CONVEX SETS OF NON-NEGATIVE MATRICES

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1. Introduction. In (**8**) M. V. Menon investigates the diagonal equivalence of a non-negative matrix A to one with prescribed row and column sums and shows that this equivalence holds provided there exists at least one non-negative matrix with these row and column sums and with zeros in exactly the same positions A has zeros. However, he leaves open the question of when such a matrix exists. W. B. Jurkat and H. J. Ryser in (**7**) study the convex set of *all* $m \times n$ non-negative matrices having given row and column sums. They compute the minimal term rank and permanent of the matrices in this convex set and also determine its extreme points. The latter then constitutes a generalization of Birkhoff's theorem for the convex set of doubly stochastic matrices. Here we show how the solution of the existence question above can be obtained from a feasibility theorem for network flows of A. J. Hoffman. This result can then be used to obtain a quicker proof of Jurkat and Ryser's formula for the minimal term rank of the class of $m \times n$ non-negative matrices with given row and column sums. Another consequence is a result of A. Horn concerning the existence of a doubly stochastic matrix with a prescribed main diagonal.

We also discuss the convex sets of non-negative matrices having prescribed row and column sums and prescribed positions for all the zeros. The extreme points for the closure of such convex sets can also be classified.

2. The existence theorem. By a *zero pattern* $\mathfrak{P}_{m,n}$ or simply $\mathfrak{P}$ we shall mean an $m \times n$ matrix of 0's and 1's. We say that the $m \times n$ non-negative matrix A is *compatible* with the zero pattern $\mathfrak{P}$ provided A has 0's wherever $\mathfrak{P}$ does and positive elements wherever $\mathfrak{P}$ has 1's. If $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ are positive vectors, then the symbol $\mathfrak{P}(R, S)$ denotes the class of all $m \times n$ non-negative matrices which are compatible with $\mathfrak{P}$ and have row sum vector R and column sum vector S. Thus the question we wish to investigate is: When is the class $\mathfrak{P}(R, S)$ non-empty? An obvious necessary condition for $\mathfrak{P}(R, S)$ to be non-empty is that

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j.$$

Also since the vectors R and S have positive components, $\mathfrak{P}$ must have a 1 in each row and column, and we implicitly assume this throughout. There is

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no loss of generality in assuming that the zero pattern $\mathfrak{P}$ is *indecomposable,* that is, the rows and columns of $\mathfrak{P}$ cannot be permuted so that it takes the form

$$\begin{bmatrix} \mathfrak{P}_1 & 0 \\ 0 & \mathfrak{P}_2 \end{bmatrix}$$

where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are non-vacuous 0, 1-matrices. For if $\mathfrak{P}$ can be permuted to the above form, then the non-emptiness question for $\mathfrak{P}(R, S)$ reduces to the non-emptiness question for two "smaller" classes with zero patterns $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and appropriate row and column sum vectors.

THEOREM 2.1. *If $\mathfrak{P}$ is an indecomposable zero pattern and $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ are given positive vectors, then $\mathfrak{P}(R, S)$ is non-empty if and only if the following condition* (*) *is fulfilled:*

(*) *Whenever the rows and columns of $\mathfrak{P}$ can be permuted to the form*

$$\begin{bmatrix} \mathfrak{P}_1 & 0 \\ \mathfrak{P}_{21} & \mathfrak{P}_2 \end{bmatrix} \tag{2.1}$$

where $\mathfrak{P}_1$ is a non-vacuous 0, 1*-matrix formed from rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_q$ of $\mathfrak{P}$ and $\mathfrak{P}_2$ is non-vacuous, then*

$$s_{j_1} + \ldots + s_{j_q} > r_{i_1} + \ldots + r_{i_p}. \tag{2.2}$$

Proof.[1] The necessity of condition (*) is immediate. For, if $A \in \mathfrak{P}(R, S)$, then summing columns $j_1, \ldots, j_q$ and rows $i_1, \ldots, i_p$ and using the fact that $\mathfrak{P}_{21}$ does not consist of all 0's, we obtain the inequality in (2.2).

