f||$, of a function f in $C_b(S)$ will mean the supremum norm of f, and if K is a compact subset of S and f is in $C(S)$, then $||f||_K$ means the supremum norm of $f|_K$, the restriction of f to K. If K is a compact subset of S, then $C(K)$ denotes the Banach space with supremum norm of continuous real-valued functions on K. For each $r > 0$, B_r denotes the set of all f in $C_b(S)$ such that $||f|| \leqslant r$.

The compact open topology (the topology of uniform convergence on compact sets) on $C(S)$ will be denoted by γ, and its restriction to $C_b(S)$ will be denoted by γ'. The strict topology, see **(1)**, on $C_b(S)$ will be denoted by β and has a local neighbourhood basis at the origin the sets $V_\psi = \{f: ||f\psi|| \leqslant 1\}$ for ψ in $C_0(S)$. An equivalent neighbourhood basis, by **(4)**, consists of the sets

$$V_{\{K_n,\epsilon_n\}} = \{f: ||f||_{K_n} \leqslant \epsilon_n \quad \text{for} \quad n = 1, 2, 3, \ldots\},$$

where $\{\epsilon_n\}$ is a strictly increasing sequence of positive numbers approaching infinity, and $\{K_n\}$ is an increasing sequence of compact sets. Conway **(3)** has shown that if S is paracompact, then $(C_b(S), \beta)$ is a Mackey space, i.e., there is no locally convex linear topology on $C_b(S)$ which properly contains β and yields the same continuous linear functionals as β.

The bounded strict topology on $C_b(S)$ is denoted by β' and has as a local neighbourhood basis at the origin the system of all convex, balanced, absorbent sets V such that for each $r > 0$ there is a β neighbourhood W of 0 such that $W \cap B_r \subset V$. That this is a basis for a locally convex linear topology follows from **(7**, Theorem 2, p. 10**)**. This method of generating topologies is discussed by Collins in **(2**, § 5, pp. 265–268**)**.

The author will attempt, at appropriate places in the paper, to point out the reason for introducing the strict and the bounded strict topologies on $C_b(S)$. Although the bounded strict topology has a somewhat cumbersome definition, it does have several interesting connections with the strict topology and, besides, agrees with it on norm bounded sets and gives rise to the same continuous linear functionals. A linear transformation T from $C_b(S)$ into a locally convex topological vector space (E, τ) is $\beta' - \tau$ continuous if and only if its restriction to each norm bounded set is $\beta - \tau$ continuous. Also, a linear transformation T from $C_b(S)$ into $C_b(S)$ is $\beta' - \beta'$ continuous if and only if its restriction to each norm bounded set is $\beta - \beta$ continuous. These last two properties are not needed in this paper, so no proof is given. However, the last property comes close to giving the reason for the introduction of the β' topology.

THEOREM 1. *$(C_b(S), \beta')$ is sequentially complete and has the same continuous linear functionals as $(C_b(S), \beta)$. Thus $\beta' = \beta$ if $(C_b(S), \beta)$ is a Mackey space, which is the case if S is paracompact.*

Proof. Since $\beta \subset \beta'$, every β' Cauchy sequence is β Cauchy, and thus β convergent and norm bounded, by **(1)**. But β' and β agree on norm bounded sets, so every β' Cauchy sequence is β' convergent.

Since $\beta \subset \beta'$, every β continuous linear functional on $C_b(S)$ is β' continuous. Suppose L is a β' continuous linear functional on $C_b(S)$. Then L is norm continuous, so by the Riesz representation theorem there exists a unique bounded regular Borel measure μ on S such that $Lf = \int f\, d\mu$ for all f in $C_0(S)$. Let $Pf = \int f\, d\mu$ for all f in $C_b(S)$. Then P is β continuous on $C_b(S)$, by **(1)**, and also is therefore β' continuous. But $C_0(S)$ is β' dense in $C_b(S)$ since $\{\psi_\alpha g\}$ is norm bounded and β convergent to g for g in $C_b(S)$ and $\{\psi_\alpha\}$ an approximate identity for the Banach algebra $C_0(S)$. Therefore $Pf = Lf$ for all f in $C_b(S)$.

