Can. J. Math., Vol. XLI, No. 1, 1989, pp. 14-67

C-VALUATIONS AND NORMAL C-ORDERINGS

M. CHACRON

1. Introduction; basic facts about C-valuations. Let D stand for a division ring (or skewfield), let G stand for an ordered abelian group with positive infinity adjoined, and let $\omega: D \to G$. We call ω a *valuation* of D with *value group* G, if ω is an onto mapping from D to G such that

(i) $\omega(x) = \infty$ if and only if $x = 0$,

(ii) $\omega(x_1 + x_2) \geqq \min(\omega(x_1), \omega(x_2))$, and

(iii) $\omega(x_1x_2) = \omega(x_1) + \omega(x_2)$.

Associated to the valuation ω are its *valuation ring*

$$R = \{x \in D|\omega(x) \geqq 0\},$$

its *maximal ideal*

$$J = \{x \in |\omega(x) > 0\},$$

and its residue division ring $\bar{D} = R/J$. The invertible elements of the ring R are called *valuation units*. Clearly R and, hence, J are preserved under conjugation so that $1 + J$ is also preserved under conjugation. The latter is thus a normal subgroup of the multiplicative group $D^\bullet$ of D and hence, the quotient group $D^\bullet/1 + J$ makes sense (the *residue group* of ω). It enlarges in a natural way the residue division ring $\bar{D}$ (0 excluded, and addition "forgotten"). Of crucial importance for this work will be the group congruence

$$a \equiv b \pmod{1 + J}$$

(i.e., $a, b \in D^\bullet$ and $ab^{-1} \in 1 + J$). For convenience this equivalence relation will be extended to D in a standard way:

Definition 1.1. Write $a \equiv b \pmod{1 + J}$ if, and only if,

$$a(1 + J) = b(1 + J).$$

Unless otherwise specified "$a \equiv b$" will always stand for

$$a \equiv b \pmod{1 + J}.$$

Received August 24, 1987. This paper is dedicated to the memory of Yitz.

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

There follow some straightforward facts about these equivalences relations, all consequences of the strict triangle inequality which is listed first.

Fact 1.2. *For* $\{a_i\} \subset D$, *if* $i \neq j$ *implies* $\omega(a_i) \neq \omega(a_j)$ *then*

$$\omega(\sum a_i) = \min(\omega(a_i)).$$

Fact 1.3. *If* a *is a valuation unit* (*i.e.*, $\omega(a) = 0$) *then* $a \equiv b$ *if and only if* $a - b \in J$.

Fact 1.4. *To say that* $a \equiv b$ *is to say that* $a = 0$ *or* $\omega(a - b) > \omega(a)$.

Fact 1.5. ([**4**, Lemma 5.11, (2)]). $a \equiv b$, $c \equiv d$ *and* $a \not\equiv -c$ *together imply* $a + c \equiv b + d$.

I come to the specifics of this work. From now on the division ring D comes equipped with the antiautomorphism $x \to x^*$ of period 2 (except when D is a field and $x^* = x$ for all $x \in D$). We say that $x \in D$ is symmetric, skew-symmetric, or unitary if $x = x^*$, $x = -x^*$, or $x = (x^*)^{-1}$ respectively. A norm in D is a symmetric of the form xx^* where $x \in D^\bullet$. The subset of all sums of norms in D is written P. The closure of P under product and sum is written C(**-core*). It is the subset of all sums of monomials $(x_1x_1^*) \dots (x_rx_r^*)(x_i \in D^\bullet)$.

Definition 1.6 The valuation ω: $D \to G$ is a **-formally real valuation*, if

(C1) $\quad \omega(x^*) = \omega(x)$ (*-valuation)

(C2) $\quad \omega(a + b) = \min(\omega(a), \omega(b))$, for all $a, b \in P$.

As observed first in [**4**], Axiom (C1) alone means something of importance for the objects R, J, $\bar{D}$, $D^\bullet/1 + J$, and $\equiv$. We have that R is a *-valuation ring; that is, for all $x \in D^\bullet$ x^*x^{-1} is a valuation unit. Thus R is *-closed, J is *-closed, $\bar{D}$ carries the induced involution $x + J \to x^* + J$, $D^\bullet/1 + J$ carries the residue involution $x(1 + J) \to x^*(1 + J)$, and $\equiv$ is *-closed (i.e., $a \equiv b$ if and only if $a^* \equiv b^*$). There follow facts about *-formally real valuations.

Fact 1.7. *If* ω *is any *-valuation then to say that* ω *is *-formally real is to say that the involutorial residue division ring* $\bar{D}$ *has the property that* $\sum \bar{x}_i\bar{x}_i^* = 0$ *implies all* $\bar{x}_i = 0$. *Equivalently, in the valuation ring* R, $1 + \sum x_ix_i^*$ *is invertible* (*is a valuation unit*).

Fact 1.8. *For* ω *a *-formally real valuation we have* $0 \notin P$ *so that characteristic* $(D) = 0$. *In fact*, $0 \notin P(\bar{D})$ *so also characteristic* $(\bar{D}) = 0$.

Fact 1.9. (Holland) *For* ω *any *-formally real valuation*:

(i) $\quad c_j \equiv a_ja_j^*$, $j = 1, \dots, r$, *implies* $\sum c_j \equiv \sum a_ja_j^*$.

(ii) $\quad a \equiv a_i$, $i = 1, \dots, r$, *implies* $a \equiv \dfrac{1}{r}(a_1 + \dots + a_r)$.

Proof. Use 1.4 and 1.5.

The subject matter of *-formally real valuation ω is not to be undertaken here in its entirety. I shall narrow it down to the special case of a c-valuation, according to the main

Definition 1.10. By *c-valuation*, I mean a *-formally real valuation such that:

(C3) $\quad (aa^*)b \equiv b(aa^*)$, for all $a, b \in D$.

Before I address the consequences of Axiom (C3) let me introduce a few more notations. The centre of the division ring D is denoted by Z (or $Z(D)$, if there is danger of confusion), the center of $\bar{D}$ is denoted by $Z(\bar{D})$, the centre of $D^{\bullet}/1 + J$ is written $Z(D^{\bullet}/1 + J)$, and the inverse image of $Z(D^{\bullet}/1 + J)$ in $D^{\bullet}$ is denoted by $\hat{Z}$ (normal subgroup of residually central elements). The following theorem shows how (C1) does significantly enrich *-formal reality. (See, in particular, parts (2) and (4).)

THEOREM 1.11. *For ω any c-valuation of D follows*:

(1) $0 \notin C$;

(2) $\omega(a + b) = \min(\omega(a), \omega(b))$, *for all* $a, b \in C$;

(3) C (*and, hence*, P) $\subset \hat{Z}$;

(4) *For each* $c \in C$, *there is* $p \in P$ *with* $c \equiv p$;

(5) $u \in \hat{Z}$ *for all symmetric valuation units u and, hence, either $\bar{D}$ is a field, or $\bar{D}$ is a normal quaternionic division ring.*

Proof. (4) A typical element c of C is of the form $c = \sum y_i$ where

$$y_i = (x_{i1}x_{i1}^*) \dots (x_{ir}x_{ir}^*) \quad \text{with all } x_{ij} \in D^{\bullet}.$$

By Axiom (C1) follows that for $x_i = x_{i_r} \dots x_{i_1}$, we have $y_i \equiv x_i x_i^*$. By Fact 1.9 follows

$$c = \sum x_i x_i^* \in P,$$

as desired.

(1) This is a direct consequence of the conjunction of (4) and Fact 1.8.

(2) Again, by (4) and Fact 1.8.

(3) By (4) it suffices to show that $P \subset \hat{Z}$. Let

$$c = \sum x_i x_i^*, \quad x_i \in D^{\bullet}.$$

For arbitrary $a \in D^{\bullet}$,

$$a^{-1}ca = \sum a^{-1}(x_i x_i^*)a,$$

and by (C3),

$$a^{-1}(x_i x_i^*)a \equiv x_i x_i^*$$

so that by Fact 1.9 we get

$$a^{-1}ca \equiv \sum x_i x_i^* = c,$$

placing c in $\hat{Z}$.

(4) Let $u = u^*$ be a valuation unit. If $1 - u \in J$ then $u \equiv 1$ and, hence, $u \in \hat{Z}$. If, on the other hand, $1 - u \notin J$, then $1 - u$ is also a symmetric valuation unit. For arbitrary $a \in D^\bullet$, we have

$$a^{-1}u^2a \equiv u^2 \quad \text{and} \quad a^{-1}(1-u)^2a \equiv (1-u)^2$$

so that using Fact 1.3,

$$p = a^{-1}u^2a - u^2 \quad \text{and} \quad q = a^{-1}(1-u)^2a - (1-u)^2$$

both belong to J. Thus

$$p - q = 2(a^{-1}ua - u) \in J.$$

Since 2 is a valuation unit it follows that $a^{-1}ua - u \in J$ hence, $a^{-1}ua \equiv u$, placing u in $\hat{Z}$.

In particular, u maps into the centre $Z(\bar{D})$ of $\bar{D}$, for all $u = u^* \in R$. By a well-known theorem of J. Dieudonné follows that either $\bar{D}$ is a field or $\bar{D}$ is 4-dimensional with the unique involution with fixed set and subfield precisely $Z(\bar{D})$ (normal quaternionic division ring).

As an interesting consequence of the preceding theorem, P and C can be identified in $D^\bullet/1 + J$ so that, modulo $(1 + J)$, P, shares the richer property of C (normal subgroup of $D^\bullet$).

As for the motivation for Axiom (C3) here is my reasoning. When the *-core C of the division ring D excludes 0 (regardless of the valuation) I called such a division ring D a *c-orderable division ring*. It turns out that D possesses a partial order relation named *c-ordering* such that:

(01) $a > b$ implies $a^* > b^*$;

(02) $a > b$ and $c > d$ together imply $a + c > b + d$;

(03) $1 > 0$ and for each $s = s^* \in D^\bullet$ either $s > 0$ or $s < 0$;

(04) $a > 0$ implies $axx^* > 0$, for all $x \in D^\bullet$.

A basic question asks whether every c-ordering induces an order valuation ω and if, further, ω is a c-valuation of D.

Here, ω (if it exists) is the canonical valuation with valuation ring R the subset of all elements $a \in R$ such that aa^* can be bounded above by some positive integer. In [**2**], I claimed that the answer to the first question is yes and that the answer to the second question is partially yes ([**5**], Theorem 1.11). Unfortunately there is an error in the proof of the basic [**2**, Lemma 5]. I shall have to produce a counter-example here.

Definition 1.12. By *normal* c-ordering of the involutorial division ring D, I mean a c-ordering of D with the following extra axiom:

Axiom (05). *For all non-zero symmetrics s and U in D, if u can be bounded below and above by some positive rationals then sus* > 0.

For such a c-ordering, I will show here that there is an order valuation, which is a c-valuation (Theorem 3.1.9). Actually, the converse will also be demonstrated under the assumption the residue division ring $\bar{D}$ carries some archimedean c-ordering (Theorem 3.1.10).

As Axiom (C3) is purely valuation-theoretic it leads to the question asking what is the full nature of a c-valued division ring D? This question is to be addressed in Section 2 of this article. There will be fairly complete results. And the promised link between c-valuations and normal c-orderings will open the way to new results about c-orderings. Among other things a criterion for extending a valuation of the centre Z of D to a c-valuation ω of D is applied to extend a field ordering of Z to a normal c-ordering of D. This result provides a varied class of finite dimensional normally c-ordered division rings.

Finally, I should emphasize that in this work involutions need not be of the first kind and that unless otherwise assumed the carrier D will be algebraic over its centre Z (if not finite dimensional over Z).

Acknowledgements. I thank S. S. Holland (Jr.) for a number of key ideas and constructive criticisms over the period of this research. Many thanks also to J. Dauns, H. Gross, and A. Wadsworth for stimulating conversations. Partial support of NRC of Canada is acknowledged.

2. Structure of a c-valued division ring. This section begins with a quick investigation of the commutator subgroup of $D^{\bullet}$ (Section 2.0). In 2.1, residue degree, ramification index are examined. In 2.2, certain types of bases (MVP-bases) are investigated with a view to solving the extension problem for c-valuation. In 2.3, examples of c-valuations are given. All notations, facts, and definitions in Section 1 shall be freely used. The involutorial division ring D is everywhere equipped with a fixed c-valuation.

2.0. *Commutator subgroup of $D^{\bullet}$.* For this subsection only D need not be algebraic over Z. Let me write $[a, b]$ for the left commutator $a^{-1}b^{-1}ab$; $a, b \in D^{\bullet}$.

THEOREM 2.0.1. *For D any c-valued division ring, $[D^{\bullet}, D^{\bullet}]$ maps into the unitary group of the involutorial residue division ring $\bar{D}$.*

Proof. Given $a, b \in D^{\bullet}$, let $c = [a, b]$. We have to show that

$$cc^* \equiv 1 \pmod{1 + J}.$$

Now:

$$\begin{aligned} cc^* &= a^{-1}b^{-1}a(bb^*)a^*(b^{-1})^*(a^{-1})^* \\ &\equiv (bb^*)a^{-1}b^{-1}(aa^*)(b^{-1})^*(a^{-1})^* \\ &\equiv (bb^*)(aa^*)a^{-1}(b^{-1})(b^{-1})^*(a^{-1})^* \\ &\equiv (bb^*)(aa^*)(a^*a)^{-1}(b^*b)^{-1}. \end{aligned}$$

By Axiom (C3),

$$aa^* \equiv a^{-1}(aa^*)a = a^*a,$$

giving

$$cc^* \equiv (bb^*)(b^*b)^{-1} \equiv 1,$$

as desired.

THEOREM 2.0.2. *If all valuation units in D are in $\hat{Z}$, then $[D^\bullet, D^\bullet]$ maps, in fact, into the subgroup $\{1, -1\}$ of $\bar{D}$*

Proof. For let $a, b \in D^\bullet$. We have

$$\begin{aligned} 1 &\equiv [aa^*, b] \\ &\equiv ((a^*)^{-1}[a, b]a^*)[a^*, b] \\ &\equiv [a, b][a^*, b] \\ &\equiv [a, b][(a^*a^{-1})a, b] \\ &\equiv [a, b](a^{-1}[a^*a^{-1}, b]a)[a, b] \\ &\equiv [a, b][a, b]. \end{aligned}$$

Therefore $[a, b] \equiv 1$ or $[a, b] \equiv -1$.

THEOREM 2.0.3. *If not all valuation units are in $\hat{Z}$, then $[D^\bullet, D^\bullet]$ maps onto the unitary group of $\bar{D}$.*

Proof. There are two cases.

Case I. Some valuation units do not commute modulo $1 + J$. By Theorem 1.11 (5) either $\bar{D} = Z(\bar{D})$, or D is quaternionic and the involution of $\bar{D}$ is the unique involution with fixed set and subfield $Z(\bar{D})$. By hypothesis the latter case must occur. In this case, given $\bar{a} \in \bar{D}$, $\bar{a} \neq 0$, we can find $\bar{b}$ with $\bar{a}^* = \bar{b}^{-1}\bar{a}\bar{b}$ (Noether-Skolem). For $\bar{u}$ any unitary in $\bar{D}$, $\bar{u} \neq \pm 1$, we can write $\bar{u} = \bar{a}^*\bar{a}^{-1}$ for some $\bar{a}$ (via Cayley parametrisation). Then

$$\bar{u} = \bar{b}^{-1}\bar{a}\bar{b}\bar{a}^{-1} = [\bar{b}, \bar{a}^{-1}].$$

It suffices then to lift $\bar{b}$ and $\bar{a}$ to valuation units b and a in R, which gives

$$[b, a^{-1}] + J = [\bar{b}, \bar{a}^{-1}] = \bar{u}.$$

Case II. Valuation units always commute modulo $1 + J$. If for each $k = -k^* \in R$, $\omega(k) > 0$ then the induced involution is trivial and, hence, all valuation units are in $\hat{Z}$, which we agreed to rule out. This shows that there is $k_0 = -k_0^*$ with $\omega(k) = 0$. Now if $k_0 \in \hat{Z}$ it is easy to show that all valuation units are in $\hat{Z}$, which we ruled out. Thus there must be $x_0 \in D^\bullet$ with

$$x_0^{-1}k_0x_0 \not\equiv k_0 \pmod{1 + J}.$$

Now, the mapping

$$u + J \to x_0^{-1}ux_0 + J$$

is an automorphism of the field $\bar{D}$. By construction, this automorphism fixes the subfield F of symmetrics of $\bar{D}$, which has index 2 in $\bar{D}$. Thus the Galois group of $\bar{D}$ over F is $\{I, (*)\}$. Hence, the preceding mapping coincides with the identity automorphism or with the induced involution of $\bar{D}$. Because

$$x_0^{-1}k_0x_0 \not\equiv k_0 \pmod{J}$$

we get $x_0^{-1}k_0x_0 \equiv k_0^* \equiv -k_0$. Let then $\bar{u}$ be a unitary in $\bar{D}$. If $\bar{u} = -1$ then lift $\bar{u}$ to $[x_0, k] \equiv -1$. If $\bar{u} \neq \pm 1$, then $\bar{u} = \bar{a}^*\bar{a}^{-1}$ for some valuation unit $a \in R$, where $\bar{a} = a + J$. Then $x_0^{-1}ax_0 \equiv a^*$ giving

$$a^*a^{-1} \equiv x_0^{-1}ax_0a^{-1}.$$

Hence, $\bar{u}$ can be lifted to $[x_0, a^{-1}]$.

To close this subsection, it seems desirable to say something decisive about a commutator c in Z, which is a root of unity. For this situation occurs for a symbol division algebra.

THEOREM 2.0.4. *Every commutator (or a product of commutators) c which is a central root of unity must be, in fact,* 1 *or* -1.