Associate a network N with the zero pattern $\mathfrak{P}$ which has nodes

$$\{x_1, \ldots, x_m, y_1, \ldots, y_n, s, t\}.$$

There is an arc (x_i, y_j) from x_i to y_j in N if and only if the (i, j) position of $\mathfrak{P}$ is a 1. There are also arcs (s, x_i) from s (the source) to each x_i and arcs (y_j, t) from y_j to t (the sink) for each y_j. These are the only arcs of N. We put upper $c(u, v)$ and lower $l(u, v)$ bounds for the flows in the arcs (u, v) of N as follows: for the arcs (x_i, y_j) of N we put $c(x_i, y_j) = \infty$, $l(x_i, y_j) = \epsilon$ where ϵ is (for the moment) an unspecified positive number; for the arcs (s, x_i) we put $c(s, x_i) = l(s, x_i) = r_i$, while for the arcs (y_j, t) we put $c(y_j, t) = l(y_j, t) = s_j$. Applying the circulation theorem (**3**, p. 51) of A. J. Hoffman or rather its consequence for the existence of a flow f from a source s to a sink t in a network with lower and upper bounds on arc flows, we see that the relevant condition is

$$s_{j_1} + \ldots + s_{j_q} \geqslant r_{i_1} + \ldots + r_{i_p} + n(\mathfrak{P}_{21})\epsilon \tag{2.3}$$

whenever $\mathfrak{P}$ is permuted to the form (2.1) and where $n(\mathfrak{P}_{21})$ equals the number of 1's in $\mathfrak{P}_{21}$. But by (2.2) we can choose ϵ small enough so that (2.3) is always satisfied. If $f(u, v)$ denotes the value of the flow f on the arc

[1]We assume for this proof that the reader is familiar with network flows as given in (**3**).

(u, v) of N, then the $m \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = f(x_i, y_j)$ if (x_i, y_j) is an arc of N and $a_{ij} = 0$ otherwise, is seen to be a member of $\mathfrak{P}(R, S)$. This proves the theorem.

Construction.[2] In case the row sum vector $R = (r_1, \ldots, r_m)$ and column sum vector $S = (s_1, \ldots, s_n)$ have rational components, we give a method to construct a matrix in $\mathfrak{P}(R, S)$ when condition (*) is satisfied. This construction can be considered as an alternative proof of the sufficiency of condition (*) in Theorem 2.1. The construction is based on the labelling method (**3**, pp. 17–18) for constructing maximal flows in capacity-constrained networks. We go through two such labelling processes. The first constructs a matrix $A = [a_{ij}]$ having row sum vector R and column sum vector S and 0's in the positions where $\mathfrak{P}$ has 0's but possibly also in other positions. The second labelling process then seeks to create positive elements in those positions of A which have 0's where $\mathfrak{P}$ has 1's without destroying the row sum vector R and column sum vector S.

I. If (i, j) is a position where $\mathfrak{P}$ has a 0, we define $a_{ij} = 0$. Select a position (i, j) where $\mathfrak{P}$ has a 1 and define $a_{ij} = \min\{r_i, s_j\}$. If $r_i < s_j$, we define $a_{ik} = 0$ for all $k \neq j$; if $r_i > s_j$, we define $a_{lj} = 0$ for all $l \neq i$; and if $r_i = s_j$, we define $a_{ik} = 0$ for all $k \neq j$ and $a_{lj} = 0$ for all $l \neq i$. We then delete the ith row, jth column or both according to which case occurs above, diminish row and column sums accordingly, and proceed inductively. Suppose we have not succeeded in constructing a matrix with row sum vector R and column sum vector S. Then the matrix A constructed after permuting its rows and columns, has the form

$$\begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix} \tag{2.4}$$

where A_1 is a $p \times q$ non-negative matrix. The first q columns of (2.4) have the desired column sum and the last $m - p$ rows have the desired row sum, while the remaining rows and columns do not. Moreover, the zero pattern $\mathfrak{P}$ (permuted as A is) has 0's in the positions occupied by O. Let r'_i be the ith row sum of (2.4) and s'_j be the jth column sum. Finally call the positions of A for which $\mathfrak{P}$ has a 1 *admissible positions* and all other positions *inadmissible*. We first label those rows of A for which $r'_i < r_i$. If row i is such a row, we assign it the label $(-, \epsilon(i))$ where $\epsilon(i) = r_i - r'_i > 0$. Such rows are now labelled and unscanned. In general select any labelled unscanned row or column, e.g. row l. Then scan row l of A for admissible positions in unlabelled columns. If column j contains an admissible position in row l, we assign column j the label $(l^+, \epsilon(j))$ where $\epsilon(j) = \epsilon(l)$ if $s_j = s'_j$ and $\epsilon(j) = \min\{\epsilon(i), s_j - s'_i\}$ if $s_j > s'_j$. Row l is now labelled and scanned. If column j is a labelled, unscanned column, we scan the column for positions of A which have positive

[2]This construction is along lines suggested to the writer by D. R. Fulkerson.