2. Semi-groups of operators. A semi-group of operators in a linear space X is a collection $\{T_t; t \geqslant 0\}$ of linear transformations from X into X satisfying

$$T_0 = I, \text{ the identity operator on } X,$$

and

$$T_t T_s = T_{t+s} \text{ for } s, t \geqslant 0.$$

See **(5)** for the theory and terminology of semi-groups in a Banach space, and **(9)** for the theory and terminology in a topological vector space. In this section, $\{\phi_t; t \geqslant 0\}$ denotes a class (C_0) semi-group of maps in S, and $T_t f = f \circ \phi_t$ for f in $C_b(S)$ and $t \geqslant 0$. Thus $\{T_t\}$ is a semi-group of operators in $C_b(S)$, and $||T_t f|| \leqslant ||f||$ for f in $C_b(S)$ and $t \geqslant 0$.

THEOREM 2.1. *Let Φ denote the set of all f in $C_b(S)$ such that $||T_t f - f|| \to 0$ as $t \to 0$. Then Φ is a Banach algebra under the supremum norm, and $T_t \Phi \subset \Phi$ for $t \geqslant 0$, so that $\{T_t\}$ is a class (C_0) semi-group when restricted to the Banach space Φ. Also, Φ is β dense in $C_b(S)$.*

Proof. Clearly Φ is a linear space. Φ is an algebra since $T_t(fg) = (T_t f)(T_t g)$. If f is in $C_b(S)$, $\{f_n\} \subset \Phi$, and $||f_n|| \to 0$, then

$$T_t f - f = T_t(f - f_n) + (T_t f_n - f_n) + (f_n - f)$$

and

$$||T_t f - f|| \leqslant 2\,||f - f_n|| + ||T_t f_n - f_n||,$$

so that f is in Φ. If $s, t \geqslant 0$, then $T_t T_s f - T_s f = T_s(T_t f - f)$, so that $T_s \Phi \subset \Phi$.

If f is in $C_b(S)$ and $\alpha > 0$, then define f_α on S by

$$f_\alpha(x) = (1/\alpha)\int_0^\alpha f(\phi_t(x))\,dt.$$

The integral exists for each x in S because the integrand is bounded and continuous from the right. f_α is clearly bounded. Take y in S and K a compact neighbourhood of y. Let $g(t) = T_t f|_K$ for $0 \leqslant t \leqslant \alpha$. Then g is weakly continuous from the right as a function from $[0, \alpha]$ into $C(K)$ and is therefore strongly measurable. Since g is bounded, g is Bochner integrable, and

$$f_\alpha|_K = (1/\alpha)\int_0^\alpha g(t)\,dt.$$

Therefore f_α is in $C_b(S)$. For each x in S,

$$\begin{aligned}|f_\alpha(\phi_t(x)) - f_\alpha(x)| &= (1/\alpha)\left|\int_t^{\alpha+t} f(\phi_s(x))\,ds - \int_0^\alpha f(\phi_s(x))\,ds\right| \\ &= (1/\alpha)\left|\int_\alpha^{\alpha+t} f(\phi_s(x))\,ds - \int_0^t f(\phi_s(x))\,ds\right| \leqslant (2t/\alpha)||f||,\end{aligned}$$

so that f_α is in Φ. Also $f_\alpha \to f$ weakly in $(C_b(S), \beta)$ as $\alpha \to 0$, so that Φ is weakly dense in $(C_b(S), \beta)$ and therefore dense in $(C_b(S), \beta)$.

Remark. For f in Φ, the function $t \to T_t f$ is norm continuous (from both sides) since for $t > 0$ and $0 < h < t$, we have

$$T_{t+h}f - T_t f = T_h T_t f - T_t f$$

and

$$T_t f - T_{t-h} f = T_{t-h}(T_h f - f).$$

Theorem 2.1 provides perhaps the most pressing reason for the use of the β topology. The β-denseness of Φ is needed to prove Theorem 2.2, which is essential to all that follows. The β-denseness is obtainable because of the simple nature of β weak sequential convergence.