Proof. Write $c = c_1 + c_2$ with $c_1 = c_1^*$ and $c_2 = -c_2^*$ both in Z. If $c_2 \notin J$ then $\omega(c_2) = 0$ for clearly $c \in R$. Since $c \in Z$ it follows that c_2 is a central skew symmetric valuation unit. This readily gives that every skew symmetric valuation v unit must be in $\hat{Z}$ (just multiply v by c_2 to get a symmetric valuation unit). By a routine argument all valuation units must be in $\hat{Z}$, in which case, $[D^\bullet, D^\bullet]$ maps into $\{1, -1\}$ so that $c \equiv \pm 1$ and, hence, $c \equiv c^*$ whence $c_2 \equiv 0$. This shows that $c_2 \equiv 0$. Because c is a root of unity in Z it is clear that c_2 is algebraic over the rational subfield $\mathbf{Q}$ of D. Now the restriction of ω to $\mathbf{Q}$ is the trivial valuation. Since $c_2 \equiv 0$ we get $c_2 = 0$ and, hence, $c = c_1$ is symmetric. The subfield F of symmetrics of Z is clearly formally real and, hence, orderable. In F sits the root of unity c. Thus $c = 1$ or -1, as desired.

2.1. *Residue degree and ramification index*. Throughout, D stands for an involutorial division ring with a c-valuation ω (unless otherwise specified). Clearly, ω/Z, the restriction of ω to the field Z, is a valuation of Z with value group written $G_0(=\{\omega(z)|z \in Z\})$ and valuation ring $R \cap Z$. The quotient group G/G_0 (∞'s excluded) is called the *relative group* and its order, if finite, is the *ramification index* of ω (over Z) which is denoted by e (or e_ω if there is danger of confusion). Let $\bar{Z} = Z \cap R/J$. Then

$\bar{Z} \subset Z(\bar{D})$, the centre of the residue division ring $\bar{D}(=R/J)$. As a vector space over $\bar{Z}$, the division algebra $\bar{D}$ has a dimension, which is called the *residue degree* of ω over Z, and is denoted by f (or f_ω if necessary). Clearly,

Fact (1) $f = [\bar{D}:\bar{Z}] = [\bar{D}:Z(\bar{D})][Z(\bar{D}):\bar{Z}]$.

The *inertial inequality* asserts that

Fact (2) $ef \leqq [D:Z(D)]$.

By a very useful result of late Draxl as considerably generated by P. Morandi (**[16]**) follows

Fact (3) $efd = [D:Z]$,

where

$$d = \begin{cases} 1, & \text{for characteristic } (\bar{D}) = 0 \\ p^\alpha, & \text{for characteristic } (\bar{D}) = p \\ & (\alpha, \text{ some non negative integer}). \end{cases}$$

When $d = 1$, we say that the valuation ω is *defectless*. Another result of considerable use for this work is Wedderburn's:

Fact (4). *If $a \in D^\bullet$ is algebraic over Z with degree r, then there are r conjugates $a_1, a_2, \ldots, a_r$ to a in D such that the minimal polynomial p_a of a over Z can be factorized as follows*:

$$p_a(t) = (t - a_1)\ldots(t - a_r),$$

(*the factorization is in $D[t]$, the polynomial ring over D with central indeterminate t.*)

THEOREM 2.1.1. *If $a \in \hat{Z}$ then, in fact, $a \equiv z$ for some $z \in Z^\bullet$. Hence, $\hat{Z} = Z^\bullet(1 + J)$ and so, $Z^\bullet$ maps onto $Z(D^\bullet/1 + J)$.*

Proof. Under the notations in Fact (4) we have $\sum a_i = z$, where $-z$ is the coefficient of t^{r-1} in p_a. Since $a \in \hat{Z}$, it follows that $a \equiv a_i$ and since $\omega(r) = 0$,

$$ra \equiv \sum a_i = z$$

giving

$$a \equiv \frac{1}{r}z.$$

The rest of the statement of the theorem is straightforward.

COROLLARY 2.1.2. *Every symmetric valuation unit s is of the form $s \equiv z$, where $z = z^* \in Z$ is a valuation unit.*

Proof. We know from Theorem 1.11, part (4) that $s \in \hat{Z}$. Now, the minimal polynomial of s has all its coefficients central symmetrics. It suffices then to repeat the reasoning in Theorem 2.1.1.

THEOREM 2.1.3. *For D algebraic over Z and ω any $*$-formally real valuation all of whose symmetric valuation units are in $\hat{Z}$ (in particular if ω is a c-valuation) we have:*

1. $f_\omega = 1$, *if and only if, the induced involution is trivial or, to the contrary, there is a central skew symmetric valuation unit in D.*
2. $f_\omega = 1$ *or* $f_\omega = 2$ *for* $\bar{D} = Z(\bar{D})$.
3. $f_\omega = 4$ *for* $\bar{D} \neq Z(\bar{D})$.

Proof 1. If the involution of $\bar{D}$ is trivial then each valuation unit u is a symmetric modulo J so that $u \in \hat{Z}$ and, hence, $f_\omega = 1$. If on the other hand, there is $k_0 = -k_0^* \in Z$ with $\omega(k_0) = 0$, this readily gives $u \in \hat{Z}$ for all valuation units u so that, again, $f_\omega = 1$. Conversely, if $f_\omega = 1$ but the induced involution is not trivial then we can find $k_0 = -k_0^*$ a valuation unit and $z \in Z$, such that $z \equiv k_0 \pmod{J}$. Then

$$\frac{1}{2}(z - z^*) \equiv k_0$$

and so,

$$\omega\left(\frac{1}{2}(z - z^*)\right) = 0 \quad \text{with } \frac{1}{2}(z - z^*) \in Z^{\bullet}.$$

2. The subfield $Z \cap R/J$ is a $*$-closed subfield of the involutorial field $\bar{D}$, and $Z \cap R/J$ contains the subfield of symmetrics of $\bar{D}$ (Corollary 2.1.2). From this

$$f_\omega = [\bar{D}:Z \cap R/J] = 1 \text{ or } 2.$$

3. Here, $\bar{D}$ is a division ring with centre precisely its subfield of symmetrics and

$$[\bar{D}:Z(\bar{D})] = 4.$$

Now, $Z \cap R/J$ is a $*$-closed subfield of $\bar{D}$ containing $Z(\bar{D})$. If $Z \cap R/J$ strictly contains $Z(\bar{D})$ then there must be $\bar{a} \in Z \cap R/J$, $\bar{a} \notin Z(\bar{D})$. If $\bar{b} = \bar{a} - \bar{a}^*$ then

$$\bar{b} \in Z \cap R/J, \bar{b} \neq 0,$$

and $\bar{b}$ lifts to a central skew symmetric valuation unit, which is ruled out by 1.

THEOREM 2.1.4. *For ω a c-valuation of D and D algebraic over Z, the reative group G/G_0 is an elementary 2-group (G = value group of ω, $G_0 = \omega(Z)$).*

Proof. For let $\infty \neq g \in G$. There is $x \in D^{\bullet}$ such that $\omega(x) = g$. Then

$$2g = 2\omega(x) = \omega(xx^*).$$

Since $xx^* \in \hat{Z}$ it follows by Theorem 2.1.1 that $xx^* \equiv z$, same $z \in Z$. Thus

$$2g = \omega(xx^*) = \omega(z) \in G_0,$$

as desired.

THEOREM 2.1.5. *Let D be any finite dimensional c-valued division ring. Then*:

1. $[D:Z]$ *is a power of* 4.
2. *The ramification index* $e = [D:Z]/f$ *with* f *a divisor of* 4.

Proof. By Theorems 2.1.3 and 2.1.4 combined ef is a power of 2. Since characteristic $(\bar{D}) = 0$ it follows by Fact (3) that $ef = [D:z]$ so that $[D:Z]$ is a power of 2. Because $[D:Z]$ is a perfect square $[D:Z]$ is, hence, a power of 4.

Some remarks about the dimensionality of D are in order.

Remark 2.1.6. (Albert) *Indeed,* $[D:Z]$ *is a power of* 4 *as soon as the involution of D is of the first kind.*

Remark 2.1.7. *Every algebraic symmetric element a of D,* $a \notin Z$, *has even degree over Z, where D is any c-valued division ring (possibly not algebraic over Z).*

Proof. With the same notations used in Fact (4), without loss of generality, $a_1 + \ldots + a_r = 0$. Now, $\prod a_i = z = z^* \in Z$. Thus

$$r\omega(a) = \omega(a_1 \ldots a_r) = \omega(z).$$

If r were odd then $r = 2m + 1$ for some integer $m \geqq 0$. Thus by the preceding equality

$$\begin{aligned} \omega(a) &= \omega(z) - 2m\omega(a) \\ &= \omega(z) - m(\omega(aa^*)) \\ &= \omega(z(aa^*)^{-m}); \end{aligned}$$

or

$$a = uz(aa^*)^{-m},$$

for some valuation unit u. Since $a = a^*$, $z = z^* \in \hat{Z}$, and $(aa^*)^{-m} = ((aa^*)^{-m})^* \in \hat{Z}$ it follows that $u \equiv u^* \pmod{J}$. Then since $\omega(u) = 0$, $u \in \hat{Z}$ and, hence, $a \in \hat{Z}$, contrary to the equation $a_1 + \ldots + a_r = 0$.

Remark 2.1.8. *For D algebraic over Z and* $f_\omega \neq 4$, *every element a in D,* $a \notin Z$, *has even degree over Z.*

Proof. For $f_\omega = 1$, $a \in D$, $a \notin Z$, we may assume that $a_1 + \ldots + a_r = 0$. Since $f_\omega = 1$ it follows by Theorem 2.0.5, that all commutators $[a, b] \equiv \pm 1$. Thus $a_i \equiv \pm a$, for all $i = 1, \ldots, r$. We count the number r_1 of occurrences $a_i \equiv a$ and the number r_2 of occurrences $a_i \equiv -a$. Then from the preceding equality would follow $r_1 a \equiv r_2 a$ giving $r_1 \equiv r_2$ and, hence, $r_1 = r_2$ so that $r = r_1 + r_2 = 2r_1$ is even. For $f_\omega = 2$, reduce to the case a is a valuation unit with $a \equiv -a^*$. Again (Theorem 2.0.3) $a \equiv \pm a_i$ for each i so that r must be even.

As a corollary to Remark 2.1.8 one can show directly (that is, independently from the powerful Fact (3)) that if $f_\omega \neq 4$ then, again, $[D:Z]$ is a power of 4. Is there a direct proof for the case $f_\omega = 4$ as well? In this connection, one might ask the following questions.

Question 2.1.9. If D is algebraic over Z must every element $a \in D$, $a \notin Z$, be of even degree over Z? (True for $f_\omega \neq 4$.)

Question 2.1.10. If, regardless of the valuation, we assume that every symmetric $a \in D$, $a \notin Z$, has even degree over Z does it follow that every element $a \in D$, $a \notin Z$, has even degree over Z? (In the affirmative $[D:Z]$ is then a power of 4 for $[D:Z] < \infty$.)

2.2. MVP-*Bases and extending *-formally real valuations*. By MVP-*basis* $\{e_\alpha\}$ of any valued division ring D over its centre Z, I mean one which verifies the Minimum Value Principle; that is

$$\text{(MVP)} \qquad \omega\left(\sum z_\alpha e_\alpha\right) = \min(\omega(z\alpha e_\alpha))(z_\alpha \in Z).$$

The basis $\{e_\alpha\}$ need not be finite for the preceding requirement to make sense; all that is needed is that, as usual, all $z_\alpha = 0$ but for finitely many indices. Some properties of such a basis shall be investigated here in the context of a c-valuation ω. On one or two occasions an interpretation of Axiom (C3) in terms of the given basis $\{e_\alpha\}$ shall be given. This will help for the main goal of this subsection; namely, to find a criterion for a valuation of the centre Z of D to extend to a c-valuation of D, where we will be given an appropriate basis of D over Z.

THEOREM 2.2.1. *For ω any *-formally real valuation (in particular if ω is a c-valuation) and arbitrary $a, b \in D$, if $\omega(a) \neq \omega(b)$, then*

$$\text{(E)} \qquad \omega(aa^* + bb^*) < \omega(ab^* + ba^*).$$

Proof. We have

$$\begin{aligned} \omega(ab^* + ba^*) &\geqq \min(\omega(ab^*), \omega(ba^*)) \\ &= \omega(a) + \omega(b) \\ &> 2\min(\omega(a), \omega(b)) \end{aligned}$$

$$= \min(\omega(aa^*), \omega(bb^*))$$
$$= \omega(aa^* + bb^*).$$

THEOREM 2.2.2. *Let D be a finite dimensional division ring with involution of the first kind. Let ω be any *-formally real valuation such that every symmetric valuation unit is in $\hat{Z}$ (in particular if ω is a c-valuation or if ω has residue degree* 1). *Then*:

1. *There is a basis $\{e_\alpha\}$ of D over Z such that if $\alpha \neq \beta$ then for $z_\alpha, z_\beta \in Z$ not both zeros follows*

$$\text{(E}'\text{)} \qquad \omega((z_\alpha e_\alpha)(z_\alpha e_\alpha)^* + (z_\beta e_\beta)(z_\beta e_\beta)^*) < \omega((z_\alpha e_\alpha)(z_\beta e_\beta)^* + (z_\beta e_\beta)(z_\alpha e_\alpha)^*).$$

2. *Conversely, any *-formally valuation ω with basis $\{e_\alpha\}$ verifying* (E′) *is such that $\{e_\alpha\}$ is a* MVP-*basis. Here, every norm xx^* is of the form*

$$xx^* = \sum(z_\alpha e_\alpha)(z_\alpha e_\alpha)^*, \quad z_\alpha \in Z^\bullet.$$

It follows that ω is a c-valuation if and only if all $e_\alpha e_\alpha^* \in \hat{Z}$.

Proof 1. By Theorem 2.1.3 for given f we can find f valuation units $y_0, y_1, \dots, y_{f-1}$ with the following properties: (i) when $f \neq 1$, then all y_i with $i \neq 0$ are skew symmetrics modulo J; (ii) for $f = 4$, we have $y_1 \cdot y_2 + y_2 y_1 \equiv 0 \pmod J$, and $y_3 = y_1 y_2$. Choose $x_1, \dots, x_e$ in $D^\bullet$ such that the $\omega(x_i)$ form a complete set of representatives for the congruence $\equiv \pmod{G_0}$ of G. Put

$$\{e_\alpha\} = \{x_i\} \times \{y_j\}.$$

By construction, $\{e_\alpha\}$ is linearly independent over Z. Since ω is *-formally real it follows that characteristic $(\bar{D}) = 0$ so that ω is defectless and, hence, $\{e_\alpha\}$ is a basis of D over Z. I proceed to check (E′). In effect, for $\alpha \neq \beta$, and $z_\alpha, z_\beta \in Z$ put $a = z_\alpha e_\alpha$ and $b = z_\beta e_\beta$. Now (E′) is just (E) in Theorem 2.2.1. Thus if (E′) fails necessarily $z_\alpha, z_\beta \in Z^\bullet$, and $\omega(a) = \omega(b) \neq \infty$. Then for $e_\alpha = x_i y_j$, $e_\beta = x_e y_k$ we have

$$\omega(a) = \omega(z_\alpha e_\alpha) = \omega(z_\alpha) + \omega(x_i), \quad \omega(b) = \omega(z_\beta) + \omega(x_l).$$

It follows that

$$\omega(x_i) \equiv \omega(x_l) \pmod{G_0}.$$

By construction $i = l$ follows so that $j \neq k$ and ω has residue degree >1. Now,

$$\omega(a) = \omega(z_\alpha) + \omega(x_i) = \omega(b) = \omega(z_\beta) + \omega(x_i)$$

so that $\omega(z_\alpha) = \omega(z_\beta)$. A simple computation shows that

$$0 = \omega(aa^* + bb^*) - \omega(ab^* + ba^*) = \omega(y_j y_k^* + y_k y_j^*).$$

When $y_j = y_0 = 1$, then since $y_k \equiv -y_k^*$ it follows that

$$y_j y_k^* + y_k y_j^* \equiv -y_k + y_k \equiv 0,$$

which contradicts the preceding equality. This shows that both y_j and y_k are skew symmetrics modulo J; but then,

$$y_j y_k^* + y_k y_j^* \equiv -(y_k y_k + y_k y_j) \equiv 0,$$

which contradicts again the preceding equality. With this (E′) is established.

2. For $x \in D^\bullet$, we write

$$x = \sum z_\alpha e_\alpha, \quad z_\alpha \in Z.$$

Only finitely many indices α are such that $z_\alpha \neq 0$. Without loss of generality

$$x = \sum_{r=1}^{s} z_r e_r$$

with all $z_r \in Z^\bullet$. We can write xx^* in the form

$$xx^* = y_1(x) + y_2(x),$$

where

$$y_1(x) = \sum_{r=1}^{s} (z_r e_r)(z_r e_r)^*;$$

$$y_2(r) = \sum_{1 \leqq i < j \leqq s} (z_i e_i)(z_j e_j)^* + (z_j e_j)(z_i e_i)^*.$$

For each pair i, j with $i < j$ the corresponding term in $y_2(x)$ has value strictly larger than

$$\omega(y_1(x)) = \min(\omega(z_r e_r)(z_r e_r)^*).$$

It follows that $\omega(y_2(x)) > \omega(y_1(x))$ so that $xx^* \equiv y_1(x)$. Then

$$\begin{aligned} 2\omega(x) = \omega(xx^*) &= \omega(y_1(x)) \\ &= \min(\omega(z_\alpha e_\alpha)(z_\alpha e_\alpha)^*) \\ &= 2\min(\omega(z_\alpha e_\alpha)) \end{aligned}$$

or

$$\omega(x) = \min(\omega(z_\alpha e_\alpha)),$$

as desired. Finally, when ω is a c-valuation then evidently $e_\alpha e_\alpha^* \in \hat{Z}$ for all α (after all, $e_\alpha \neq 0$). Conversely, if $e_\alpha e_\alpha^* \in \hat{Z}$ for all α then by the preceding

$$xx^* \equiv \sum_{r=1}^{s} (z_r e_r)(z_r e_r)^*,$$

where $(z_r e_r)(z_r e_r)^* \in \hat{Z}$. For arbitrary $a \in D^\bullet$, we have

$$\begin{aligned} a^{-1}(xx^*)a &\equiv \left(\sum_{r=1}^{s} a^{-1}(z_r e_r)(a_r e_r)^* a\right) \\ &\equiv \sum_{r=1}^{s} (z_r e_r)(z_r e_r)^* \\ &\equiv xx^* \end{aligned}$$

so that $xx^* \in \hat{Z}$, as desired.