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entries in unlabelled rows. If row k is such a row, we assign it the label $(j^-, \epsilon(k))$ where $\epsilon(k) = \min\{\epsilon(j), a_{kj}\}$. We repeat this general procedure until either some column p for which $s'_p < s_p$ is labelled and unscanned, or until no more labels are possible and no such columns are labelled. We show that the latter caes is impossible (that is, violates condition (*)). Suppose we have not succeeded in labelling some column p for which $s'_p < s_p$. Then permute rows and columns of A so that the labelled rows are among the first row positions and the labelled columns are among the first column positions:

$$\begin{array}{r c} \text{lab.} & \multirow{2}{*}{$\begin{bmatrix} A'_1 & O_1 \\ O_2 & A'_2 \end{bmatrix}$} \\ \text{unlab.} & \\ & \begin{array}{cc} \text{lab.} & \text{unlab.} \end{array} \end{array}$$

where A'_1 is formed from rows $i_1, \ldots, i_u$ and columns $j_1, \ldots, j_v$ of A. Since no more labels are possible, the positions of O_1 are all inadmissible and thus O_1 consists of all 0's. Also the positions of O_2 may be admissible, but O_2 must consist of all 0's. Hence

$$r_{i_1} + \ldots + r_{i_u} = s_{j_1} + \ldots + s_{j_v}$$

and this contradicts condition (*). Thus the only possibility is that we have succeeded in labelling a column whose sum is less than that specified. If such a column has a label $(q^+, \epsilon(p))$, then we increase the (q, p) entry of A by $\epsilon(q)$. If the qth row is labelled $(k^-, \epsilon(q))$, then we decrease the (q, p) position of A by $\epsilon(p)$ and so on until we arrive at a row whose sum is less than that specified. Thus we have succeeded in increasing the sum of the entries of the matrix A without destroying these row or column sums which have already the specified value. We repeat the entire labelling procedure. Since the $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ have rational components and thus the constructed matrices have rational entries, after a finite number of applications of the labelling process we construct a non-negative A having the desired row and column sum vectors and 0's in at least those positions where $\mathfrak{P}$ has 0's.

II. Let A be a non-negative matrix with row sum vector R and column sum vector S and 0's in at least those positions where $\mathfrak{P}$ has 0's. Suppose there is an admissible position (i, j) of A which has an entry of 0. Label the row of this position with (j). Row i is now labelled but unscanned. In general select any labelled but unscanned row or column. If row k is labelled but unscanned, scan row k of A for positive entries in unlabelled columns and label such columns by (k). Row k is now labelled and scanned. If column l is labelled but unscanned, scan column l of A for admissible positions in unlabelled rows and label such rows by (l). Column k is now labelled and scanned. Continue this labelling process until the column of the (i, j) position has been labelled or until no further labels are possible, and this column is not labelled. We again show that the latter case violates condition (*). For, if the rows and columns of A are permuted so that A has the form

$$\begin{array}{rcc} \text{lab.} & \left[\begin{array}{c} A''_1 \\ O_2 \end{array}\right. & \left.\begin{array}{c} O_1 \\ A''_2 \end{array}\right] \\ \text{unlab.} & & \\ & \text{lab.} & \text{unlab.} \end{array}$$

Now since no more labels are possible, all positions of O_2 must be inadmissible positions, while all entries of O_1 must be 0's. As before, this contradicts condition (*). Hence we must succeed in labelling the column of position (i, j). Let column j be labelled (i_k), row i_k labelled (j_k), column j_k labelled $(i_{k-1}), \ldots,$ column j_1 labelled (i), and consider the positions

$$(i, j), \quad (i, j_1), \quad (i_1, j_1), \quad \ldots, \quad (i_{k-1}, j_k), \quad (i_k, j_k), \quad (i_k, j).$$

The first, third, etc., of these are admissible positions, while the second, fourth, etc., are positions in which A has a positive entry. Let C be the $m \times n$ matrix having $+1$'s in positions $(i, j), (i_k, j_k), \ldots, (i_1, j_1)$, -1's in positions $(i, j_k), \ldots, (i_2, j_1), (i_1, j)$, and 0's elsewhere. Then we may choose ϵ sufficiently small (e.g. one half the minimum of the indicated positive entries) so that $A + \epsilon C$ has row sum vector R, column sum vector S, and has at least one less admissible position having a 0 than A. We may repeat the argument until we construct a matrix in $\mathfrak{P}(R, S)$.