We give an example in which Φ is not all of $C_b(S)$, and the setting cannot be easily reduced to a simpler one. Take S to be real Euclidean 4-space E_4, and let

$$\phi_t(x_1, x_2, x_3, x_4) = (x_1/(1 + |x_1|\,t),\, x_2 e^t,\, x_3 \cos t + x_4 \sin t,\, -x_3 \sin t + x_4 \cos t).$$

Then $f(x_1, x_2, x_3, x_4) = \sin x_2$ is not in Φ. The problem cannot be reduced to linear semi-groups in E_4 because of the first term, and the maps ϕ_t have no apparent continuous extension to any reasonable compactification of S in such a way that the extended maps would form a separately continuous semi-group. The generator of $\{\phi_t\}$ is an extension of

$$-x_1|x_1|(\partial/\partial x_1) + x_2(\partial/\partial x_2) + x_4(\partial/\partial x_3) - x_3(\partial/\partial x_4).$$

THEOREM 2.2. *If $\{\phi_t\}$ is separately continuous, then $\{\phi_t\}$ is doubly continuous.*

Proof. Suppose x is in S and $t > 0$ (the case $t = 0$ requires only a trivial modification). Suppose U is an open set in S having compact closure U^- and containing $\phi_t(x)$. Take V open in S with $\phi_t(x)$ in V and $V^- \subset U$. Take f in $C_b(S)$ with $f(S) \subset [0, 1], f(\phi_t(x)) = 0$, and $f(y) = 1$ for y in $S \setminus V$.

Since Φ is β dense in $C_b(S)$, Φ is certainly γ' dense in $C_b(S)$. Take g in Φ such that $||g - f||_{U^-} < 1/8$ Take $\delta > 0$ such that $||T_s g - T_t g|| < 1/8$ for $|s - t| < \delta$. Let W denote a neighbourhood of x such that $\phi_t(y)$ is in V and $g(\phi_t(y)) < 1/4$ for y in W.

If $|s - t| < \delta$, and y is in W, then

$$g(\phi_s(y)) = g(\phi_s(y)) - g(\phi_t(y)) + g(\phi_t(y)) < 3/8,$$

so that $\phi_s(y)$ is in $V \cup (S \setminus U^-)$, because $g(z) > 7/8$ for z in $U^- \setminus V$. Now fix y in W, and let $h(s) = \phi_s(y)$ for $|s - t| < \delta$. The range of h is connected, and $h(t)$ is in V, so $h(s)$ is in V for all $|s - t| < \delta$, since V and $S \setminus U^-$ are separated.

THEOREM 2.3. *If S is σ-compact, then $\{\phi_t\}$ is separately continuous.*

Proof. For each f in $C(S)$ and $t \geqslant 0$, let $P_t f = f \circ \phi_t$. Then each P_t is a continuous linear operator from $(C(S), \gamma)$ into $(C(S), \gamma)$, because

$$||P_t f||_K = ||f||_{\phi_t(K)}$$

for $t \geqslant 0$ and K compact. Clearly $\{P_t\}$ is a semi-group of operators in $C(S)$, and $P_t f = T_t f$ for $t \geqslant 0$ and f in $C_b(S)$. The function $t \to P_t f$ is norm continuous, and thus γ measurable on $[0, \infty)$ for each f in Φ. Since Φ is β dense in $C_b(S)$, Φ is γ dense in $C(S)$. Since S is σ-compact, $(C(S), \gamma)$ is a Fréchet space, so that every function f in $C(S)$ is the limit of a sequence of functions in Φ, and thus $\{P_t; t \geqslant 0\}$ is a strongly measurable semi-group of continuous operators in $(C(S), \gamma)$. Therefore, by **(8)**, $\{P_t\}$ is strongly continuous on $(0, \infty)$.