As a corollary of the preceding theorem if ω totally ramified, ω is *-formally and if the involution is of the first kind then ω is a c-valuation, a result which will be explored in [**3**]. What about the case of an involution of the second kind? There is a serious obstacle for ensuring (E′), and the argument as given in the proof of Theorem 2.2.2 Part 1, could badly break in the mixed case where $z_\alpha = z_\alpha^*$ and $z_\beta = -z_\beta^*$. An interesting situation where (E′) is still valid even for an involution of the second kind is for a basis $\{e_\alpha\}$ of D over Z, which is a q-basis (in the sense given by Amitsur, Rowen, and Tignol in [**1**]).

THEOREM 2.2.3. *Every q-basis $\{e_\alpha\}$ of D over Z is a* MVP *basis, where D is any c-valued division ring (the basis may be infinite). In fact, $\{e_\alpha\}$ verifies* (E′) *as in Theorem* 2.2.2.

Proof. I shall break the argument in three steps.

Step 1. *Let ω be any *-formally real valuation all of whose symmetric valuations units are in $\hat{Z}$. If u is a valuation unit such that $c^{-1}uc \equiv -u$ for some $c \in D$ then $u \equiv -u^*$* (mod J) *provided* $[c^*c, u] \equiv 1$.

Subproof. Deny the conclusion. Then $u = u_1 + u_2$ with $u_1 = u_1^*$ and $u_2 = -u_2^*$ are in R, and $u_1 \not\equiv 0$ (mod J). By hypothesis, $u_1 \in \hat{Z}$. Thus modulo J

$$-u_1 - u_2 \equiv c^{-1}(u_1 + u_2)c \equiv u_1 + c^{-1}u_2 c.$$

Now,

$$\begin{aligned} (c^{-1}u_2 c)^* &= -c^* u_2 c^{-1})^* \\ &= -(c^*c)(c^{-1}u_2 c)(c^*c)^{-1} \\ &\equiv -(c^{-1}u_2 c). \end{aligned}$$

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

In other words, $c^{-1}u_2c$ is a skew symmetric modulo J so that $u_2 + c^{-1}u_2c$ is a skew symmetric modulo J and, hence, $-2u_1$ is a skew symmetric modulo J, which is nonsense. This shows that $u_1 \equiv 0$ and so, $u \equiv u_2$ is a skew symmetric modulo J.

Step 2. *If a, $b \in D^\bullet$ are such that for some $c \in D^\bullet$,*

$$[cc^*, ba^{-1}] \equiv 1, \quad \textit{and } [c, ba^{-1}] \equiv -1,$$

then $\omega(ab^* + ba^*) > \omega(aa^* + bb^*)$ where ω is as in Step 1.

Subproof. Deny the inequality. Then $\omega(a) = \omega(b)$. Thus if we put $u = ba^{-1}$ then u is a valuation unit. By Step 1, $u \equiv -u^*$ (mod J). Now $u = ba^{-1}$ gives $b = ua$ so that

$$\begin{aligned} ab^* + ba^* &= aa^*u^* + uaa^* \\ &= aa^*(u^* + (aa^*)^{-1}uaa^*), \end{aligned}$$

and, hence,

$$\omega(ab^* + ba^*) = \omega(aa^*) + \omega(u^* + (aa^*)^{-1}uaa^*).$$

The relations, $u^* \equiv -u$ and $(aa^*)^{-1}u(aa^*) \equiv u$ together imply

$$u^* + (aa^*)_u^{-1}(aa^*) \equiv 0 \text{ (mod } J),$$

so that

$$\omega(ab^* + ba^*) > \omega(aa^*) = \omega(bb^*) = \omega(aa^* + bb^*).$$

Step 3. *Every q-basis $\{e_\alpha\}$ of D over Z verifies* (E′) *and, consequently, $\{e_\alpha\}$ is a* MVP-*basis, where I assume that the valuation ω is a c-valuation.*

Subproof. Let $\alpha \neq \beta$ and let z_α, $z_\beta \in Z$. Put $a = z_\alpha e_\alpha$, $b = z_\beta e_\beta$. If $z_\alpha = 0$ or if $z_\beta = 0$ then (E′) readily follows. For z_α, $z_\beta \in Z^\bullet$, we have

$$ab^{-1} = z_\alpha z_\beta^{-1} e_\alpha e_\beta^{-1}.$$

Clearly $ab^{-1} \notin Z$ (for, otherwise, $e_\alpha = z'e_\beta$, where $z' \in Z$, contrary to linear independence). Thus there must be γ such that

$$[ab^{-1}, e_\gamma] = [e_\alpha e_\beta^{-1}, e_\gamma] \neq 1.$$

Since $[e_\alpha, e_\gamma] = \pm 1$, $[e_\beta, e_\gamma] = \pm 1$, it follows that $[e_\alpha e_\beta^{-1}, e_\gamma] = \pm 1$ so that $[e_\alpha e_\beta^{-1} e_\gamma] = -1$ and, hence, $[ab^{-1}, e_\gamma] = -1$. By construction,

$$[ab^{-1}, e_\gamma e_\gamma^*] = [ab^{-1}, e_\gamma^* e_\gamma] \equiv 1.$$

By Step 2 follows the inequality (E′). It suffices then to repeat the proof of Theorem 2.2.2 Part 2.

As a side observation, the given of any q-basis $\{e_\alpha\}$ for a c-valued division ring D is rich enough so as to yield directly the fact that such a valuation is defectless (without making use of Fact 3)). Another interesting aspect of such a basis is underscored in the

THEOREM 2.2.4. (Structure of a c-valued division ring.) *Let D be any finite dimensional c-valuated division ring, which is a tensor product of quaternionic division rings. Then*

$$D \approx D_0 \bigotimes_{Z} D_0',$$

where D_0 is a central division subalgebra of D over its centre Z of D, at most, 4-dimensional such that ω/D_0 has a residue degree precisely f_ω, and D_0', the centralizer of D_0 in D, is totally ramified as a valued division ring.

Proof. When $f = 1$ we simply take D_0 to be Z. In the opposite case, we may choose a q-basis $\{e_\alpha\}$ of D over Z, which is an "armature", that is, $e_\alpha e_\beta \in ze_\gamma$, for some index γ and $z \in Z$. Because $\{e_\alpha\}$ is a MVP-basis, if we set

$$F = \{\alpha | \omega(e_\alpha) \equiv 0 \ (\text{mod}\ G_0)\}$$

then $|F| = f$. Without loss of generality the index set for the basis has a first member written 0, and $e_0 = 1$. Moreover, when $\alpha \in F$ then $\omega(e_\alpha) = 0$. For such an $\alpha \neq 0$, we can readily show that

$$e_\alpha \equiv -e_\alpha^* \ (\text{mod}\ J).$$

(See Theorem 2.2.3 Step 1). It follows that either $|F| = 2$ or else $|F| = 4$ and then $\bar{D}$ is 4-dimensional with all its symmetrics central so that the three nonconstant base vectors e_1, e_2, e_3 anticommute pairwise. Since $[e_i, e_j] = \pm 1$ in D we get that, in fact, $[e_i, e_j] = -1$ if $i \neq j \leqq 3$. We now choose D_0 as follows. When $f = 2$, pick any e_α with $[e_1, e_\alpha] = -1$ and set $D_0 = Z[e_1, e_\alpha]$. When $f = 4$, set

$$D_0 = Z[e_1, e_2, e_3].$$

Since $e_1 e_2$ is a valuation unit it follows that $D_0 = Z[e_1, e_2]$. Thus in both cases D_0 is 4-dimensional with centre precisely Z. By construction, for $f = 2$, $\{1, e_1, e_\alpha, e_1 e_\alpha\}$ is a MVP-basis of D_0 over Z, which gives that

$$f_{\omega/D_0} = 2 = f_\omega.$$

When $f = 4$ then, again,

$$f_{\omega/D_0} = 4 = f.$$

Hence in both cases $f_{\omega/D_0} = f$. Now, if D_0' is the centralizer of D_0 in D, then D_0' carries the q-basis formed by all e_α, that commute with all of D_0. By construction, for all such e_α, we have

$$\omega(e_\alpha) \not\equiv 0 \ (\text{mod}\ G_0).$$

The latter sub-basis of D_0' is a MVP-basis so that the residue degree $f_{\omega/D_0'} = 1$. Because ω/D_0' is defectless it follows that $(D_0'; \omega/D_0')$ is totally ramified, as desired.

It should be pointed out that the preceding theorem is not really of involutorial nature as nothing guarantees that D_0 and, hence, D_0' are *-closed. Of course, if we insist that D_0 is a tensor product of involutorial quaternionic division rings we can exploit a q-basis all of whose members are symmetrics or skew symmetrics, in which case, D_0 and, hence, D_0' are *-closed so that ω/D_0 and ω/D_0' become c-valuations and Theorem 2.2.4 takes on a full involutorial meaning.

For the last objective for this subsection let me make the carrier D merely an involutorial ring with centre Z, a field. (This will prove practical for the constructions of c-valuations to follow.) I shall make a definitely strong choice of the basis $\{e_\alpha\}$ of D over Z; namely, $\{e_\alpha\}$ is a "*-armature basis" of D over Z, according to the

Definition 2.2.5. By **-armature basis* $\{e_\alpha\}$ I mean a basis $\{e_\alpha\}$ of D over Z such that

(A1) The index set $\{\alpha\}$ is totally ordered with first member written 0, $e_0 = 1$, and $e_\alpha^2 \neq 0$, for all α.

(A2) Given indices α, β there are $z = z^* \in Z$ and γ such that $e_\alpha e_\beta = ze_\gamma$.

(A3) Each e_α is symmetric or skew symmetric so that one can assign to each e_α integer $\epsilon_\alpha = \pm 1$ depending on whether $e_\alpha = e_\alpha^*$ or $e_\alpha = -e_\alpha^*$.

There follow some straightforward observations and notations needed for the extension problem; namely, to extend a *-valuation ω_0 of the field Z to a c-valuation of D.

Remark 2.2.6. *Every *-armature basis $\{e_\alpha\}$ of D consists solely of invertible elements e_α with $[e_\alpha, e_\beta] = \pm 1$, for all pairs α and β, so that all $e_\alpha e_\alpha^* = \pm e_\alpha^2$ are nonzero central elements.*

Remark 2.2.7. *Every tensor product of quaternionic involutorial division rings with involutions of the first kind equipped with the tensor product involution has a *-armature basis.*

Notation 2.2.8. For G_0 the value group of the *-valuation ω_0 of Z let

$$G_0^{(+1)} = \{\omega(z) | z = z^* \in Z)$$

and let

$$G_0^{(-1)} = \{\omega(z) | z = -z^* \in Z\}.$$

Notation 2.2.9. If for a given e_α we have

$$\omega_0(e_\alpha e_\alpha^*) = \omega_0(e_\alpha^2) \notin 2G_0^{(\epsilon_\alpha)},$$

I shall denote this relation by NCR$[e_\alpha]$ (read: non congruence relation $\omega_0(e_\alpha^2) \not\equiv 0 \pmod{2G_0^{(\epsilon_\alpha)}}$). When NCR$[e_\alpha]$ holds true for all $\alpha \neq 0$, I shall write ENCR$[e_\alpha]$ (read: enough non congruence relations $\omega_0(e_\alpha^2) \neq 0 \pmod{2G_0^{(\epsilon_\alpha)}}$).

Notation 2.2.10. For G_0 the value group of ω_0 and $\{e_\alpha\}$ a *-armature basis of D, let

$$G = G_0\left[\frac{1}{2}\omega_0(e_\alpha^2)\right]$$

stand for the extended order abelian group obtained by adjoining formal halves of all the $\omega_0(e_\alpha^2) \in G_0$ with $\omega_0(e_\alpha^2) \notin 2G_0$.

Notation 2.2.11. Let $P_0 = P(Z)$ be the set of sums of norms zz^*, $z \in Z^\bullet$. Let $P_0[e_\alpha e_\alpha^*]$ stand for the overset of sums of products of norms of the form zz^* or $e_\alpha e_\alpha^*$. When

$$\omega_0(a + b) = \min(\omega_0(a), \omega_0(b)) \text{ for all } a, b \in P_0[e_\alpha e_\alpha^*],$$

I shall write EFR$[e_\alpha]$ (read: extended *-formal reality of ω_0 to $P_0[e_\alpha e_\alpha^*]$).

I can now establish an important result. This is the

THEOREM 2.2.12. (Main Theorem) *Let D be an involutorial ring with centre Z a field, let ω_0 be a *-valuation of the involutorial field Z, and let $\{e_\alpha\}$ be a *-armature basis of D over Z. The following requirements are equivalent*:

1. *ω_0 extends to a c-valuation of D so that D is a division algebra.*
2. *Both* EFR$[e_\alpha]$ *and* ENCR$[e_\alpha]$ *hold true.*

Proof. $1 \Rightarrow 2$. That EFR$[e_\alpha]$ holds true this is evident. For ENCR$[e_\alpha]$ let $0 \neq \alpha$. Negate the corresponding relation

$$\omega_0(e_\alpha^2) \not\equiv 0 \pmod{2G_0^{(\epsilon_\alpha)}}.$$

Thus

$$\omega_0(e_\alpha^2) \equiv 0 \pmod{2G_0}.$$

Equivalently,

$$\omega(e_\alpha) \equiv 0 \pmod{G_0};$$

that is, there is $z = \epsilon_\alpha z^*$ with $\omega(e_\alpha) = \omega(z)$ so that for $u = z^{-1}e_\alpha$ we have $\omega(u) = 0$ with $u = u^*$. However, since $\alpha \neq 0$, there is β with $[e_\alpha, e_\beta] = -1$ giving $[u, e_\beta] = -1$; but, since $u = u^*$ is a valuation unit it follows that $u \in \hat{Z}$ so that $[u, e_\beta] \equiv 1$, which brings a contradiction.

$2 \Rightarrow 1$. The argument is broken in four steps.

Step 1. *For $\alpha \neq \beta$, arbitrary γ, and z_α, $z_\beta \in Z$ not both zeros follows*

$$\text{(E}''\text{)} \qquad \omega_0(aa^* + bb^*) < \omega_0(\mathrm{proj}_\gamma(ab^* + ba^*)) + \frac{1}{2}\omega_0(e_\gamma^2)$$

where $a = z_\alpha e_\alpha$ *and* $b = z_\beta e_\beta$. (proj_γ *is the linear form, which assigns to* $x \in D$ *the coefficient of* e_γ *after expansion of* x *in terms of the basis*).

Subproof. If $z_\alpha = 0$ or if $z_\beta = 0$ then $ab^* + ba^* = 0$, while by EFR[e_i] follows

$$\begin{aligned} g_2 &= 2\omega_0(\epsilon_\beta z(z_\alpha z_\beta^* + \epsilon_\gamma z_\beta z_\alpha^*)) + \omega_0(e_\gamma^2) \\ &= \omega_0((ze_\gamma)^2) + 2\omega_0(z_\alpha z_\beta^* + \epsilon_\gamma z_\beta z_\alpha^*) \\ &\geqq \omega_0((ze_\gamma)^2) + 2\omega_0(z_\alpha z_\beta^*) \\ &= \omega_0(e_\alpha^2 e_\beta^2) + 2\omega_0(z_\alpha b_\beta) \\ &= \omega_0((z_\alpha e_\alpha)^2) + \omega_0((z_\beta e_\beta)^2) \\ &= \omega_0((z_\alpha e_\alpha)(z_\alpha e_\alpha)^*) + \omega_0((z_\beta e_\beta)(z_\beta e_\beta)^*) \\ &\geqq 2\min(\omega_0(aa^*), \omega_0(bb^*)) \\ &= g_1. \end{aligned}$$

If then $g_2 > g_1$ fails then by the preceding inequalities follow:

(i) $\qquad \omega_0(z_\alpha z_\beta^*) = \omega_0(z_\alpha z_\beta^* + \epsilon_j z_\beta z_\alpha^*)$; and

(ii) $\qquad \omega_0(aa^*) = \omega_0(bb^*)$.

By (ii) follows

$$2\omega(z_\alpha) + \omega_0(e_\alpha^2) = 2\omega(z_\beta) + \omega_0(e_\beta^2).$$

Combining this with the equality $e_\alpha e_\beta = ze_\gamma$, which readily gives

$$\omega_0(e_\alpha^2) + \omega_0(e_\beta^2) = \omega_0(z^2) + \omega_0(e_\gamma^2),$$

we can eliminate $\omega_0(e_\beta^2)$ to get

$$2\omega(z_\alpha) + 2\omega_0(e_2^2) = 2\omega(z_\beta) + \omega_0(e_\gamma^2) + \omega_0(z^2).$$

Then

$$\begin{aligned} &2(\omega(z_\alpha) + \omega_0(e_\alpha^2) - \omega(z_\beta)) = \omega_0((ze_\gamma)^2); \\ &2(\omega_0(z_\alpha z_\beta^{-1} e_\alpha^2) = \omega_0((ze_\gamma)^2); \\ &2\omega_0(z^{-1}(z_\alpha z_\beta^{-1} e_\alpha^2) = \omega_0(e_\gamma^2); \\ &2\omega_0(z^{-1}(z_\alpha z_\beta^{-1})e_\alpha e_\alpha^*) = \omega_0(e_\gamma e_\gamma^*). \end{aligned}$$

Put

$$c = z^{-1}(e_\alpha e_\alpha^*) z_\alpha z_\beta^{-1}.$$

We have $c \in Z$. Working in the valued field Z with respect to the congruence $\equiv(\mathrm{mod}(1 + J_0)$, where J_0 is the maximal ideal of ω_0, if $c \not\equiv -\epsilon_\gamma c^*$ then for $d = c + \epsilon_\gamma c^*$ we have $d = \epsilon_\gamma d^*$ and $\omega(d) = \omega(c)$ so that

$$\omega_0(e_\gamma e_\gamma^*) = 2\omega(d), \quad \text{with } d \equiv \epsilon_\gamma d^*,$$

which violates ENCR$[e_i]$. This shows that $c \equiv -\epsilon_\gamma c^*$. Now

$$c = (z^{-1}e_\alpha e_\alpha^*)(z_\alpha z_\beta^{-1}) \quad \text{with } (z^{-1}e_\alpha e_\alpha^*)^* = (z^{-1}e_\alpha e_\alpha^*) \text{ in } Z.$$

It follows that

$$z_\alpha z_\beta^{-1} \equiv -\epsilon_\gamma(z_\alpha z_\beta^{-1})^* \quad \text{or, } z_\alpha z_\beta^* \equiv -\epsilon_\gamma z_\beta z_\alpha^*$$

so that

$$\omega(z_\alpha z_\beta^* + \epsilon_\gamma z_\beta z_\alpha^*) > \omega(z_\alpha z_\beta^*),$$

contrary to (i). This shows that $g_2 > g_1$.