COROLLARY TO THEOREM 2.1 (A. Horn **6**). *Let $d_1, d_2, \ldots, d_n$ be n numbers with $0 \leqslant d_i \leqslant 1$, $i = 1, \ldots, n$. Then a necessary and sufficient condition that $(d_1, d_2, \ldots, d_n)$ be the main diagonal of an $n \times n$ non-negative doubly stochastic matrix is that*

$$\sum_{k=1}^{n} d_k - 2 \min_{1 \leqslant i \leqslant n} d_i \leqslant n - 2.$$

Proof. First observe that we can assume each $d_i < 1$, for otherwise the problem reduces to that for a smaller-order matrix. The corollary then follows from Theorem 2.1 by observing that it is equivalent to finding a non-negative matrix having row and column sum vector $(1 - d_1, \ldots, 1 - d_n)$ and 0's on the main diagonal. (The matrix can be taken to have positive off-diagonal entries.)

3. The class $\mathfrak{P}(R, S)$. Let $\mathfrak{P}$ be a zero pattern such that $\mathfrak{P}(R, S)$ is non-empty. Since R and S are positive vectors, there must be a 1 in every row and column of $\mathfrak{P}$. With the zero pattern $\mathfrak{P}$ we associate a (undirected, bipartite) graph, which on occasion will also be denoted by $\mathfrak{P}$, as follows. Corresponding to row i we introduce a vertex x_i, $i = 1, \ldots, m$, and corresponding to column j we introduce a vertex y_j, $j = 1, \ldots, n$. There is an edge $\{x_i, y_j\}$ joining x_i and y_j if and only if the (i, j) entry of $\mathfrak{P}$ is a 1, and these are the only edges. (The connection between this graph and the directed network defined in § 2 should be noted.) It is easily seen that the pattern $\mathfrak{P}$ is indecomposable if and only if the graph $\mathfrak{P}$ is connected. If the graph $\mathfrak{P}$ is not connected, it splits up into disjoint connected components, say $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$, and thus the pattern $\mathfrak{P}$, upon permutation of its rows and columns, assumes the form

$$\begin{bmatrix} \mathfrak{P}_1 & O & \ldots & O \\ O & \mathfrak{P}_2 & \ldots & O \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ O & O & \ldots & \mathfrak{P}_k \end{bmatrix} \tag{3.1}$$

where $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$ are non-vacuous indecomposable zero patterns corresponding to the connected components of the graph $\mathfrak{P}$. We call (3.1) the *canonical form* of $\mathfrak{P}$ and $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$ the indecomposable components of $\mathfrak{P}$. They are uniquely determined up to permutations of their rows and columns.

It is clear that any cycle of this graph consists of an even number of edges. If γ is a cycle of this graph, we associate with γ two $m \times n$ matrices whose entries are 0, 1, -1 in the following way. If γ consists of the edges $\{x_{i_1}, y_{j_1}\}$, $\{x_{i_2}, y_{j_1}\}$, $\}x_{i_2}, y_{j_2}\}, \ldots, \{x_{jk}, y_{jk}\}$ with $x_{i_1}, \ldots, x_{ik}$ and $y_{j_1}, \ldots, y_{jk}$ distinct, then one of these matrices has a 1 in the (i_1, j_1) position, a -1 in the (i_2, j_1) position, a 1 in the (i_2, j_2) position, ..., a -1 in the (i_1, j_k) position, and 0's elsewhere. The other is the negative of this matrix. If C is either of these matrices and θ is any non-negative number, then obviously θC has all row and column sums equal to 0. Thus adding θC to any matrix A does not alter the row or column sums of A. We shall refer to such matrices θC as *cycle matrices* for the pattern $\mathfrak{P}$. Finally, if A and B are any two matrices in $\mathfrak{P}(R, S)$, we write $A \sim B$ provided there exists a cycle matrix θC such that $B = A + \theta C$. Note that $A \sim B$ if and only if $B \sim A$, and $A \sim A$.