Suppose that x is in S, $t > 0$, and U is an open set containing $\phi_t(x)$. Take f in $C(S)$ such that $f(S) \subset [0, 1], f(\phi_t(x)) = 0$, and $f(y) = 1$ for y in $S \setminus U$. Take $\delta > 0$ such that

$$||P_s f - P_t f||_{\{x\}} < 1/2$$

for $|s - t| < \delta$. Then $f(\phi_s(x)) < 1/2$ for $|s - t| < \delta$ so that $\phi_s(x)$ is in U.

THEOREM 2.4. *If $\{\phi_t\}$ is separately continuous, and $k > 0$, then the family $\{T_t; 0 \leqslant t \leqslant k\}$ is an equi-continuous family of operators from $(C_b(S), \beta)$ into $(C_b(S), \beta)$.*

Proof. Let $V_{\{K_n, \epsilon_n\}}$ be a β neighbourhood of 0. Let $K_n' = G([0, k] \times K_n)$ for $n = 1, 2, 3, \ldots,$ where $G(t, x) = \phi_t(x)$ for $t \geqslant 0$ and x in S. Then

$$T_t V_{\{K'_n, \epsilon_n\}} \subset V_{\{K_n, \epsilon_n\}} \qquad \text{for} \quad 0 \leqslant t \leqslant k.$$

THEOREM 2.5. *If* $\{\phi_t\}$ *is separately continuous, and* $\alpha > 0$, *then*

$$\{e^{-\alpha t} T_t; t \geqslant 0\}$$

is an equi-continuous family of operators from $(C_b(S), \beta')$ *into* $(C_b(S), \beta')$.

Proof. Let $Q_t = e^{-\alpha t}T_t$ for $t \geqslant 0$, and let V denote a convex balanced β' neighbourhood of 0. For each $r > 0$, let V_r denote a convex balanced β neighbourhood of 0 such that $V_r \cap B_r \subset V$, take $r' > 0$ such that $r' < r$ and $B_{r'} \subset V_r$, and take $k_r > 0$ such that $e^{-\alpha k_r} < (r'/r)$. Let W_r denote a convex balanced β neighbourhood of 0 such that $T_t W_r \subset V_r$ for $0 \leqslant t \leqslant k_r$. Then

$$Q_t(B_r \cap W_r) \subset (B_r \cap V_r) \subset V \qquad \text{for } t \geqslant 0.$$

Let

$$W = \bigcap_{t \geqslant 0} Q_t^{-1}(V).$$

Then $B_r \cap W_r \subset W$ for $r \geqslant 0$, so that W is a β' neighbourhood of 0.

Remark. (Theorem 2.5 shows the reason for the use of the β'-topology. In order to use the semi-group theory as given in **(9)**, it is necessary that the operators form an equi-continuous collection in the topological vector space in which they are considered, and $(C_b(S), \beta')$ is the only satisfactory space the author can find.

THEOREM 2.6. *If* $\{\phi_t\}$ *is separately continuous, and* $\alpha \geqslant 0$, *then* $\{e^{-\alpha t} T_t\}$ *is strongly continuous on* $[0, \infty)$ *as a semi-group of operators in* $(C_b(S), \gamma')$, $(C_b(S), \beta)$, *or* $(C_b(S), \beta')$.

Proof. It suffices to prove strong continuity in $(C_b(S), \gamma')$, since γ', β, and β' agree on norm bounded sets. Also, it suffices to take $\alpha = 0$. The strong continuity in $(C_b(S), \gamma')$ follows by a routine argument based on the double continuity of $\{\phi_t\}$.