Step 2. *For* $r \in D$, $x = \sum z_\alpha e_\alpha$ *with* $z_\alpha \in Z$ *put*

$$\omega(x) = \min(\omega_0(z_\alpha) + \frac{1}{2}\omega_0(e_\alpha^2)\,).$$

Then:

(i) $\omega(x)$ *exists in* G *so that* ω: $x \to \omega(x)$ *maps* D *onto* G;
(ii) ω *extends* ω_0;
(iii) $\omega(x^*) = \omega(x)$, *for all* $x \in D$;
(iv) $\omega(x + y) \geqq \min(\omega(x), \omega(y)\,)(x, y \in D)$;
(v) $\omega(x) = \infty$ *if, and only if,* $x = 0$.

Subproof. This step is straightforward.

Step 3. ω *is a* $*$-*valuation of* D.

Subproof. First, I will establish that for $x \in D$, $x \neq 0$, follows xx^* is of the form

$$xx^* = c + \sum_{\alpha \neq 0} c_\alpha e_\alpha,$$

where $c \in Z$, and $\omega(c) = \omega_0(c) < \omega(c_\alpha e_\alpha)$ for each $\alpha \neq 0$. For put:

$$x = \sum z_\alpha e_\alpha, \quad z_\alpha \in Z, \quad \text{and}$$

$$x_\alpha = z_\alpha e_\alpha; \quad y_1 = \sum x_\alpha x_\alpha^*; \quad y_2 = \textstyle\sum_{\beta<\gamma} x_\beta x_\gamma^* + x_j x_\beta^*.$$

For a fixed pair β and γ with $\beta < \gamma$ and not both k_β and k_j are zeros we have for every δ (Step 1):

$$\omega_0(\mathrm{proj}_\delta(x_\beta x_\gamma^* + x_j x_\beta^*)\,) + \frac{1}{2}\omega_0(e_8^2)$$

$$> \omega_0(x_\beta x_\beta^* + x_j x_j^*) \geqq \min(\omega_0(x_\alpha x_\alpha^*))$$
$$= \omega_0(y_1).$$

Thus

$$\omega(x_\beta x_\gamma^* + x_\gamma x_\beta^*) > \omega_0(y_1).$$

Now,

$$xx^* = y_1 + y_2,$$

where $y_1 \in Z$. Here, each term in y_2 is of the form

$$x_\beta x_\gamma^* + x_\gamma x_\beta^* \in Ze_{\delta'},$$

where $e_\beta e_\gamma \in Ze_{\gamma'}$. By construction, $e_{\delta'} \neq 1$. Thus xx^* can be represented as desired with $c = y_1$. Let now $0 \neq y \in D$. We have

$$y(xx^*)y^* = c(yy^*) + \sum c_\alpha y e_\alpha y^*.$$

I contend that

$$\omega(ye_\alpha y^*) \geqq \omega(y) + \omega(e_\alpha) + \omega(y^*).$$

For write

$$y = \sum y_\beta, \quad \text{where } y_\beta = t_\beta e_\alpha,\ t_\beta \in Z.$$

Then

$$ye_\alpha y^* = \sum y_\beta e_\alpha y_{\beta'}^*.$$

Here,

$$\omega(y_\beta e_\alpha y_{\beta'}^*) = \omega(t_\beta t_{\beta'}^* e_\beta e_\alpha e_{\beta'}).$$

Since $e_\beta e_\alpha e_{\beta'}$ is a base vector it is clear that

$$\omega(e_\beta e_\alpha e_{\beta'}) = \frac{1}{2}\omega_0(\,(e_\beta e_\alpha e_{\beta'})(e_\beta e_\alpha e_{\beta'})^*)$$

$$= \frac{1}{2}\omega_0(e_\beta e_\alpha e_{\beta'}(e_{\beta'})^* e_\alpha^* e_\beta^*)$$

$$= \frac{1}{2}\omega_0(\,(e_{\beta'} e_{\beta'}^*)(e_\alpha e_\alpha^*)(e_\beta e_\beta^*)\,)$$

$$= \omega(e_\beta) + \omega(e_\alpha) + \omega(e_{\beta'}).$$

Thus

$$\omega(y_\beta e_\alpha y_{\beta'}^*) = \omega(t_\beta t_{\beta'}^* + \omega(e_\beta) + \omega(e_\alpha) + \omega(e_{\beta'})$$
$$= \omega(t_\beta e_\beta) + \omega(e_\alpha) + \omega(t_{\beta'} e_{\beta'})$$

so that

$$\omega(ye_\alpha y^*) \geqq \omega(e_\alpha) + \omega(y) + \omega(y^*),$$

as desired. It follows that

$$\omega\left(\sum_{\alpha \neq 0} c_\alpha y e_\alpha y^*\right) \geqq \omega(c_\alpha) + \omega(y) + \omega(y^*).$$

Now, $\omega(y) = \omega(y^*)$. Because of the form of the norm yy^* we have

$$\omega(yy^*) = 2\omega(y)$$

so that

$$\begin{aligned}\omega(c_\alpha) + \omega(y) + \omega(y^*) &= \omega(c_\alpha) + \omega(yy^*)\\ &> \omega(c) + \omega(yy^*)\\ &= \omega(cyy^*).\end{aligned}$$

Therefore,

$$\begin{aligned}\omega(y(xx^*)y^*) &= \omega(cyy^*)\\ &= \omega(c) + \omega(yy^*)\\ &= \omega(xx^*) + \omega(yy^*)\\ &= 2(\omega(x) + \omega(y)).\end{aligned}$$

It follows that $yxx^*y^* \neq 0$ so that $yx \neq 0$. Then

$$\begin{aligned}\omega(y(xx^*)y^*) &= \omega((yx)(yx)^*)\\ &= 2\omega(yx).\end{aligned}$$

From the preceding equality follows

$$\omega(yx) = \omega(x) + \omega(y) = \omega(y) + \omega(x).$$

It is now clear that the algebra D has no divisors of zero. That D is a division algebra, in fact, D is a locally finite division algebra over its centre Z, this readily follows from the commutation rules $[e_\alpha, e_\beta] = \pm 1$.

Step 4. *ω is a c-valuation.*

Subproof. From the form of a norm xx^* follows

$$xx^* \equiv \sum z_\alpha z_\alpha^* e_\alpha e_\alpha^* (x = \sum z_\alpha e_\alpha).$$

By EFR$[e_\alpha]$ readily follows that ω is *-formally real. Actually

$$\sum z_\alpha z_\alpha^* e_\alpha e_\alpha^* \in Z$$

so that certainly $xx^* \in \hat{Z}$.

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

COROLLARY 2.2.13. *Let D be as in Theorem* 2.2.12 *and suppose that the $*$-armature basis $\{e_\alpha\}$ is such that $\alpha \neq 0$ implies*

$$\omega_0(e_\alpha^2) \not\equiv 0 \pmod{2G_0}.$$

Then D is a locally finite division algebra and ω_0 extends to a c-valuation ω, which is totally ramified.

Proof. By construction ENCR$[e_\alpha]$ holds true. Because of the submultiplicative property of $\{e_\alpha\}$ we have $\omega_0(e_\alpha^2) \not\equiv \omega_0(e_\beta^2)$ for all α, β with $\alpha \neq \beta$, which gives EFR$[e_\alpha]$.

COROLLARY 2.2.14. *If D has an involution of the first kind then the extensibility of ω_0 simply means that if $e_\alpha = e_\alpha^* \neq 1$ then*

$$\omega_0(e_\alpha^2) \not\equiv 0 \bmod 2(G_0)$$

(EFR$[e_\alpha]$ *unchanged*).

COROLLARY 2.2.15. *If, to the contrary, there is $z = -z^* \in Z$ with $\omega_0(z) = 0$ then the extensibility of ω_0 forces $\omega_0(e_\alpha^2) \notin 2G_0$ for all $\alpha \neq 0$, in which case, the extension ω is totally ramified.*

For Corollary 2.2.14 just observe that $G_0^{(+1)} = G_0$, while $G_0^{(-1)} = \{\infty\}$ so that in the requirement ENCR$[e_\alpha]$ the skew symmetrics base vectors automatically verify their corresponding non congruence relations, while the non constant symmetric base vectors e_α must now be subjected to the full requirement $\omega_\alpha(e_\alpha^2) \notin 2G_0$. For Corollary 2.2.15, just observe that by hypothesis $G_0^{(+1)} = G_0^{(-1)} = G_0$ so that the relation NCR$[e_\alpha]$ assumes its full version

$$\omega_0(e_\alpha^2) \not\equiv 0 \pmod{2G_0},$$

where $\alpha \neq 0$. From this we get

$$\omega(e_\alpha^2) \not\equiv \omega_0(e_\beta^2) \pmod{2G_0}$$

for all pairs $\alpha \neq \beta$, which ensures EFR$[e_\alpha]$.

Recall that the Structure Theorem (Theorem 2.2.4) asserts that every c-valued division ring D with q-basis is of the form

$$D \approx D_0 \bigotimes_Z D_0',$$

where D_0 absorbs the residue degree while D_0' is totally ramified. For a converse, let me then replace the rôle of Z in Theorem 2.2.12 by a $*$-closed division subring D_0 with same centre Z and with a fixed c-valuation ω_0. We define a $*$-armature basis $\{e_\alpha\}$ of D over D_0, to be a $*$-armature basis of D_0', the centralizer of D_0 in D. We write G_0 for the value group of ω_0, $G_0^{(+1)}$, $G^{(-1)}$ as in the preceding case, and use similar notations ENCR$[e_\alpha]$ and EFR$[e_\alpha]$ relative to D_0. By similar arguments (although fairly longer) one can show the

THEOREM 2.2.16. *Let D be any involutorial ring with centre Z a field, let D_0 be a *-closed division subring with c-valuation ω_0, which has same centre Z, and suppose that D_0', centralizer of D_0 in D, has a *-armature basis $\{e_\alpha\}$. Then the following requirements are equivalent.*

1. *ω_0 extends to a c-valuation of D.*
2. *Both* $\mathrm{ENCR}[e_\alpha]$ *and* $\mathrm{EFR}[e_\alpha]$ *hold true relative to* D_0.

2.3. *Construction of c-valuations.* As a first special construction of c-valued division ring take $D = F[a, b]$, where F is a formally real field and a, b are two anti-commuting square roots of -1. Let

$$\{e_\alpha\} = \{1, a, b, ab\}.$$

For the unique involution of D with fixed set F at which thus $a = -a^*$, $b = -b^*$ it is clear that $\{e_\alpha\}$ is a *-armature basis. Let ω_0 be any formally real valuation of the field F. Since $e_\alpha = -e_\alpha^*$ if $\alpha \neq 0$, $\mathrm{NCR}[e_\alpha]$ is vacuously satisfied. Since $e_\alpha e_\alpha^* = 1$, $\mathrm{EFR}[e_\alpha]$ holds trivially true. Thus ω_0 extends to the c-valuation ω with value group

$$G = G_0\left[\frac{1}{2}\omega_0(e_\alpha e_\alpha^*)\right] = G_0.$$

Here,

$$\omega(x) = \frac{1}{2}\omega(xx^*).$$

In effect, $xx^* \in Z(D)$, for all $x \in D$. This is the

THEOREM 2.3.1. *The classical normal quaternionic division ring $D = F[a, b]$, where a and b are anti-commuting roots of -1, and F is a formally real field with formally real valuation ω_0, carries an extended c-valuation ω with full residue degree* 4, *and, hence, same value group G_0. Here, the residue division ring is $\bar{F}[a, b]$, where $\bar{F}$ is the residue field of F.*

For the next constructions of c-valued divisions rings I will assume familiarity of the reader with the notion of Hilbert division ring $D\langle t; \varphi\rangle$, where D is any division ring, t is an indeterminate, and the period of φ modulo the normal subgroup of inner automorphisms of D is r, with φ^r induced by an element $u_0 \in D$ fixed by φ and with $u_0 = v_0^r$ for v_0 fixed by φ. Then

$$Z(D\langle t; \varphi\rangle) = Z^{(\varphi)}\langle (v_0 t)\rangle$$

where $Z^{(\varphi)}$ is the fixed subfield of $Z(D)$ relative to φ.

We begin with $D_1 = D\langle x_1\rangle$ (i.e., x_1 is a central indeterminate) and choose the automorphism φ_1 of D_1 at which $\varphi_1(x_1) = -x_1$, and $\varphi_1(a) = a$, for all $a \in D$. We pass to

$$D^{(1)} = D_1\langle y_1;\ \varphi_1\rangle,$$

which we write $D\langle\ (x_1;\ y_1)\ \rangle$.

Here,

$$Z(D^{(1)}) = Z(D)\langle x_1^2\rangle\langle y_1^2\rangle$$

and $D^{(1)}$ has the armature basis $\{1, x_1, y_1, x_1y_1\}$ over $D\langle x_1^2, y_1^2\rangle$. We define inductively

$$\begin{aligned} D^{(n)} &= D_{n-1}\langle y_i;\ \varphi_i\rangle \\ &= D^{(n-1)}\langle\ (x_i;\ y_i)\ \rangle. \end{aligned}$$

We also define by a limiting process D^{∞}.

THEOREM 2.3.2. *Let D be any finite dimensional c-valued division ring, let $D^{(n)}$ be the iterated Hilbert division ring with 2n indeterminates*

$$(x_i;\ y_i)y_ix = -x_iy_i, \quad x_iy_j = y_jx_i \quad \text{and} \quad x_ix_j = x_jx_i \quad \text{for } i \neq j,$$

all x_i, y_i commute with all of D), and let $D^{(\infty)} = \cup D^{(n)}$. Then

1. *$D^{(n)}$ carries an extended involution at which $x_i = \pm x_i^*$, $y_i = \pm y_i^*$ (at will).*
2. *$D^{(n)}$ has the *-armature basis*

$$\{e_\alpha\} = \prod\{1, x_i, y_i, x_iy_i\} \quad \text{over } D\langle x_2^1, y_1^2, \dots, x_n^2, y_n^2\rangle$$

3. *Both* EFR$[e_\alpha]$ *and* ENCR$[e_\alpha]$ *hold true relative to the standard extension $\omega_{(n)}$ of ω to $D\langle x_1^2, \dots, y_n^2\rangle$ so that $\omega_{(n)}$ extends to the c-valuation $\omega^{(n)}$ of $D^{(n)}$*
4. *$Z(D^{(n)}) = Z(D)\langle x_1^2, \dots, y_n^2\rangle$)*
5. *For $[D{:}Z] < \infty$, we have*

$$e_{\omega^{(n)}} = 4^n e_\omega; \quad f_{\omega^{(n)}} = f_\omega.$$

and, hence, $[D^{(n)}{:}Z^{(n)}] = 4^n[D{:}Z]$.

Proof. The standard extension is the valuation with value group the lexicographic product $G_\omega \times 2\mathbf{Z} \times 2\mathbf{Z}$ at which

$$\omega_{(1)}(x_1^2) = (0, 2, 0), \quad \text{and} \quad \omega_1(y_1^2) = (0, 0, 2).$$

Here

$$\omega_{(1)}(a) = (\omega(a), 0, 0), \quad \text{for all } a \in D.$$

Generally, $\omega_{(n)}$ has value group $G_\omega \times 2\mathbf{Z} \times \dots \times 2\mathbf{Z}$, while $\omega^{(n)}$ has value group $G_\omega \times \mathbf{Z} \times \dots \times \mathbf{Z}$ ($\omega^{(n)}$ is also the standard extension of ω to all of $D^{(n)}$). It is now fairly easy to verify 1 to 5 making use of Theorem 2.2.

If instead of $D^{(n)}$ we take $D^{(\infty)}$ then all properties 1 to 4 in the preceding theorem remain valid. Of course, $D^{(\infty)}$ is locally finite, $f_{\omega(\infty)} = f_\omega$, and $e_{\omega(\infty)} = \infty$.

As a special case of the preceding theorem we can take D to be a field F with trivial involution and trivial valuation, where we assume F to be formally real. In that case $F^{(n)}$ is then totally ramified of dimension 4^n. On the other hand, if we apply the construction to the classical division ring $D = F[a, b]$ described in Theorem 2.3.1, then $D^{(n)}$ has dimension 4^{n+1} with full residue degree 4 and ramification index 4^n.

I am left with one more type of example; namely, a c-valued division ring with residue degree 2, which is not normal quaternionic. A prospective example was given by Holland. Here is essentially his example.

Start with any formally real field F with formally real valuation ω_F. Pass to the complexification $F[\sqrt{-1}]$ of F and extend ω_F to $F[\sqrt{-1}]$ in the usual way:

$$\omega_F(a + b\sqrt{-1}) = \frac{1}{2}\omega_F(a^2 + b^2).$$

Let φ be the Galois automorphism of $F[\sqrt{-1}]$ over F and let

$$D = F[\sqrt{-1}]\langle t; \varphi\rangle.$$

Here, $Z = Z(D) = F\langle t^2\rangle$ and $\{1, \sqrt{-1}, t, \sqrt{-1}t\} = \{e_\alpha\}$ is a $*$-armature basis of D over Z, where the involution (of the first kind) extends that of $F[\sqrt{-1}]$, and is such that $t = t^*$. Then ω_F extends in a standard way to $Z = F\langle t^2\rangle$ with value group $G_F \times 2\mathbf{Z}$. We obviously have ENCR$[e_\alpha]$ as well as EFR$[e_\alpha]$ relative to the extension of ω_F to Z. Thus ω_F extends to a c-valuation with value group $G_F \times \mathbf{Z}$. Here, $e_\omega = f_\omega - 2$, and $\bar{D} = \bar{F}[\sqrt{-1}]$, the complexification of the residue field of F. Hence the

THEOREM 2.3.3. *Let $D = F[\sqrt{-1}]\langle t; \varphi\rangle$ be the Hilbert division ring over the complexification of a formally real field F, relative to the Galois automorphism of $F[\sqrt{-1}]$ over F. Let ω_F be any formally real valuation of F, and let $D^{(n)}$ be the iterated Hilbert division ring as in the preceding theorem (or, even, $D^{(\infty)}$). Then ω_F extends to a c-valuation of $D^{(n)}$ of residue degree 2 (and $\overline{D^{(n)}} = \overline{D^{\infty}} = \bar{F}[\sqrt{-1}]$, the complexification of the residue field of $\bar{F}$.*

Finally, other types of construction of c-valued division rings shall be offered in the referred joint work with Professor A. Wadsworth. And lastly, there is a cheap way to make the forgoing constructions with involutions of the second kinds without disturbing their nature. Thus if D is any c-valued division ring, so is the Hilbert division ring $D\langle t\rangle$, where we declare the indeterminate t a central skew symmetric. Then ω extends in a standard way to a c-valuation with same residue degree and same ramification index, and, hence,

$$\overline{D\langle t\rangle} = \bar{D}.$$

3. C-ordered division rings: new results. In this concluding section, I assume that the reader is fairly familiar with part of my work [**2**] and, mostly, with the results of Sections 1 and 2. In 3.1, relationship between normal c-ordering and c-valuation is established. In 3.2, I will investigate the centre of a c-ordering and in 3.3, solutions to the problem of extending a field ordering to a normal c-ordering are given with examples.