THEOREM 3.1. *Let A and B be any two matrices in $\mathfrak{P}(R, S)$. Then there exist matrices $A_1 = A, A_2, \ldots, A_k = B$, all in $\mathfrak{P}(R, S)$, such that*

$$A = A_1 \sim A_2 \sim \ldots \sim A_k = B.$$

Proof. Let A be different from B. Then there is some position in which A and B both have positive elements and disagree. Thus, for instance, $a_{ij} > b_{ij} > 0$. Since A and B are both in $\mathfrak{P}(R, S)$, there must be a $k \neq i$ such that $0 < a_{kj} < b_{kj}$. Then there must be an $l \neq j$ such that $a_{kl} > b_{kl} > 0$. Continuing in this way, we must eventually return to a previous position. This will then give rise to a cycle $\{x_{i_1}, y_{j_1}\}, \{y_{j_1}, x_{i_2}\}, \ldots$ of the graph $\mathfrak{P}$ such that

$$a_{i_1 j_1} > b_{i_1 j_1} > 0, \qquad 0 < a_{i_2 j_1} < b_{i_2 j_1}, \ldots .$$

Let

$$\theta = \min\{a_{i_1 j_1} - b_{i_1 j_1}, b_{i_2 j_1} - a_{i_2 j_1}, \ldots\}$$

and C be the cycle matrix associated with the above cycle which has a -1 in the (i_p, j_p) position if the minimum θ is assumed at $a_{i_p j_p} - b_{i_p j_p}$ or, if not, a $+1$ in position (i_p, j_q) if the minimum θ is assumed at $b_{i_p j_q} - a_{i_p j_q} > 0$.

Then it is easily verified that $A_2 = A_1 + \theta C \in \mathfrak{P}(R, S)$ and A_2 agrees with B in at least one more position than $A_1 = A$ did. Continuing this process proves the theorem.

THEOREM 3.2. *If $\mathfrak{P}$ is an arbitrary zero pattern, then $\mathfrak{P}(R, S)$ contains either no, precisely one, or infinitely many matrices. If $\mathfrak{P}(R, S)$ is non-empty, then it contains precisely one matrix if and only if the associated graph $\mathfrak{P}$ contains no cycles, that is, the graph is a tree or a forest.*

Proof. By Theorem 3.1 if the graph $\mathfrak{P}$ has no cycles, then there cannot be more than one matrix in $\mathfrak{P}(R, S)$. However, if the graph does have a cycle and C is one of the associated matrices, then $A \pm \epsilon C$ will be in $\mathfrak{P}(R, S)$ for $A \in \mathfrak{P}(R, S)$ and all suitable small positive numbers ϵ.

Suppose $\mathfrak{P}$ is a zero pattern such that $\mathfrak{P}(R, S)$ contains a unique matrix. The graph $\mathfrak{P}$ then consists of, say, p disjoint trees. Since the number of vertices of the graph is $m + n$ and the number N of edges is the number of 1's in $\mathfrak{P}$, then (**1**, p. 27) $N = (m + n) - p$. Since $p \geqslant 1$, we have $N \leqslant m + n - 1$ with equality if and only if the associated graph $\mathfrak{P}$ is a tree.

4. The partially ordered set $L(R, S)$. Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ again be positive vectors with $r_1 + \ldots + r_m = s_1 + \ldots + s_n$. We denote by $L(R, S)$ the collection of all $m \times n$ zero patterns $\mathfrak{P}$ such that $\mathfrak{P}(R, S)$ is non-empty. The class $L(R, S)$ is always non-empty for if $\mathfrak{J}$ denotes the $m \times n$ zero pattern consisting of all 1's, then $\mathfrak{J}(R, S)$ is non-empty for (see **7** or **4**, p. 19) the matrix $A = [a_{ij}]$ with

$$a_{ij} = r_i s_j \Big/ \sum_{k=1}^{m} r_k$$

is in $\mathfrak{J}(R, S)$. We define a partial order on $L(R, S)$ as follows. If $\mathfrak{P} \neq \mathfrak{Q}$ are in $L(R, S)$, then $\mathfrak{P} < \mathfrak{Q}$ provided $\mathfrak{P}$ has a 0 in every position that $\mathfrak{Q}$ has a 0. We write $\mathfrak{P} \leqslant \mathfrak{Q}$ provided $\mathfrak{P} < \mathfrak{Q}$ or $\mathfrak{P} = \mathfrak{Q}$. With this definition $L(R, S)$ becomes a partially ordered set.