THEOREM 7. *Suppose* $\{Z_t; t \geqslant 0\}$ *is a semi-group of linear operators in* $C_b(S)$. *Then the following statements are equivalent:*

(i) *there exists a unique separately continuous semi-group* $\{\theta_t; t \geqslant 0\}$ *of maps in S such that* $Z_t f = f \circ \theta_t$ *for f in* $C_b(S)$ *and* $t \geqslant 0$;

(ii) (a) *each* Z_t *is a non-zero algebraic homomorphism on* $C_b(S)$;

(b) *each* Z_t *is either* β *continuous,* β' *continuous, or* γ' *continuous;*

(c) *for each f in* $C_b(S)$, *the function* $t \to Z_t f$ *is either* β *continuous,* β' *continuous, or* γ' *continuous on* $[0, \infty)$;

(iii) *each* Z_t *is a non-zero algebraic homomorphism on* $C_b(S)$, *and for each* $\alpha > 0$, $\{e^{\alpha t} Z_t; t \geqslant 0\}$ *is an equi-continuous semi-group of class* (C_0) *in* $(C_b(S), \beta')$, see **(9**, p. 234**)**.

Proof. That (i) implies (iii) has already been established. That (iii) implies (ii) is apparent. We shall show that (ii) implies (i).

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Suppose $t \geqslant 0$ and x is in S. Let

$$Lf = [Z_t f](x) \qquad \text{for each } f \text{ in } C_b(S).$$

Then L is a non-zero multiplicative linear functional on $C_b(S)$. Therefore, there is a unique point y in S^-, the Stone–Čech compactification of S, such that $Lf = f^-(y)$ for all f in $C_b(S)$, where f^- denotes the continuous extension of f to S^-. But by (b), **(1)**, and Theorem 1, y is in S. Therefore, for each $s \geqslant 0$, there exists a unique function θ_s from S into S such that $Z_s f = f \circ \theta_s$ for all f in $C_b(S)$. The fact that each θ_s is a map follows from the complete regularity of S and the fact that $f \circ \theta_s$ is continuous for each f in $C_b(S)$. Clearly, $\{\theta_s \colon s \geqslant 0\}$ is a semi-group of maps in S. We now have $\|Z_s f\| = \|f\|$ for f in $C_b(S)$ and $s \geqslant 0$, so the three types of strong continuity described in (c) are all equivalent. The separate continuity of $\{\theta_s\}$ follows by an argument like the last part of the argument for Theorem 2.3.

THEOREM 2.9. *Let A be a linear transformation from a subspace $D(A)$ of $C_b(S)$ into $C_b(S)$. Then the following statements are equivalent:*

(i) *there exists a unique separately continuous semi-group $\{\theta_t; t \geqslant 0\}$ of maps in S such that f is in $D(A)$ if and only if*

$$\lim (f \circ \theta_t - f)/t \qquad (t \to 0)$$

exists in the β' topology (or, equivalently, in the β topology) and

$$Af = \lim (f \circ \theta_t - f)/t \qquad (t \to 0)$$

for all f in $D(A)$;

(ii) *$D(A)$ is dense in $(C_b(S), \beta')$ (or equivalently, in $(C_b(S), \beta)$), A is a derivation (i.e., fg is in $D(A)$ and $A(fg) = f(Ag) + g(Af)$ for f, g in $D(A)$), and for each $\alpha > 0$, the collection*

$$F_\alpha = \{[I - n^{-1}(A - \alpha)]^{-m}\}_{m,n=1}^{\infty}$$

is an equi-continuous collection of operators in $(C_b(S), \beta')$;

(iii) *$D(A)$ is dense in $(C_b(S), \beta')$, A is a derivation, and for some $\alpha > 0$, F_α is an equi-continuous collection of operators in $(C_b(S), \beta')$.*

Proof. First, let us remark that the equivalence of the β and β' denseness of $D(A)$ is a consequence of Theorem 1 and the fact that weak density and density of a subspace are equivalent.

Suppose that (i) holds. Then A is clearly a derivation. Suppose $\alpha > 0$, and let

$$M_t f = e^{-\alpha t} f \circ \theta_t$$

for f in $C_b(S)$ and $t > 0$. Then $\{M_t\}$ is an equi-continuous semi-group of class (C_0) in $(C_b(S), \beta')$, by Theorem 2.7. Clearly, the infinitesimal generator of $\{M_t\}$ is $A - \alpha$, so that (ii) follows from **(9**, p. 246**)**.