3.1. *The order valuation of a normal c-ordering.* First, let me review some of the correct results in [**2**] about a general c-ordering.

Fact 3.1.1. *The subset R of all elements $a \in D$ such that $aa^* < n$ for some natural number n is a $*$-closed valuation ring in D.*

Fact 3.1.2. (i) *The unique maximal ring ideal and 2-sided ideal J of R is the subset of infinitesimals j; that is, $jj^* < 1/n$ for every natural number n.*

(ii) *The group of invertible elements U of the ring R is the subset of elements u such that $1/n < uu^* < n$ for some natural number n.*

(iii) $\bar{D} = R/J$ *has all its symmetrics central in* $\bar{D}$.

Fact 3.1.3. *Let $M = \{a \in D | a = a^* > 0\}$ and let $\bar{M} = (M \cap U)/J$. For $\bar{a}, \bar{b} \in \bar{D} = R/J$ write $\bar{a} > \bar{b}$ if and only if $\bar{a} - \bar{b} \in \bar{M}$. This is an archimedean field ordering of the subfield of symmetrics of $\bar{D}$ at which $\bar{a}\bar{a}^* > 0$ for all $\bar{a} \in \bar{D}$, $\bar{a} \neq 0$.*

Next, let me answer the general question to know whether R induces a c-valuation of D. The valuation in question ω is the canonical valuation with valuation ring R. Its existence is then ensured by R is preserved under conjugation in D. As for its desired type; namely, ω is a c-valuation, we have something in this direction:

Fact 3.1.3′. (i) *If R is preserved under conjugation in D then, in fact, for every $a \in D^\bullet$, $a^{-1}a^*$ is in U. Equivalently, $\omega(a^*) = \omega(a)$.*

(ii) *The residue division ring $\bar{D}$ is evidently $*$-formally real and, consequently, ω is a $*$-formally real valuation.*

Proof. (i) We pass from the element $a^{-1}a^*$ to its inverse $a^{*-1}a$ by conjugation by a followed by taking the involution. From this $a^{-1}a^*$ must be in U.

(ii) This is evident.

Fact 3.1.3′ tells us that we can say that our c-ordered division ring D has a $*$-formally real order valuation. I proceed to an example of c-ordered division ring D with order valuation definitely not a c-valuation.

THEOREM 3.1.4. *There is a 4-dimensional c-ordered division ring D with order valuation ω not a c-valuation.*

Proof. Let $F = \mathbf{Q}[\sqrt{2}]$ and let $D = F\langle t\rangle$ be the Hilbert division ring over F where the indeterminate t is subjected to the commutation rule

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

$$t(a + b\sqrt{2}) = (a - b\sqrt{2})t(a, b \in \mathbf{Q}).$$

Clearly D is a 4-dimensional division ring with centre $Z = \mathbf{Q}\langle t^2\rangle$ and generators $\sqrt{2}$ and t over Z. Here, D has the discrete valuation ω which assigns to the Laurent power series

$$\sum_{i \geqq n_0} a_i t^i$$

its "value" n_0, where we assume that all $a_i \in F$ and $a_{n_0} \neq 0$.

We turn D to an involutorial division ring by declaring $\sqrt{2}$ and t both symmetrics. Here,

$$(\sum a_i t^i)^* = \sum \bar{a}_i^{(i)} t^i,$$

where

$$\overline{a + \sqrt{2b}} = a - \sqrt{2}b \quad (a, b \in \mathbf{Q}).$$

A general series $a = \sum a_i t^i$ gives rise to the norm

$$aa^* = \sum a_i t^i t^j a_j = a_n^2 t^{2n} + \dots \quad (a_j \in F).$$

Any monomial in norms is of the form

$$c = b_{2^m}^2 t^{2m} + \dots \quad (b_{2m} \in F).$$

If C is the *-core of D and if C_1 is the set of sums of the form $\sum z_i c_i$, where $c_i \in C$ and $z_i \in Z = Q\langle t^2\rangle$ has a positive lowest rational coefficient then $0 \notin C_1$. By a version of [**2**, Theorem 2] there is a c-ordering of D such that if $x > 0$ in D then $xc > 0$, where $c \in C_1$. In particular, t^2 is an infinitesimal at the c-ordering and, hence, t is an infinitesimal. From this $F[[t]] \subset R$, the order subring of the c-ordering. Because $(\sqrt{2})^2 = 2$ it follows that $\sqrt{2} \in R$. Thus $F[[t]][t, \sqrt{2}]$, the subring of D generated by t and $\sqrt{2}$ over $F[[t]]$ is contained in R. However, $F[[t]][t, \sqrt{2}]$ is the valuation ring of the discrete valuation ω. By maximality follows

$$R = F[[t]][t, \sqrt{2}].$$

Hence, the c-ordering has order valuation ω. However, we have $\sqrt{2}$ is a symmetric valuation unit with $t\sqrt{2} = -\sqrt{2}t$. Therefore ω is not a c-valuation.

Theorem 3.1.4 shows that, [**2**, Theorem 7] is wrong even when the residue division ring $\bar{D}$ is a field. In effect, there is an error of sign that occurs in the course of the proof of Lemma 5 of the cited paper in page 509, line 8 from the bottom. Actually, even the asserted fact that the order subring R is preserved under conjugation cannot be salvaged from the proof as given. There follows a necessary and sufficient condition for this preservation requirement to hold true. This is the

THEOREM 3.1.5. *For any $*$-closed valuation ring R in D with 2 invertible in R to be preserved under conjugation it is necessary and sufficient that for all pairs $s = s^*$ and $k = -k^*$ follows $sk \notin 1 + J$ or, equivalently, $ks \notin 1 + J$.*

Proof. For if R is preserved under conjugation but $sk - 1 \in J$ then

$$-ks - 1 = (sk - 1)^* \in J.$$

Also

$$ks - 1 = s^{-1}(sk - 1)s \in J.$$

Thus

$$-2 = (-ks - 1) + (ks - 1) \in J,$$

which is ruled out.

Conversely, let $s = \pm s^*$ and $d = -d^*$ be in $D^\bullet$. Let $g = d^{-1}s^{-1}ds$. If $g \in J$ then $g \neq \pm 1$, and, hence, $sd - ds \neq 0$ and $sd + ds \neq 0$. Now

$$\begin{aligned}&(sd - ds)^{-1}(sd + ds)\\ &= (sd(1 - d^{-1}s^{-1}ds))^{-1}(sd(1 + d^{-1}s^{-1}ds))\\ &= (sd(1 - g))^{-1}(sd(1 + g)) = (1 - g)^{-1}(sd)^{-1}(sd)(1 + g)\\ &= (1 - g)^{-1}(1 + g).\end{aligned}$$

Because $g \in J$ we have $(1 - g)^{-1}(1 + g) - 1 \in J$ and, hence,

$$(sd - ds)^{-1}(sd + ds) - 1 \in J,$$

which obviously contradicts the hypothesis. This shows that $g \notin J$. Similarly $g^{-1} \notin J$ and, hence, $g \in U$, the group of invertible elements of R.

From this for every $s = \pm s^* \in D^\bullet$, we have $s^{-1}us \in U$, for all $u \in U$. Thus $s^{-1}Rs \subset R$. For general $x \in D^\bullet$, if $x^*x^{-1} \in J$ then

$$\begin{aligned}&(x + x^*)^{-1}(x^*x^{-1})(x + x^*)\\ &= ((1 + x^*x^{-1})x)^{-1}(x^*x^{-1})(1 + x^*x^{-1})x\\ &= x^{-1}(x^*x^{-1})x = x^{-1}x^* \in J.\end{aligned}$$

Thus

$$(x^{-1}x^*)^* = xx^{*-1} = (x^*x^{-1})^{-1} \in J;$$

but we have $x^*x^{-1} \in J$. This shows that $x^*x^{-1} \notin J$. Similarly, we can show that $(x^*x^{-1})^{-1} \notin J$ giving $x^*x^{-1} \in U$, for all $x \in D^\bullet$. By an observation of Holland [**4**], since every commutator $aba^{-1}b^{-1}$ is a product of elements of the form x^*x^{-1} it follows that $aba^{-1}b^{-1} \in R$, and R is preserved under conjugation.

THEOREM 3.1.6. *Let D be any c-ordered division ring. The following requirements are equivalent*:

1. *The order is compatible*; *that is, we have the Axiom* (06) (*Compatibility Axiom*). *If* $s = s^* > 0$ *and if* $j \in J$, *then*

$$s(1 + j) + (1 + j^*)s > 0.$$

2. *The order valuation* ω *exists and* ω *is order compatible*; *that is, for all symmetrics* s *and* d *in* D:

$$s > d > 0 \Rightarrow \omega(s) \leqq \omega(d).$$

Proof. 1 implies 2. For if $s = s^*$ and $k = -k^*$ were such that $ks - 1 \in J$ then there is $j \in J$ with $k = (1 + j)s^{-1}$. If, say, $s^{-1} > 0$ then

$$0 = k + k^* = (1 + j)s^{-1} + s^{-1}(1 + j^*);$$

but,

$$s^{-1}(1 + j^*) + (1 + j)s^{-1} > 0.$$

Similarly, if $s^{-1} < 0$ then

$$-k = (1 + j)(-s^{-1})$$

gives

$$0 = (-s^{-1}(1 + j^*) + (1 + j)(-s^{-1}) > 0.$$

This shows that $ks - 1 \notin J$. Since 2 is evidently invertible in R we get that R is preserved under conjugation in D. If ω is the canonical valuation with valuation ring R we know from Fact 3.1 that ω is a *-formally real valuation. Finally, if $s = s^* > d = d^* > 0$ but $\omega(s) > \omega(d)$ then there is $j \in J$ with $dj = s > d$ giving

$$d(j - 1) + (j^* - 1)d = 2(s - d) > 0.$$

However, by hypothesis $d > 0$ implies

$$d(1 - j) + (1 - j^*)d > 0,$$

which contradicts the preceding inequality.

2 implies 1. This is straightforward.

Conversely, if we are granted that the order valuation ω of a c-ordered division ring exists and ω is order compatible then evidently Axiom (06) holds true. One might ask whether this axiom will render the valuation ω a c-valuation? In effect, the example found in Theorem 3.1.4 yields an order compatible order valuation so, the answer to the question is definitely in the negative. As a matter of fact, later on in Theorem 3.2.7, I will establish that the order valuation if it is to be real (i.e., archimedean value group) it must be order compatible, in particular for a discrete order valuation (as in Theorem 3.1.4).

On the positive side there are partial results true for general c-ordering (no need for normality) to be established in the next theorems. They flow from the following fairly precise result, which is definitely not purely algebraic.

THEOREM 3.1.7. *Let $u \in R$ be such that $uu^* - 1 \in J$ (unitary modulo J). If*

$$u^{2^n} + u^{*2^n} \geqq 0 \quad \textit{for all integers } n = 0, 1, 2, \dots$$

then $u - 1 \in J$.

Proof. Put

$$u_n = u^{2^n} = a_n + b_n$$

where $a_n a_n^*$ and $b_n = -b_n^*$. Clearly

$$u_{n+1} = u_n^2 = (a_n^2 + b_n^2) + (a_n b_n + b_n a_n)$$
$$= a_{n+1} + b_{n+1}.$$

From $u_0 + u_0^* \geqq 0$ follows $a_0 = a_0^* > 0$. Since

$$u_0 u_0^* = (a_0 + b_0)(a_0 - b_0) = (a_0^2 - b_0^2) + (b_0 a_0 - a_0 b_0)$$
$$\equiv 1 \pmod J$$

it follows that

$$a_0^2 - b_0^2 \equiv 1 \pmod J.$$

If $b_0^2 \in J$ then $b_0 \in J$ giving $u_0^2 \equiv u_0 u_0^* \equiv 1$ and, hence, $u_0 \equiv \pm 1$. Because $a_0 > 0$ we must have $u_0 \equiv a_0 \equiv 1$. Assume to the contrary that $b_0^2 \notin J$. From

$$a_0^2 - b_0^2 \equiv 1 \pmod J$$

and $-b_0^2 \in U$ is positive follows $a_0^2 < q \leqq 1$ for some positive rational q. Then

$$u_1 = (a_0^2 + b_0^2) + (a_0 b_0 + b_0 a_0) = a_1 + b_1$$

has symmetric part $a_1 = a_1 a_0^2 + b_0^2 < q$. Thus $u_2 = a_2 + b_2$ has symmetric part $a_2 < q^2$. By induction n, we get that u_n has symmetric part $a_n < q^n$. Hence, for n large enough $a_n < 1/2$.

However, since $uu^* \equiv 1$ we get $u_n u_n^* \equiv 1$. Thus $a_n^2 - b_n^2 \equiv 1$. From $u_{n+1} + u_{n+1}^* \geqq 0$ follows

$$a_n^2 + b_n^2 \geqq 0 \quad \text{or} \quad a_n^2 \geqq -b_n^2.$$

By the previous congruence relation follows thus $a_n^2 \geqq 1/2$, which contradicts the inequality $a_n < 1/2$.

THEOREM 3.1.8. *Let D be any c-ordered division ring with order subring R. Then*

1. *For every* $c \in C$ *we have* $c^*c^{-1} - 1 \in J$.

2. *For every* $c \in C$ *we have* $cRc^{-1} \subset R$. *In fact*, $cuc^{-1} - u \in J$ *for all* $u \in U$.

Proof. 1. Let $u = c^{-1*}cc^*c^{-1}$. Then $u = u^* \in C$ and, hence, $u > 0$. If $q > 1$ is a rational such that $u > q$ we get $cc^* > qc^*c$. Then

$$c > qc^*c(c^*)^{-1}.$$

Then $c^* > qcc^*c^{-1}$ and so,

$$c^*c > qcc^* > q^2c^*c$$

giving

$$(1 - q^2)cc^* > 0;$$

but, $cc^* > 0$ and $1 - q^2 < 0$. This shows that $u < q$ for all rationals $q > 1$. By symmetry $u^{-1} < q$ for all such rationals. Therefore $u - 1 \in J$. If $v = c^{-1*}c$ then

$$vv^* = u \equiv 1 \pmod J.$$

Because $v \in C$ we have

$$v^{2^n} + v^{*2^n} > 0.$$

By the preceding theorem $v \equiv 1 \pmod J$ and, hence, $c^*c^{-1} \equiv 1 \pmod J$ or $c^*c^{-1} - 1 \in J$.

2. Let $u = u^* \in U$ and let $c \in P$. Then $cu^2 \in C$. By 1,

$$(cu^2)^{-1}(cu^2)^* = u^{-2}c^{-1}u^2c \equiv 1 \pmod J.$$

Since $u \in U$ we get

$$c^{-1}u^2c \equiv u^2 \pmod J.$$

Replacing u by $1 + u$ and eliminating gives

$$c^{-1}uc \equiv u \pmod J.$$

For general $s = s^* \in R$ if $s \in J$ then $1 + s = (1 + s)^* \in U$. By the preceding

$$1 + s \equiv 1 + c^{-1}sc \quad \text{or} \quad c^{-1}sc \in J.$$

For $k = -k^* \in R$ if $k \in U$ then $-k^2 \in U$ is symmetric. Thus $c^{-1}k^2c \equiv k^2$. Hence, $ckc^{-1} \in U$. If, on the other hand, $k \in J$ then $k^2 \in J$ giving $c^{-1}k^2c \in J$ and, hence, $c^{-1}kc \in J$.

Because $c^{-1}sc \in R$ and $c^{-1}kc \in R$ for all $s = s^*$, $k = -k^* \in R$ we get $c^{-1}Rc \subset R$. It remains to show that

$$c^{-1}uc \equiv u \pmod J \quad \text{for all } u \in U.$$

As $c^{-1}su \equiv s \pmod{J}$ for all $s = s^* \in R$ it suffices to show that

$$c^{-1}kc \equiv k \pmod{J} \quad \text{for all } k = -k^* \in U.$$

Let $v = ckc^{-1}k^{-1}$. Then $v \in U$. Because the residue division ring D has all its symmetrics central it follows that

$$vv^* \equiv v^*v \pmod{J}.$$

From this v^*v^{-1} is a unitary modulo J. Hence, $v^*v^{-1} - 1 \in J$ or $v^* \equiv v$. Thus

$$k^{-1}c^{-1}kc = v^* \equiv v.$$

Since $v \equiv v^*$ and since $k \in U$ we have

$$v \equiv k^{-1}vk = k^{-1}ckc^{-1}.$$

Thus

$$k^{-1}c^{-1}kc \equiv k^{-1}ckc^{-1}$$

and so,

$$c^{-1}kc \equiv ckc^{-1}.$$

From this

$$(ckc^{-1})k^{-1} \equiv k^{-1}(ckc^{-1}) \equiv k^{-1}(c^{-1}kc).$$

Accordingly

$$\begin{aligned}(ckc^{-1}k^{-1})^2 &= ((ckc^{-1})k^{-1})^2 \\ &\equiv (ck^2c^{-1})k^{-2} \\ &\equiv 1.\end{aligned}$$

Thus $ckc^{-1}k^{-1} \equiv \pm 1$. From $ckc^{-1}k^{-1} \in C$, we derive that $ckc^{-1}kk^{-1} \equiv 1$.

THEOREM 3.1.9. *For any c-ordered division ring D the following requirements are equivalent*:

1. *The normality Axiom* (05) *holds true*: (*i.e., if $s = s^* \in D^\bullet$ and $u = u^*$ is such that $q_1 > u > q_2$ for some positive rationales q_1 and q_2 then* $sus > 0$).