THEOREM 4.1. *Let $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_k$ be those zero patterns such that $\mathfrak{T}_1(R, S)$ contains a unique matrix $i = 1, \ldots, k$. Then $\mathfrak{T}_1, \ldots, \mathfrak{T}_k$ are the minimal elements of the partially ordered set $L(R, S)$.*

Proof. Let $\mathfrak{P} \in L(R, S)$ and suppose $\mathfrak{P}(R, S)$ contains more than one matrix, so that the associated graph has a cycle γ. Let $A \in \mathfrak{P}(R, S)$ and let $\theta > 0$ be the minimum of the entries of A in the positions corresponding to the edges of the cycle. Let C be the 0, 1, -1 matrix associated with γ which has a -1 in a position where the minimum θ is attained. Then $A - \theta C \in \mathfrak{Q}(R, S)$ with $\mathfrak{Q} < \mathfrak{P}$. We may continue until we obtain a $\mathfrak{T}_j$ with $\mathfrak{T}_j < \mathfrak{P}$.

Conversely, consider the possibility of a $\mathfrak{Q} \in L(R, S)$ with $\mathfrak{Q} < \mathfrak{T}_j$ for some $j = 1, \ldots, k$. Since the graph of $\mathfrak{T}_j$ is a forest and thus a disjoint collection of trees and since trees are minimally connected (removal of any edge disconnects it), it follows that we may permute rows and columns of $\mathfrak{T}$ and $\mathfrak{Q}$ in the same way to obtain

$$\mathfrak{T} = \begin{bmatrix} \mathfrak{T}' & O \\ O & \mathfrak{T}'' \end{bmatrix}, \qquad \mathfrak{Q} = \begin{bmatrix} \mathfrak{Q}' & O \\ O & \mathfrak{Q}'' \end{bmatrix}$$

where $\mathfrak{T}'$ and $\mathfrak{Q}'$ are non-vacuous $e \times f$ zero patterns with the graph of $\mathfrak{T}'$ a tree and with possible $\mathfrak{T}''$ (and thus $\mathfrak{Q}''$) being vacuous, and where $\mathfrak{Q}'$ is obtained from $\mathfrak{T}'$ by changing one or more 1's of $\mathfrak{T}'$ to 0's. It then follows that the rows and columns of $\mathfrak{T}'$ and $\mathfrak{Q}'$ can be permuted in the same way to obtain

$$\mathfrak{T}' = \begin{bmatrix} \mathfrak{T}'_1 & O \\ \mathfrak{T}'_{12} & \mathfrak{T}'_2 \end{bmatrix}, \qquad \mathfrak{Q}' = \begin{bmatrix} \mathfrak{Q}'_1 & O \\ O & \mathfrak{Q}'_2 \end{bmatrix}$$

where $\mathfrak{T}'_1$ and $\mathfrak{Q}'_1$ are $e' \times f'$ zero patterns and $\mathfrak{T}'_{12}$ does not consist of all 0's. If $i_1, \ldots, i_{e'}$ and $j_1, \ldots, j_{f'}$ are the rows and columns corresponding to $\mathfrak{T}'_1$ and $\mathfrak{Q}'_1$, then since $\mathfrak{T}(R, S)$ is non-empty,

$$s_{j_1} + \ldots + s_{j_{f'}} > r_{i_1} + \ldots + r_{i_{e'}},$$

and since $\mathfrak{Q}(R, S)$ is non-empty,

$$s_{j_1} + \ldots + s_{j_{f'}} = r_{i_1} + \ldots + r_{i_{e'}}.$$

We obviously have a contradiction and thus $\mathfrak{T}_j$ is a minimal element of $L(R, S)$, $j = 1, \ldots, k$.