That (ii) implies (iii) is clear. Suppose (iii) is true, and let α denote a positive number such that the collection F_α is equi-continuous.

Let $\{N_t\}$ denote the semi-group generated by $A - \alpha$. By (9, Remark, p. 248),

$$N_t f = \lim_{n\to\infty} (exp[t(A - \alpha)(I - n^{-1}(A - \alpha))^{-1}])f \qquad \text{for each } f \text{ in } C_b(S).$$

Let $Z_t = e^{\alpha t} N_t$ for $t \geqslant 0$. Then $\{Z_t\}$ is a semi-group of operators in $C_b(S)$. Moreover, $\{Z_t\}$ is strongly β' continuous on $[0, \infty)$ and β' equi-continuous on each interval $[0, k)$. Also

$$(d/dt)Z_t f = A(Z_t f) = Z_t (Af)$$

for f in $D(A)$ and $t \geqslant 0$. We now prove that each Z_t is a non-zero algebraic homomorphism.

Take f, g in $D(A)$, and let

$$H(t) = (Z_t f)(Z_t g) \qquad \text{for } t \geqslant 0.$$

Then

$$\begin{aligned} H'(\mathrm{t}) &= (Z_t f)A(Z_t g) + (Z_t g)A(Z_t f), \\ H'(s) - AH(s) &= 0, \\ Z_{t-s} H'(s) - Z_{t-s} AH(s) &= 0, \\ (d/ds)Z_{t-s} H(s) &= 0, \\ Z_{t-s} H(s) &= Z_t H(0) = Z_0 H(t), \\ H(t) &= Z_t(fg). \end{aligned}$$

Since $D(A)$ is dense in $(C_b(S), \beta')$ and each Z_t is continuous, each Z_t is an algebraic homomorphism. Since $Z_t f \to f$ as $t \to 0$ for all f in $C_b(S)$, Z_t is certainly non-zero for small t. Suppose J_n is a sequence of continuous non-zero algebraic endomorphisms on $(C_b(S), \beta')$ and $J_n f \to 0$ as $n \to \infty$ for all f in $C_b(S)$. Then, by the argument for Theorem 2.7, there is a unique sequence $\{\psi_n\}$ of maps in S such that $J_n f = f \circ \psi_n$. Fix an x in S. Then $\{\psi_n(x)\}$) must cluster at some point y in S^-, the Stone–Čech compactification of S, and $f(\psi_n(x))$ must cluster at $f^-(y)$ for every f in $C_b(S)$, where f^- denotes the extension of f to S^-. Therefore $f^-(y) = 0$ for all f in $C_b(S)$, a contradiction. Therefore, each Z_t is non-zero.

Therefore, by Theorem 2.7, there is a unique separately continuous semi-group of maps $\{\theta_t\}$ such that $Z_t f = f \circ \theta_t$ for f in $C_b(S)$ and $t \geqslant 0$. Thus A is defined as in (i).

Remark. In connection with Theorem 2.8, we mention that f is in D(A) if there exists a g in $C_b(S)$ such that $(f(\theta_t(x)) - f(x))/t \to g(x)$ as $t \to 0$ for all x in S. Suppose there is such a function g. Then for each x, the function $t \to f(\theta_t(x))$ has right derivative $g(\theta_t(x))$ at t for all $t \geqslant 0$. Since this right derivative is continuous, it is also the derivative (see, for instance, (9, pp. 239, 240)). Thus, for each x in S, we have

$$f(\theta_t(x)) - f(x) = \int_0^t g(\theta_\xi(x))d\xi$$

and

$$[f \circ \theta_t - f](x) = \left[\int_0^t g \circ \theta_\xi \, d\xi \right](x),$$

so that

$$(f \circ \theta_t - f)/t = (1/t) \int_0^t g \circ \theta_s \, ds,$$

and $Af = g$, the last two integrals being taken as integrals of continuous functions from $[0, t]$ into $(C_b(S), \beta')$.

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