2. *The order valuation ω exists and ω is a c-valuation.*

Proof. Assume the assertion in 2. For s and u as in the statement if $sus < 0$ then

$$sus^{-1} = (sus)s^{-2} < 0.$$

Also, $s^{-1}us < 0$. Thus

$$sus^{-1} + s^{-1}us < 0.$$

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

Now, since ω is a c-valuation we have

$$sus^{-1} = u(1 + j) \quad \text{for some } j \in J.$$

Thus

$$u(1 + j) + (1 + j^*)u < 0 \quad \text{or } -(uj + j^*uj) > 2u > 2q_2.$$

However, from $u \in U$ and $j \in J$ follows $-(uj + j^*u) \in J$, a contradiction.

Assume the assertion in 1. Let $u = u^* \in U$ and let

$$v = \frac{1}{1 + u^2}.$$

Then $v \in U$ and

$$vu^2 + v = \frac{u^2}{1 + u^2} + \frac{1}{1 + u^2} = 1.$$

For $s = s^* \in D^\bullet$ we have

$$s^{-1}vs + s^{-1}(vu^2)s = 1.$$

Since $vu^2 \in C$ it follows that $s^{-1}(vu^2)s > 0$ giving

$$s^{-1}vs = 1 - s^{-1}(vu^2)s < 1.$$

From this

$$(s^{-1}vs)(svs^{-1}) < 1 \text{ for } s^{-1}vs \in C.$$

Hence $s^{-1}vs \in R$. Thus $s^{-1}vs + svs^{-1} \in R$. Let

$$t = 2v - (s^{-1}vs + svs^{-1}).$$

If $t \in U$ and $t > 0$ then $t > q$ for some positive rational q. Thus $sts^{-1} > q_1$ for every positive rational $q_1 < q$ for then $t - q_1$ remains a positive symmetric in U so that $s(t - q_1)s > 0$ or $s(t - q_1)s^{-1} > 0$; $sts^{-1} > q_1$. Thus

$$sts^{-1} = 2svs^{-1} - (b + s^2vs^{-2}) > q_1.$$

Now by the preceding theorem $s^2vs^{-2} - v \in J$. It follows that

$$2svs^{-1} - 2v > q_2$$

for every positive rational q_2 less than q_1. Hence

$$2(svs^{-1} + s^{-1}vs) - 4v > 2q_2)$$

resulting in $-t > q_2$, which is impossible as $t > 0$. This shows that $t > 0$ implies

$$t \in J \quad \text{or } 2v - (s^{-1}vs + svs^{-1}) \in J.$$

By symmetry, if $t < 0$ then $t \in J$ as well or

$$2v \equiv s^{-1}vs + svs^{-1}.$$

Now $s^{-1}vs - svs^{-1} \in J$ for $s^{-2}vs^2 - v \in J$. Thus

$$2v \equiv 2s^{-1}vs \quad \text{or } v \equiv s^{-1}vs \pmod{J}.$$

Because $v = 1/(1 + u^2)$ is invertible in R we get

$$s^{-1}(1 + u^2)s \equiv 1 + u^2$$

and, hence,

$$s^{-1}u^2s \equiv u^2 \pmod{J}.$$

Replacing u by $1 + u$ and eliminating gives

$$s^{-1}us \equiv u \pmod{J}$$

for all $s = s^*$ and $u = u^* \in U$.

I proceed to show that for all $x \in D^\bullet$, $x^{-1}x^*$ is a unitary modulo J. To begin with if s and k are such that $ks - u \in J$ with $u = u^* \in U$ then we would get $ks - sk - 2u = j$ for some $j \in J$. Thus

$$\begin{aligned} 2u + j &\equiv s^{-1}(2u + j)s = s^{-1}(ks)s - s^{-1}(sk)s \\ &= s^{-1}(ks)s - ks. \end{aligned}$$

Because $ks - u \in J$ we have $ks \in U$. Since $s^{-1}xs \equiv sxs^{-1}$ for all $x \in U$ we have

$$s^{-1}(ks)s \equiv s(ks)s^{-1} \equiv sk.$$

Thus

$$sk - ks \equiv 2u + j = ks - sk,$$

a contradiction. This shows that $ks - u \notin J$ for all $k = -k^*$ and $s = s^*$. From this the valuation ring R is certainly preserved under conjugation. Thus $x^{-1}x^*$ is an invertible R. Put $x = s + k$, where $s = s^*$ and $k = -k^*$. If $k^{-1}s = u$ with $u \in U$ but $u \not\equiv -u^*$ then

$$k^{-1}s - sk^{-1} = u + u^* \in U.$$

As in the preceding we can dismiss this situation. Thus $u \equiv -u^*$. Now

$$\begin{aligned} x^{-1}x^* &= x(s + k)^{-1}(s - k) \\ &= (k(k^{-1}s + 1))^{-1}k(k^{-1}s - 1) \\ &= (k^{-1}s + 1)^{-1}(k^{-1}s - 1). \end{aligned}$$

Because $u = k^{-1}s \equiv -u^*$ it follows that $x^{-1}x^*$ maps onto a unitary in $\bar{D}$.

If, on the other hand, $k^{-1}s \notin U$ then for $k^{-1}s \in J$ we get

$$x = s + k = k(k^{-1}s + 1) \equiv k$$

so that certainly $x^*x^{-1} \equiv -1$ is a unitary modulo J. For $s^{-1}k \in J$ we get

$$x \equiv s(1 + s^{-1}k) \equiv s$$

and, again, $x^*x^{-1} \equiv 1$ is a unitary modulo J.

Having shown that x^*x^{-1} is always a unitary modulo J, we derive that all commutators are unitaries modulo J. For $c \in C$ and $x \in D^{\bullet}$, $cxc^{-1}x^{-1}$ is thus a unitary modulo J. In view of Theorem 3.1, $cxc^{-1}x^{-1} \equiv 1$ and ω is a c-valuation.

THEOREM 3.1.11. *Conversely, if ω is a c-valuation of D then ω can be realized as the order valuation of some normal c-ordering if, and only if, the residue division ring $\bar{D}$ carries some archimedean c-ordering.*

Proof. For the "only if" part observe that in $\bar{D}$ we have an archimedean field ordering of the subfield $\bar{S}$ of symmetrics of $\bar{D}$ at which $\bar{x}\bar{x}^* > 0$ for all $\bar{x} \in \bar{D}$, $\bar{x} \neq 0$. This is evidently an archimedean c-ordering.

For the "if" part the restricted c-ordering to $\bar{S}$ is an archimedean q-ordering (in the sense given by Prestel) and, hence, by Prestel's **[9]** this is a field ordering of $\bar{S}$, which is evidently an archimedean c-ordering of $\bar{D}$. Let

$$P_1 = \{a = a^* \in U, a + J > o \text{ in } \bar{D}\}$$

and let $P = \{a + j, j \in J\}$. By construction, P is closed under sums, products, taking inverse, and taking (*). Also, because every $a = a^* \in U$ is residually central we get that P is preserved under conjugation in D. In short P behaves as the *-core C of D and then

$$P.C = \{\sum p_i c_i | p_i \in P, c_i \in C\}$$

inherits all properties of C but possibly $0 \notin P.C$. A general element p_i in C is of the form

$$p_i = (\sum x_{ij}x_{ij}^*)(1 + \epsilon_i), \quad \text{where } \epsilon_i \in J.$$

Moreover, we can express p_i in the form

$$p_i = y_i y_i^*(1 + b_i),$$

where $v_i \equiv v_i^* \in R$. Thus

$$p_i c_i = y y_i y_i^*(1 + v_i)d_i,\ d_i = (1 + \epsilon_i)c_i \in P.$$

Hence, a general element of $P.C$ is of the form

$$x = \sum y_i y_i^*(1 + v_i)d_i.$$

If, say, $\omega(y_1 y_1^*) \leqq \omega(y_i y_i^*)$ for all i, then

$$(y_i y_i^*)^{-1}x = (1 + v_i)d_1 + \sum_{i \pm 1} z_i z_i^*(1 + v_i)d_i.$$

By construction, $v_i > 0$ in D so that $1 + v_i > 0$ in $\bar{D}$. Thus $(y_1 y_1^*)^{-1}x$ maps onto a positive element in $\bar{D}$. Therefore $x \neq 0$.

Put $C_1 = C.P$. I proceed to show that C_1 enlarges to a domain of positivity of some c-ordering of D with order valuation precisely ω. For let M be any subset of D which is maximal for the conjunction of the following requirements: (i) $0 \notin M$, (ii) $M = M^*$, (iii) $M + M \subset M$, (iv) $C_1 \subset M$, and (v) $C_1 M \subset M$. By a trivial version of [**1**, Theorem 2, p. 499] follows that if we declare $a > b$ in D if and only if $a - b \in M$ then we get a c-ordering of D. It remains to show that the order valuation associated to this c-ordering is equivalent with the initial c-valuation ω. This reduces to establishing that for each u in D we have $\omega(u) = 0$ if and only if uu^* can be bounded below and above by rationals. Now, $\omega(u) = 0$ implies $\omega(uu^*) = 0$ so that uu^* maps onto a norm in $\bar{D}$. By construction, $uu^* + J > q + J$ in $\bar{D}$ for some positive rational. Because $M \supset C \supset \pi$ it follows that $uu^* - q \in M$ so that $uu^* > q$ in D. Also, since the residue ordering in $\bar{D}$ is archimedean a similar argument shows that $uu^* < q'$ in D for some rational q', as desired. Conversely, if $q < uu^* < q'$ for some positive rationals we are to show that $\omega(u) = 0$. Because $uu^* > q$ it follows that $\omega(u) \not> 0$. For, otherwise, since $q > 0$ in $\bar{D}$ and since $q - uu^*$ maps onto q in $\bar{D}$ we would get $q - uu^* \in \pi \subset M$ or $q > uu^*$, a contradiction. Because $q' > uu^*$ it follows that $\omega(u) \not< 0$. Therefore $\omega(u) = 0$, which concludes the proof.

3.2. *The centre of a normal c-ordering.* A central feature for a c-ordered division ring D lies in the fact that every norm $aa^* \in D^*$ is order multipliable according to the

Definition 3.2.1. (1) Call $a \in D$ *order multipliable* if $b > 0$ implies $ba > 0$ and $ab > 0$.

(2) Let $\hat{C}$ stand for the subset of all elements a, which are order multipliable. Refer to $\hat{C}$ as the *centre* of the c-ordering.

THEOREM 3.2.2. *The centre $\hat{C}$ of any c-ordering (possibly non normal) has the following properties*:

1. $C \subset \hat{C} \subset M = \{x \in D | x > 0\}$.
2. *$\hat{C}$ is a $*$-closed multiplicative subgroup of D, all of whose elements c are such that* $c^*c^{-1} - 1 \in J$.
3. *$\hat{C}$ is closed under sums.*
4. *For all a, b $\in \hat{C}$ we have* $\omega(a + b) = \min\{\omega(a), \omega(b)\}$.

Proof. 1 and 3. It is fairly straightforward to show that $\hat{C}$ is closed under sums and products. Since $xx^* \in \hat{C}$ for all $x \in D^\bullet$ it follows that $C \subset \hat{C}$. Evidently $\hat{C} \subset M$.

2. If $a \in \hat{C}$ and $b \in M$ then since $b^* \in M$ we get $ab^* \in M$ and $b^*a \in M$ and, hence, ba^* and $a^*b \in M$ giving $a^* \in \hat{C}$. Because $(a^*a)^{-1} \in M$ we obtain $a(a^*a)^{-1} \in \hat{C}$ or $(a^*)^{-1} \in \hat{C}$ and, hence, $a^{-1} \in \hat{C}$. Also, as for the case $c \in C$ one can show that if $c \in \hat{C}$ then c^*c^{-1} maps onto a unitary in $\bar{D}$. In view of Theorem 3.1.7 one gets that $c^*c^{-1} - 1 \in J$.

4. For $a, b \in \hat{C}$ we have $a, b \in M$ and, hence, $a + b > b > 0$. For $c = a + b$ we have $c > b > 0$ with $c \in \hat{C}$. Then $c^* \in \hat{C}$, and, hence, $cc^* > bc^* > 0$. Since $c^* > b^*$ we have $bc^* > bb^*$ and, consequently, $cc^* > bb^*$. If now $\omega(a + b) = \omega(c) > \omega(b)$ then $c = bj$ for some $j \in J$. Thus $cc^* = bjj^*b^*$. Since $q - jj^* > 0$ for all positive rationals we get

$$b(q - jj^*)b^* > 0 \quad \text{or} \quad bjj^*b^* < qbb^*;$$

that is, $cc^* < qbb^*$, contrary to the relation $cc^* > bb^*$. This shows that

$$\omega(a + b) \leqq \omega(b).$$

Similarly

$$\omega(a + b) \leqq \omega(a).$$

Because

$$\omega(a + b) \geqq \min\{\omega(a), \omega(b)\}$$

we end up with the quality

$$\omega(a + b) = \min\{\omega(a), \omega(b)\},$$

as desired.

To strengthen the analogy between $\hat{C}$ and C let me establish the

THEOREM 3.2.3. *Let D be any c-ordered division ring. Then*:
1. *For every $c \in \hat{C}$ and $x \in R$ we have $cxc^{-1} - x \in J$*;
2. *If the c-ordering is a normal c-ordering and if the compatibility Axiom* (06) *as in Theorem* 3.1.6 *holds true then $\hat{C} \subset \hat{Z}$.*

Proof. 1. For let $x = x^* \in U$. Then $x^2c \in \hat{C}$. In view of the preceding theorem we get

$$(x^2c)^*(x^2c)^{-1} - 1 = c^*x^2c^{-1}x^{-2} - 1 \in J.$$

Because $c^*c^{-1} \in J$ we derive that

$$cx^2c^{-1}x^{-2} - 1 \in J.$$

As in an earlier situation this gives

$$cxc^{-1}x^{-1} - 1 \in J \quad \text{for all } x \in U.$$

2. We have

$$cc^* \equiv c^2 \pmod{1 + j}.$$

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

Since the c-ordering is normal it follows that $c^2 \in \hat{Z}$ and so, $c^2 \in \hat{Z}$. I contend that if $x \in D^\bullet$ is such that

$$[c, x] = cxc^{-1}x^{-1} \not\equiv 1 \pmod{J}$$

then, in fact,

$$[c, xc - cx] \equiv -1 \pmod{J}.$$

For clearly $xc - cx \in D^\bullet$. Now

$$c(xc - cx)c^{-1} = cx - c^2xc^{-1}.$$

Because $c^2 \in \hat{Z}$ we can write c^2x in the form $c^2x = (1 + j)xc^2$, where $j \in J$. Thus

$$c(xc - cx)c^{-1} = (x - (1 + j)xc)$$
$$= (cx - xc) - jxc.$$

Because

$$\omega(jxc) > \omega(xc) \quad \text{and}$$
$$\omega(cx - xc) = \omega(\,(cxc^{-1}x^{-1} - 1)xc) = \omega(xc)$$

we get that

$$(cx - xc) - jxc = (cx - xc)(1 + j'),$$

where $j' \in J$. Hence

$$c(xc - cx)x^{-1} \equiv -(xc - cx) \pmod{1 + j},$$

as desired.

If now $s = -s^* \in D^\bullet$ is such that

$$[c, s] = csc^{-1}s^{-1} \not\equiv 1 \pmod{J}$$

then $[c, cs - sc] \equiv -1$. Because $c^* \equiv c \pmod{1 + J}$,

$$[cs - sc^*] \equiv -1.$$

Hence for $d = d^* = cs - sc^*$ we have

$$cdc^{-1} = -(1 + j_2)d$$

where $j_2 \in J$. It follows that

$$cdc = -(1 + j_2)dc^2.$$

If $d > 0$ then so must be cdc and, hence,

$$-(1 + j_2)d > 0.$$

However, since $j_2 \in J$ we have

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

$$(1 + j_2)d + d(1 + j_2^*) > 0,$$

a contradiction to the assumption $[c, s] \not\equiv 1$.

On the other hand for $s = s^* \in D^\bullet$ if $[c, s] \not\equiv 1$, then

$$[c, cs - sc] \equiv -1;$$

but, since $c \equiv c^* \pmod{1 + j}$ we have

$$cs - sc \equiv -(cs - sc)^* \pmod{1 + j},$$

which contradicts the preceding.

All in all, $[c, x] \equiv 1 \pmod J$, whenever $x = x^*$ or $x = -x^*$. Since by 1, $[c, u] \equiv 1$ for all $u \in U$, we can conclude that $c \in \hat{Z}$ making use of the formal identities

$$x = \begin{cases} (1 + x^*x^{-1})^{-1}(x + x^*) & \text{for } x^*x^{-1} \not\equiv 1 \\ (1 - x^*x^{-1})^{-1}(x - x^*) & \text{for } x^*x^{-1} \not\equiv -1. \end{cases}$$

For a general c-ordering a further property of the *-core C lies in the fact that $xCx^* \subset C$ and, consequently, one can say that for all $c \in C$ we have $xcx^* > 0$. Can one say as well that $xcx^* > 0$ for all $c \in \hat{C}$? Under the hypothesis of the preceding theorem in part 2, we can write

$$xcx^* = xx^*vc(1 + j),$$

where $j \in J$. For $c = c^* \in C$ we have $xx^*c > 0$. Hence, if we assume $xcx^* < 0$ we get a contradiction. For general $c \in \hat{C}$ if we strengthen Axiom (06) by assuming $1 + J \subset \hat{C}$ a similar argument shows that $xcx^* \in C$ for all $x \in D^\bullet$. Better still, the preceding stronger axiom will force $\hat{C}$ to be preserved under conjugation in D and so,

$$xcx^* = xcx^{-1}(xx^*) \in \hat{C}.$$

Hence, the

THEOREM 3.2.4. (1) *If the c-ordering is a normal c-ordering such that the compatibility Axiom* (06) *holds true then for all* $c \in \hat{C}$ *and* $x \in D^\bullet$ *follows* $xcx^* > 0$.

(2) *If the c-ordering is a normal c-ordering and if Axiom* (06) *is strengthened to the extent* $1 + J \subset \hat{C}$ *then* $\hat{C}$ *becomes a normal subgroup such that* $xcx^* \in C$ *for all* $c \in \hat{C}$ *and* $x \in D^\bullet$.

As a last point of investigation for this part let me characterize the extreme case $\hat{C} = M$; i.e., M is multiplicatively closed. Recall that if, further, M happens to be a normal subgroup of D then the c-ordering was given the name of *strong ordering* in **[5]**.

THEOREM 3.2.5. *Any c-ordering of D which has a multiplicatively closed domain of positivity is such that*:

1. *The c-ordering is a normal c-ordering.*
2. *Axiom* (06) *holds true.*
3. *D carries a strong ordering.*
4. *D contains no algebraic elements over Z except when* $D = Z$ *or D is a normal quaternionic division ring.*

Proof. 1. Given $s = s^*$ and any $u = u^* > 0$ we have

$$sus = (-s)u(-s).$$

Thus $sus > 0$ and Axiom (05) holds evidently true.

2. If $s = s^* > d = d^* > 0$ then $s^2 > ds > d^2$. From this $\omega(s) \leqq \omega(d)$.

3. Let $\hat{M} = M(1 + J)$. It can be readily verified that the order relation defined by

$$a > b \Leftrightarrow a - b \in \hat{M}$$

is a strong ordering of D.

4. By Theorem 3.2.5 follows $\hat{C} = M \subset \hat{Z}$. Thus for all $s = s^* \in D^{\bullet}$ and $x \in D^{\bullet}$ we have

$$[s, x] = sxs^{-1}x^{-1} \equiv 1.$$

I proceed to show a more general result: Let ω be any *-valuation of D such that (1) $\omega(n) = 0$ for all integers $n \neq 0$, (2) $[a, b] \equiv 1$ for all $a = a^*$, $b = b^* \in D^{\bullet}$. Then except when $D = Z$ or D is a normal quaternionic division ring, D contains no algebraic elements over Z (other than the elements in Z).

Step 1. $a \in \hat{Z}$ *for all* $a^* \in D^{\bullet}$.

Subproof. It suffices to show that $[a, x] \equiv 1$ for all $x = -x^* \in D^{\bullet}$ and $a = a^* \in D^{\bullet}$. If $[a, x] \not\equiv 1$ then since x^2 is symmetric we get $[a, x^2] \equiv 1$. From this

$$[x, xa - ax] \equiv -1.$$

Because $x = -x^*$ and $xa - ax = (xa - ax)^*$, if we put

$$b = x(xa - ax) - (xa - ax)x$$

then $b = b^*$ so that $[a, b] \equiv 1$. Since $[x, xa - ax] \equiv -1$ we have

$$b \equiv 2x(xa - ax) \pmod{1 + J}.$$

From $xa - ax = (xa - ax)^*$ follows $[xa - ax, a] \equiv 1$ and from $[b, a] \equiv 1$ follows thus $[x, a] \equiv 1$, which we agreed to rule out.

Step 2. *The conclusion holds true for the involution of D of the second kind.*

Subproof. By routine argument $[a, b] \equiv 1$ for all pairs a, b with $a = \pm a^*$, $b = \pm b^*$. Thus all $a = \pm a^* \in \hat{Z}$. From this all valuation units are in $\hat{Z}$, which readily gives $D^{\bullet} \subset \hat{Z}$. In that case, it is easy to deduce that $D^{\bullet}$ contains no proper algebraic elements over Z using a trace argument.

Step 3. *The conclusion holds true for D algebraic over Z.*

Subproof. It suffices to show that $a \in Z$ for all $a = a^* \in D^{\bullet}$. In effect, if $a \notin Z$ then we can find $b = a - z = b^*$ with $b \notin Z$ and some conjugates to b adding to 0. However, because $b \in \hat{Z}$ we reach a contradiction.

Step 4. *The conclusion holds true.*

Subproof. By the preceding steps we may assume that the involution of D is of the first kind with $[D:Z] = \infty$. Let then $a \in D$, $a \notin Z$ be algebraic over Z. We shall reach a contradiction. After modifying a we may assume that there are conjugates to a adding to 0 so that $a \notin \hat{Z}$. By Noether-Skolem's, we can find $t \in D^{\bullet}$ such that $t^{-1}at = a^*$. If $t + t^* \neq 0$ then by routine calculations

$$(t + t^*)^{-1}a(t + t^*) = a^*.$$

Since $t + t^* \in D^{\bullet}$ we get $a \equiv a^* \pmod{1 + J}$ and, hence, $a \in \hat{Z}$, which we agreed to rule out. This shows that $t + t^* = 0$. Let A be the centralizer of a^* in D. For $y \in A$ follows

$$(ty)^{-1}a(ty) = y^{-1}t^{-1}aty = y^{-1}a^*y = a^*$$

so that $ty + (ty)^* = 0$, this for every $y \in A$. A simple computation shows that A is commutative so that $A = Z[a]$ is a maximal subfield finite-dimensional over Z and, hence,

$$[D:Z] = [Z[a]:Z]^2 < \infty,$$

a contradiction.

In the theorem we just proved we assumed that the order valuation ω is compatible with the given c-ordering. This requirement follows, of course, from the requirement $1 + J \subset \hat{C}$. It implies the compatibility Axiom (06). Here is the announced case of interest where this axiom holds true.

THEOREM 3.2.7. *If the value group G of the order valuation ω is archimedean (real order valuation) then ω is *-compatible. It follows that $\hat{C} \subset \hat{Z}$.*

Proof. I will establish a slightly more general property of ω, to know:

$$(\mathscr{P}) \begin{cases} a > b \equiv b^* \pmod{1 + J} \\ b > 0 \end{cases} \Rightarrow \omega(a) \leqq \omega(b).$$

Property $(\mathscr{P})$ will be checked by stages beginning with case $a \in C$.

Step 1. *If $a \in C$, then $(\mathscr{P})$ holds true.*

Subproof. Put $b = ac$. Since $b > 0$ and a is order multiplicable it follows that $c > 0$. From $b \equiv b^*$ and $a \equiv a^* \in \hat{Z}$ follows $c \equiv c^*$. Here, $a > b = ac$ implies $1 > c$ so that $1 > (c + c^*)/2$ placing $(c + c^*)/2$ in R and, hence, $c \equiv (c + c^*)/2$ is in R so that

$$\omega(b) = \omega(a) + \omega(c) \geqq \omega(a).$$

Step 2. *It is to be shown that whenever* $\omega(b) \equiv 0 \pmod{2G}$ *then* $(\mathscr{P})$ *holds.*

Subproof. For one can write b in the form some $0 \neq x$ in D, and some valuation unit u. Put $a = bd$. Then from

$$a = bd = (xx^*u)d > xx^*u = b > 0$$

follows $ud > u > 0$. Here

$$u = (xx^*)^{-1}b \quad \text{with } b \equiv b^* \pmod{1 + J}$$

so that $u \equiv u^* \pmod{1 + J}$. If now $\omega(a) > \omega(b)$ then

$$\omega(d) = \omega(b^{-1}a) = \omega(a) - \omega(b) > 0.$$

Accordingly, $d \in J$ follows. Since u is a valuation unit, it follows that $ud \in J$. From this, and

$$\frac{1}{2}(ud + d^*u^*) > \frac{1}{2}(u + u^*) > 0$$

follows $(u + u^*)/2 \in J$; but

$$\frac{1}{2}(u + u^*) \equiv u \pmod{J},$$

and u is a valuation unit. This shows that $\omega(b) \geqq \omega(a)$.

Step 3. *If the conclusion of* $(\mathscr{P})$ *fails then if we put* $d = b^{-1}a$, *and* $g = \pm\omega(b)$, *depending on whether* $\omega(b) > 0$ *or* $\omega(b) < 0$, *then*

$$g > n\omega(d) > 0,$$

for all $n = 1, 2, 3, \ldots$.

Subproof. First, $\omega(d) = \omega(a) - \omega(b) > 0$. Next, by the contrapositive to the preceding step follows $\omega(b) \equiv 0 \pmod{2G}$, so that $\omega(b) \neq 0$ and, hence, either $\omega(b) > 0$ or $\omega(b) < 0$, so that $g = \pm\omega(b)$ is uniquely determined.

Case $g > 0$. Clearly:

$$\begin{aligned} 0 &\leqq (b - d^*)(b - d^*)^* \\ &= (b - d^*)(b^* - d) \\ &= bb^* - (bd + d^*b^*) + d^*d. \end{aligned}$$

Thus

$$(bd + d^*b^*) - bb^* \leqq d^*d.$$

Since $bd = b(b^{-1}a) = a > b$, it follows that $bd + d^*b^* > b + b^*$, so that

$$d^*d \geqq (bd + d^*b^*) - bb^* \geqq b + b^* - bb^*.$$

Since $\omega(b) = g > 0$, and $b > 0$, it is clear that $b + b^* > qbb^*$, where q is any positive rational. For, otherwise, one would get for some rational q_0

$$bb^* \geqq q^{-1}(b + b^*) > 0.$$

Here bb^* is in C. By Step 1, follows

$$\omega(bb^*) \leqq \omega(q^{-1}(b + b^*)).$$

Thus

$$\begin{aligned} 2\omega(b) = 2g &\leqq \omega(q^{-1}(b + b^*)) \\ &= \omega(b + b^*) \\ &= \omega(b) \\ &= g, \end{aligned}$$

for from $b \equiv b^* \pmod{1 + J_\omega}$ follows

$$\omega(b + b^*) = \omega(b).$$

Since $g > 0$, one gets a contradiction. This shows that $b + b^* > qbb^*$ for all positive rationales q. In particular,

$$b + b^* > 2(bb^*).$$

Thus:

$$\begin{aligned} d^*d &\geqq (bd + d^*d^*) - bb^* \\ &\geqq (b + b^*) - bb^* \\ &= \frac{1}{2}(b + b^*) + \left(\frac{1}{2}(b + b^*) - bb^*\right) \\ &\geqq \frac{1}{2}(b + b^*) > 0. \end{aligned}$$

By Step 1, follows

$$2\omega(d) = \omega(d^*d) \leqq \omega(b + b^*) = \omega(b) = g.$$

Since $g \not\equiv 0 \pmod{2G}$, the preceding inequality is a strict inequality; that is,

$$g > 2\omega(d).$$

To feedback this inequality proceed as follows: Let $0 \neq c$ in D be such that

$$\omega(cc^*b) \geqq 0$$

and, hence,

$$\omega(cc^*b) > 0.$$

From $a > b > 0$ follows $cc^*a > cc^*b > 0$. Again,

$$cc^*b \equiv (cc^*b)^* \pmod{1 + J_\omega}.$$

Also, since $\omega(a) > \omega(b)$ it follows that

$$\omega(cc^*a) > \omega(cc^*b).$$

Now, the element $d = b^{-1}a$ in the preceding becomes

$$d' = (cc^*b)^{-1}(cc^*a) = b^{-1}(cc^*)^{-1}(cc^*)a = d.$$

By the preceding argument follows

$$g' = \omega(cc^*b) > 2\omega(d') = 2\omega(d),$$

or

$$g = \omega(b) > 2\omega(d) - 2\omega(c).$$

If we set $c = d^{-1}$, then indeed

$$\omega(cc^*b) = \omega(d^{-1}(d^{-1})^*b) = \omega(b) - 2\omega(d) = g - 2\omega(d) > 0.$$

For such a choice of c one gets

$$\begin{aligned} g = \omega(b) &> 2\omega(d) - 2\omega(d^{-1}) \\ &= 4\omega(d). \end{aligned}$$

Step by step, one can show that $g > 2^m\omega(d)$, $m = 1, 2, 3, \ldots$, which shows that $g > n\omega(d)$, for all integers n.

Case $g < 0$. From the basic inequality

$$a > b > 0$$

follows

$$(bb^*)^{-1}a > (bb^*)^{-1}b > 0$$

or

$$(bb^*)^{-1}a > (b^*)^{-1} > 0.$$

Again,

$$(b^*)^{-1} \equiv ((b^*)^{-1})^* \pmod{1 + J_\omega},$$

and

$$\begin{aligned} \omega(((bb^*)^{-1})a) &= \omega(a) - 2\omega(b) \\ &= (\omega(a) - \omega(b)) + \omega((b^*)^{-1}) \\ &> \omega((b^*)^{-1}). \end{aligned}$$

Here,

$$\omega((b^*)^{-1}) = -\omega(b) = -g > 0.$$

The element d in the preceding case is now the element d'':

$$\begin{aligned} d'' &= ((b^*)^{-1})^{-1}(bb^*)^{-1}a \\ &= b^*b^{*-1}b^{-1}a \\ &= b^{-1}a = d. \end{aligned}$$

It follows that

$$-\omega(g) > n\omega(d),$$

$n = 1, 2, \dots$, as desired.

Step 4. ($\mathscr{P}$) *holds true.*

Subproof. For a and b as in the statement of ($\mathscr{P}$) if the conclusion $\omega(a) \leqq \omega(b)$ fails we showed that the element $g_0 = \omega(b^{-1}a)$ is much larger than $g = |\omega(b)|(|\omega(b)| = \omega(b)$ for $\omega(b) \geqq 0$, $|\omega(b)| = -\omega(b)$ for $\omega(b) < 0)$, which contradicts the archimedean assumption.

Step 5. $\hat{C} \subset \hat{Z}$.

This was established in Theorem 3.2.1.

3.3. *Extending a field ordering.* In this closing section we are given an involutorial division ring D with ordered centre Z (or a more general ordering). We wish to examine when does the field ordering of Z extend to some normal c-ordering of D. Obviously, if the problem has an affirmative solution then the order valuation of the field Z will extend to a c-valuation of D; namely, the order valuation of the extended c-ordering. Hence, this problem is at first glance harder than its purely valuation theory counterpart. In the facts, we shall attack the considered problem by means of a valuation argument. We will build our ordering by making pre-use of its order valuation.

Definition 3.3.1. By *strong c-ordering* of Z, I mean a field ordering of the subfield of symmetrics of Z such that $xx^* > 0$ for all $x \in Z$, $x \neq 0$.

THEOREM 3.3.2. *Let D be any involutorial ring with centre Z carrying a strong c-ordering, and suppose that D has a *-armature basis $\{e_\alpha\}$ over Z. Then the ordering of Z extends to some normal c-ordering of D if and only if:* (i) *each norm $e_\alpha e_\alpha^*$ is positive in Z; and,* (ii) *for every non constant base element $e_\alpha = e_\alpha^*$ ($e_\alpha = -e_\alpha^*$) the elements $e_\alpha e_\alpha^*$ and $z_\alpha z_\alpha^*$ are never archimedean equivalent in Z, this for every $z_\alpha = z_\alpha^*$ ($z_a = -z_\alpha^*$) in Z.*

Proof. Suppose that the ordering of Z extends as desired.

If ω_Z is the order valuation of the ordering of Z, and if ω is the order valuation of the extended c-ordering then ω is a c-valuation of the extended c-ordering, which extends ω_Z, a *-formally real valuation of Z. In view of Theorem 2.3, we get ENCR$[e_\alpha]$, which is precisely the requirement (ii). As for requirement (i), this is evidently true since $e_\alpha e_\alpha^* \in Z$ for all α, and since $e_\alpha e_\alpha^* > 0$ in D.

Conversely, assume both (i) and (ii). To begin with, we have MVP$[e_\alpha]$. For since each $e_\alpha e_\alpha^*$ is positive, and since the ordering of Z is such that every positive element is order multipliable, it follows that

$$\omega_Z(a + b) = \min\{\omega_Z(a), \omega_Z(b)\}$$

holds true for all pairs a, b in the cone $C(Z)[e_\alpha e_\alpha^*]$, where $C(Z)$ is the *-core of the involutorial field Z. As observed in the preceding, we also have ENCR$[e_\alpha]$. By Theorem 2.2.12, ω_Z extends to a c-valuation ω of D. Let

$$\tilde{C} = \{\sum z_i c_i(1 + x_i) | z_i \in M_Z, x_i \in J, c_i \in C\},$$

where M_Z is the domain of positivity of the ordering of Z, J is the maximal ideal of the valuation, and C is the *-core of D. I contend that $0 \notin \tilde{C}$. For as established in Theorem 2.2.12, a typical element c in C is of the form

$$c = (\sum z_\alpha z_\alpha^* e_\alpha e_\alpha^*)(1 + x),$$

where $x \in J$. Hence, we can represent a typical d element in $\tilde{C}$ in the form

$$d = \sum z_i z_{\alpha i} z_{\alpha i}^* e_\alpha e_\alpha^*(1 + x_i),$$

where $z_i \in M_Z$, and $x_i \in J$. Because all $z_i z_{\alpha i} z_{\alpha i}^* e_\alpha e_\alpha^*$ are nonnegative elements of Z we get

$$\omega_Z(\sum z_i z_{\alpha i} z_{\alpha i}^* e_\alpha e_\alpha^*) = \min\{\omega_Z(z_i z_{\alpha i} e_\alpha e_\alpha^*)\}.$$

Since $x_i \in J$, and ω is a valuation extending ω_Z, it follows that

$$\omega(d) = \min\{\omega(z_i z_{\alpha i} z_{\alpha i}^* e_\alpha e_\alpha^*(1 + x_i))\}.$$

From this $d \neq 0$, for all $d \in \tilde{C}$.

Observe that $\tilde{C}$ has the same properties for C: $\tilde{C}$ is additive, $\tilde{C}$ is multiplicative, $\tilde{C}$ is preserved under conjugation, and $\tilde{C}$ is *-closed. A maximizing process yields a c-ordering of D with domain of positivity M such that $\tilde{C}M \subset M$. Since $M_Z \subset \tilde{C}$, it follows that this c-ordering of D extends that of Z.

In the construction of the normal c-ordering of D we used a maximizing process, which is somewhat non-canonical. All the more, one is interested to know what are the properties of this ordering which do not depend on that process. In particular, what will be the residue ordering of $\bar{D}$, and what is the centre $\hat{C}$? These questions are answered in the

THEOREM 3.3.3. *The normal c-ordering found in Theorem* 3.3.2 *is such that*:

1. *The residue ordering of $\bar{D}$ coincides with the residue ordering of $\bar{Z}$.*
2. *The centre $\hat{C}$ of the c-ordering of D is precisely $M_Z(1 + J)$, where M_Z is the domain of positivity of the ordering of Z.*

Proof. By construction, the residue ordering of $\bar{D}$ has domain of positivity

$$\bar{M} = \{a + J | a = a^* \in R, a \notin J, a > 0\}.$$

If $a = a^* \in R$, $a \notin J$, then $a - z \in J$ for some $z = z^* \in Z$. Because $1 + J \subset \hat{C}$ we get $a > 0$ if, and only if, $z > 0$ on Z. Therefore

$$\bar{M} = \{z + J | z = z^* \in Z, z \notin J, z > 0 \text{ in } Z\}.$$

Equivalently, $\bar{M}$ coincides with the domain of positivity of the ordering of Z.

2. By construction, $M_Z \subset \hat{C}$. Since $1 + J \subset \hat{C}$ we get

$$M_Z(1 + J) \subset \hat{C}.$$

Conversely, if $a \in \hat{C}$ then by Theorem 2.2 we have $a - z_1 \in Z$ for some $z_1 \in Z$, $z_1 > 0$. Thus

$$a \in M_Z(1 + J), \quad \text{and} \quad \hat{C} = M_Z(1 + J).$$

More can be said about the invariance of the extended c-ordering of D in the case the extended valuation ω is totally ramified. Using the fact that $\hat{C}$ contains $1 + J$ and that $M_Z \subset \hat{C}$ one can break down the positivity of $x = \sum z_\alpha e_\alpha$ as follows: If β_0 is the unique index such that

$$\omega(x) = \omega(z_{\beta_0} e_{\beta_0})$$

then (1) $e_\beta = e_\beta^*$, and $z_\beta e_\beta > 0$; that is, z_β and e_β have the same signs. Conversely, if we start off with a totally ramified valuation ω of D and if we require that the involution of D is of the first kind then we know that ω is a c-valuation of D. The preceding positivity of $x = \sum z_\alpha e_\alpha$ can then be shown to induce a normal c-ordering. Hence, the

THEOREM 3.3.4. *Let D be any involutorial division ring with an ordered centre Z and with involution of the first kind. Assume that $\{e_\alpha\}$ is a basis of D over Z such that*

(i) *Each $e_\alpha e_\alpha^*$ is positive in Z.*

(ii) *$e_\alpha e_\alpha^*$ and z^2 are never archimedean equivalent in Z for $e_\alpha \neq e_0$, the constant base element, and $z \in Z$.*

Then the ordering of Z extends to a normal c-ordering of D with domain of positivity all linear combinations $x = \sum z_\alpha e_\alpha$ where for β such that

$$\omega(z_\beta e_\beta \cdot z_\beta e_\beta^* = \min\{\omega(z_\alpha e_\alpha z_\alpha e_\alpha^*)\},$$

we have $z_\beta > 0$ in Z, and $e_\beta = e_\beta^*$.

In the foregoing theorems can one replace the centre of Z of D by a suitable tensor factor of D? Of course, the basic assumption Z carries a strong c-ordering will have to be relaxed if we want a nontrivial tensor factor. What is quite legitimate to requiring (in view of Theorem 3.3.2) is that the centre $\hat{C}_0$ of the c-ordering of D_0 be of the form

$$\hat{C}_0 = M_Z(1 + J_0),$$

where J_0 is the maximal ideal of the order subring of D_0. Using a reasoning similar to the proof of Theorem 3.3.2 one can show the

THEOREM 3.3.5. *Let D be any ring with involution with centre Z a field, and let D_0 be a *-closed division subring of D with same centre Z and with a c-ordering with centre $\hat{C}_0 = M_Z(1 + J_0)$, where M_Z is the domain of positivity of the restricted c-ordering to Z, and J_0 is the maximal ideal of the order valuation ω_0 of D_0. Let $\{e_\alpha\}$ be a *-armature basis of D over D_0. For D to carry a c-ordering with centre $\hat{C}_0(1 + J)$ it is necessary and sufficient that* (i) *$e_\alpha e_\alpha^* \in \hat{C}_0$, and* (ii) *for each nonconstant base elements e_α, $e_\alpha e_\alpha^*$ and aa^* are never archimedean equivalent, where $a = a^* \in D_0$ for $e_\alpha = e_\alpha^*$ and $a = -a^* \in D_0$ for $e_\alpha = -e_\alpha^*$.*

To close, let me look at examples of normal c-ordered division rings D where the centre Z is an ordered field, and the involution is of the first kind. The method of construction of these examples will rest on quaternionic examples.

Consider the quaternionic division ring $D = Z[a, b]$, where Z is an ordered field and, a, b are anti-commuting generators each declared symmetric or skew symmetric. It is to be found necessary and sufficient condition for D to carry an extended normal c-ordering where the involution of D is the linear transformation of D over Z at which $a^* = \epsilon_1 a$, $b^* = \epsilon_2 b$, and $(ab)^* = b^*a^* = -\epsilon_1\epsilon_2 ab$; where $\epsilon_i = \pm 1$ (at will).

Put $e_0 = 1$, $e_1 = a$, $e_2 = b$, and $e_3 = e_1 e_2 = ab$. Clearly $\{e_i\}$ is a *-armature basis of D over Z. Applying Theorem 3.3.2 to this basis we get that the field ordering of Z will extend as desired if and only if we have:

(E1) *Each of the elements* $\epsilon_1 a^2$, $\epsilon_2 b^2$, *and* $\epsilon_1\epsilon_2 a^2 b^2$ *must be positive in* Z; *and*

(E2) *If* $e_i = e_i^* \neq 1$ *then* e_i^2 *is archimedean equivalent to no square in* Z.

To say that $e_i e_i^*$ is positive in Z is to say that if $e_i = e_i^*$ then $e_i^2 > 0$, while if $e_i = -e_i^*$ then $e_i^2 < 0$ in Z. For $e_1 = a$ we get that $\epsilon_1 = 1$ or -1 depending on whether $a^2 > 0$ or $a^2 < 0$ in Z. Likewise for $e_2 = b$ we have $\epsilon_2 = 1$ or -1 depending on whether $b^2 > 0$ or $b^2 < 0$. Since our basic ordering of Z is a field ordering the conjunction of the requirements $\epsilon_1 a^2 > 0$ and $\epsilon_2 b^2 >$ implies

$$\epsilon_1\epsilon_2 a^2 b^2 > 0.$$

Thus requirement (E1) reduces to ensuring that $\epsilon_1 a^2$ and $\epsilon_2 b^2$ are both positive in Z, which I will assume in what will follow. I am left with requirement (E2). Distinguish three cases:

Case both a^2 *and* b^2 *are negative in* Z. Here, all three base elements e_1, e_2 and e_3 must be skew symmetrics, and (E2) is then vacuously verified (normal quaternionic division ring D).

Case both a^2 *and* b^2 *are positive in* Z. Here $e_1 = e_1^*$, $e_2 = e_2^*$, and $e_3 = -e_{\alpha_3}^*$. Thus requirement (E2) simply means that both a^2 and b^2 are archimedean equivalent to no square in Z.

Case a^2 *and* b^2 *have opposite signs in* Z. We have either $a = a^*$, $b = -b^*$ or $a = -a^*$ and $b = b^*$. In both cases the element ab is symmetric. When $a^2 = aa^*$ or, equivalently, $a^2 > 0$ requirement (E2) means that a^2 and $a^2(-b^2)$ are both archimedean equivalent to no square in Z. In the case $-b^2$ is archimedean equivalent to a square in Z this requirement reduces to a^2 archimedean equivalent to no square in Z. In the opposite case where $-b^2$ is archimedean equivalent to no square in Z the requirement assumes its full strength (totally ramified order valuation). In both cases the requirement translates into the following: The positive square from the pair $\{a^2, b^2\}$ is archimedean equivalent to no square in Z, and it is archimedean non equivalent to the other member affected with the sign minus. To sum up the discussion:

THEOREM 3.3.6. *Consider the quaternionic division ring* $D = Z[a, b]$, *where* Z *is an ordered field and* a, b *are anti-commuting generators. Assign to* a *and* b *the elements* $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$ (*at will*), *and take the involution of* D *which fixes all elements of* Z *at which* $a^* = \epsilon_1 a$, $b^* = \epsilon_2 b$ *and, hence,* $(ab)^* = -\epsilon_1\epsilon_2 ab$. *For the field ordering of* Z *to extend to some c-ordering of* D *it is necessary and sufficient that*:

(E1) $\epsilon_1 a^2 > 0$ *and* $\epsilon_2 b^2 > 0$ *in* Z; *i.e., if* $a^2 > 0$ *in* Z *then and only then* $a = a^*$, *and similarly if* $b^2 > 0$ *in* Z *then and only then* $b = b^*$.

(E2) *If* a^2 *and* b^2 *are both positive in* Z *then both* a^2 *and* b^2 *are archimedean equivalent to no square in* Z.

https://doi.org/10.4153/CJM-1989-002-6 Published online by Cambridge University Press

(E3) *If a^2 and b^2 are of opposite signs in Z then up to sign a^2 and b^2 are archimedean not equivalent in Z and the positive member a^2 or b^2 is archimedean equivalent to no square in Z.*

COROLLARY 3.3.7. *Let F be any ordered field and let $D = F[a, b]$ be the quaternionic division ring extension of F where a and b are anti-commuting square roots of negative elements in F. Then the field ordering of F extends to a c-ordering of D with domain of positivity precisely the domain of positivity of F (normal quaternionic division ring extension).*

COROLLARY 3.3.8. *Let F be any ordered field and let $D = F[a, b]$ be the quaternionic division ring extension of F where a and b are anti-commuting square roots of positive elements in F. Under the involution of D at which both a and b are symmetrics the field ordering extends to a normal c-ordering of D if and only if a^2 and b^2 are both archimedean equivalent to no square in Z.*

As an example as in Corollary 3.3.8 take $D = \Phi\langle t_1, t_2\rangle$, the iterated Hilbert division ring with anti-commuting indeterminate t_1 and t_2 over the ordered field Φ. For $F = \Phi\langle t_1^2, t_2^2\rangle$ and $a = t_1$, $b = t_2$, we have $D = F[a, b]$. If we turn F to an ordered field in the usual manner with both t_1^2 and t_2^2 positive then we meet the situation described in the preceding corollary. Hence D carries an extended c-ordering. I record this in the

COROLLARY 3.3.9. *The iterated Hilbert division ring $D = \Phi\langle t_1, t_2\rangle$ over an ordered field Φ carries a normal c-ordering relative to the involution which fixes all elements in Φ, and at which $t_1 = t_1^*$, $t_2 = t_2^*$. Here, the residue ordering is the residue ordering of the residue field Φ, and the residue degree of D is* 1.

Two more cases of quaternionic division ring extension are in order:

COROLLARY 3.3.10. *Let F be an ordered field and let $D = F[a, b]$ be the quaternionic division ring extension of F where a and b are anti-commuting square roots of positive and negative elements in F respectively. Under the involution of D at which $a = a^*$ and $b = -b^*$ the field ordering extends to a normal c-ordering if and only if a^2 and $-b^2$ are not archimedean equivalent in Z and a^2 is archimedean equivalent to no square in Z.*

COROLLARY 3.3.11. *The Hilbert division ring $D = \Phi[\sqrt{-1}]\langle t\rangle$, where Φ is an ordered field carries a normal c-ordering relative to the involution which reverses $\sqrt{-1}$ and fixes t.*

Turning to examples of higher dimension let us start off with any normally c-ordered division ring D with centre $\hat{C}$ of the form $\hat{C} = M_Z(1 + J)$, where M_Z is the domain of positivity of the restricted ordering of Z. We pass to

$$\Delta = D\langle t_{11}, t_{12}, t_{21}, t_{22}, \dots, t_{n1}, t_{n2}\rangle,$$

the iterated Hilbert division ring over D which was described in earlier stages in Section 2.3. Here Δ has a *-armature basis generated by the t_{ij} over $D\langle t_{ij}^2\rangle$. The latter carries a standard normal c-ordering at which all $t_{ij}^2 > 0$.

None of the products of the t_{ij} is archimedean equivalent to a norm in $D\langle t_{ij}^2\rangle$. By Theorem 3.3.5 the c-ordering of D extends to a normal c-ordering of Δ. Hence, the

THEOREM 3.3.12. *Start off with any quaternionic division ring $D = F[a, b]$, where F is an ordered field. Assume that the field ordering of F extends to a normal c-ordering of D. Then the iterated Hilbert division ring*

$$\Delta = D\langle t_{11}, t_{12}; \dots ; t_{n1}, t_{n2}\rangle$$

carries an extended normal c-ordering at which all $t_{ij} = t_{ij}^*$ are declared positive.

The construction which follows was suggested to me (in conversation) by H. Gross. It concerns the Clifford algebra of an orthogonal space over an ordered field F equipped with its main involution (see [**12**, p. 239, last paragraph]). Let $a_i \in F$, $a_i \neq 0$ and let $A = A(a_i)$ stand for the Clifford algebra of the n-dimensional orthogonal space over F with orthogonal basis $\{u_i\}$ such that $\langle u_i, u_i\rangle = a_i$. Identifying the u_i with elements in A and $\langle u, v\rangle$ with $uv + vu$ in A, we get a *-armature basis of A over its centre F with typical member

$$e_{(r)} = u_1^{r_1} \dots u_n^{r_n} \quad (r_i = 0 \text{ or } 1).$$

Generally A is a central simple, if and only if, the dimension n of V is even, which I will assume.

THEOREM 3.3.13. *Let $A = A(a_1, \dots, a_n)$ be the Clifford algebra of the non isotropic orthogonal space V of dimension n over the ordered field F, where $\{u_j\}$ is an orthogonal basis of V over F with $\langle u_j, u_j\rangle = a_j$, $j = 1, 2, \dots, n$. Equip A with its main involution and suppose that n is even. For A to be a division ring with extended normal c-ordering it is necessary and sufficient that:*

1. *All $a_j > 0$ in F; i.e., the form is positive definite.*
2. *For every subset $E = \{r_1, \dots, r_k\}$ of $\{1, 2, \dots, n\}$ with $k = 4m$ or $k = 4m + 1$ ($m \geqq 0$) for some m, $\prod a_{r_i}$ is archimedean equivalent to no square in F.*

Proof. We apply Theorem 3.3.2 to the *-armature basis $\{e_{(r)}\}$ generated by the u_j. The positivity requirement $e_{(r)}e_{(r)}^* > 0$ in F simply means all $u_j^2 > 0$ as $u_j = u_j^*$. For the requirements $e_{(r)} = e_{(r)}^* \neq 1$ implies $e_{(r)}^2$ is archimedean equivalent to no square in F we find those (r) such that $e_{(r)} = e_{(r)}^*$. Counting the number k of 1's in (r) we can show that k is of the prescribed form $k = 4m$ or $k = 4m + 1$ ($m \geqq 0$), which proves the theorem.

Holland asked me (in private) whether c-orderability "scales"; that is, if a division ring D is c-orderable and if the involution is replaced with a co-gradient involution must D remain c-orderable?

THEOREM 3.3.14. *Consider the Hilbert division ring* $D = F[\sqrt{-1}]\langle t\rangle$ *where* F *is an ordered field. Then*:

1. *If the involution of* D *is the one fixing* t *and reversing* $\sqrt{-1}$ *then* D *is* c*-orderable.*

2. *If the involution of* D *is the one fixing* t *and* $\sqrt{-1}$ *then* D *is not* c*-orderable and this involution is co-gradient to the one in* 1.

Proof. 1. This follows from Corollary 3.3.11, where the prescribed $a = 1$.

2. Consider the involution $(\bar{\ })$: $x \rightarrow t^{-1}x^*t$, where $(^*)$ is the involution in 1. Here

$$\bar{t} = t, \quad \text{and} \quad \overline{\sqrt{-1}} = t^{-1}(-\sqrt{-1})t = \sqrt{-1}.$$

We have

$$\sqrt{-1} \cdot \overline{\sqrt{-1}} = -1.$$

Thus D is certainly not c-orderable.

Finally of the various questions about c-orderings left untouched here let me record three of utmost importance (to me).

Question 3.3.15. Must the order subring of any c-ordered division ring be preserved under conjugation?

Question 3.3.16. Must a c-ordered division ring D decompose as a tensor product of quaternionic division rings for, at least, a normal c-ordering?

Question 3.3.17. What is the structure of the order valuation of any c-ordering?

We hope to come back on some of these questions in the referred joint work [**3**].

REFERENCES

1. S. S. Amitsur, L. Rowen and Tignol, *Division algebras of degree* 4 *and* 8 *with involution*, Israel J. Math. *33* (1979), 133-148.

2. M. Chacron, *c-orderable division rings with involution*, J. of Algebra *75* (1982), 495-521.

3. M. Chacron and A. Wadsworth, *On decomposing c-valued division rings*. In preparation.

3′. T. Craven, *Approximation properties for orderings on *-fields*, Preprint.

4. S. S. Holland (Jr.), **-valuations and ordered *-fields*, Trans. Amer. Math. Soc. *262* (1980), 219-243.

5. ——— *Strong orderings of *-fields*, J. of Algebra *101* (1986), 16-46.

6. ——— *Baer ordered *-fields of the first kind*, Israel J. of Mathematics *57* (1987), 365-374.

7. I. Idris, **-valuated division rings, orderings, and elliptic heritian spaces*, Ph.D. thesis, Carleton University, Ottawa, Canada (1986).

8. T. Y. Lam, *Orderings, valuations and quadratic forms*, Amer. Math. Soc. Publ. V. *52* (1983).

9. A. Prestel, *Lectures on formally real fields*, Springer Lecture Notes *1093* (Springer-Verlag, Berlin, 1984).

10. ——— *Quadratishe Semi-Ordnumgen und quadratishe Formen*, Math. Z. *133* (1973), 319-342.

11. O. Schilling, *The theory of valuations*, Math. Surveys *4* (Amer. Math. Soc., Providence, R.I., 1950).

12. M. Nagata, *Field theory*, Pure and applied mathematics (Marcel Dekker Inc., New York and Basel, 1979).

13. N. Jacobson, *Basic algebra II* (W. H. Freeman and Company, San Francisco, 1980).

14. H. Gross, *Quadratic forms in infinite dimensional vector spaces*, Progress in Mathematics Bikhausen *22* (1979).

15. P. M. Cohn, *Algebra*, Vol. 2 (John Wiley & Sons, London, 1977).

16. P. Morandi, *The Henselization of a valued division algebra*, to appear in J. Algebra.

17. B. Jacob and A. R. Wadsworth, *Division algebras over Henselian fields*, to appear in J. Algebra.

Carleton University,
Ottawa, Ontario