By definition of the partial order if $\mathfrak{P} \in L(R, S)$ then $\mathfrak{P} \leqslant \mathfrak{J}$, so that $\mathfrak{J}$ is *the* maximal element of the partially ordered set $L(R, S)$. From the preceding theorem it follows that if we extend $L(R, S)$ to $\hat{L}(R, S)$ by an additional element σ and define $\sigma < \mathfrak{P}$ for all $\mathfrak{P} \in L(R, S)$, then $\hat{L}(R, S)$ is a p.o. set with maximal element $\mathfrak{J}$ and minimal element σ. The atoms of the p.o. set are then $\mathfrak{T}_1, \ldots, \mathfrak{T}_k$. As is usual for a p.o. set, $\mathfrak{P}$ is said to *cover* $\mathfrak{Q}$ provided $\mathfrak{Q} \leqslant \mathfrak{R} \leqslant \mathfrak{P}$ implies $\mathfrak{P} = \mathfrak{R}$ or $\mathfrak{Q} = \mathfrak{R}$. The atoms of the p.o. set $\hat{L}(R, S)$ are then the elements that cover the minimal element σ. The following theorem discusses the remaining possibility of an element of $\hat{L}(R, S)$ covering another element different from σ.

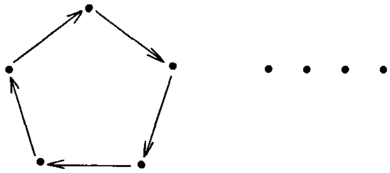
THEOREM 4.2. *Let $\mathfrak{Q} \in L(R, S)$. Then a zero pattern $\mathfrak{P}$ covers $\mathfrak{Q}$ if and only if one of the following is true:*

(1) *$\mathfrak{P}$ is obtained from $\mathfrak{Q}$ by changing a* 0 *in an indecomposable component of $\mathfrak{Q}$ to a* 1.

(2) *If the canonical form of $\mathfrak{Q}$ is*

$$\begin{bmatrix} \mathfrak{Q}_1 & O & \ldots & O \\ O & \mathfrak{Q}_2 & \ldots & O \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ O & O & \ldots & \mathfrak{Q}_k \end{bmatrix}$$

then $\mathfrak{P}$ is obtained from $\mathfrak{Q}$ by changing at most one 0 *of each of the off-diagonal zero blocks O to a* 1 *in such a way that the directed graph with vertices $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k$ and with a directed arc from $\mathfrak{Q}_i$ to $\mathfrak{Q}_j$, $i \neq j$, if and only if the (i, j)-zero block has been changed consists of precisely one directed cycle along with isolated vertices:*



Proof. Let $\mathfrak{P}$ and $\mathfrak{Q}$ satisfy (1). Then since $\mathfrak{Q}(R, S)$ is non-empty, the zero pattern satisfies the condition (*) of Theorem 2.1. But it is easy to verify that if $\mathfrak{Q}$ satisfies (*) so does $\mathfrak{P}$. Hence $\mathfrak{P} \in L(R, S)$ and clearly $\mathfrak{P}$ covers $\mathfrak{Q}$.

Suppose $\mathfrak{P}$ and $\mathfrak{Q}$ satisfy (2). Then after simultaneous permutations of the blocks of $\mathfrak{Q}$ and after relabelling, we may assume $\mathfrak{P}$ has the form

$$\left[\begin{array}{cccc|ccc}
\mathfrak{Q}_1 & \mathfrak{Q}_{12} & \cdots & & & & \\
 & \mathfrak{Q}_2 & \cdots & & & & \\
 & \vdots & & \vdots & & & \\
 & & & \mathfrak{Q}_{l-1,l} & & & \\
\mathfrak{Q}_{l1} & & & \mathfrak{Q}_l & & & \\
\hline
 & & & & \mathfrak{Q}_{l+1} & \cdots & \\
 & & & & \vdots & & \vdots \\
 & & & & & & \mathfrak{Q}_k
\end{array}\right]$$

where each of $\mathfrak{Q}_{12}, \ldots, \mathfrak{Q}_{l-1,1}, \mathfrak{Q}_{l1}$ has precisely one 1 and everything unspecified consists of all 0's. Since $\mathfrak{Q} \in L(R, S)$, $\mathfrak{Q}$ satisfies condition (*) of Theorem 2.1. It is again easy to verify that then $\mathfrak{P}$ must also satisfy (*) and hence $\mathfrak{P} \in L(R, S)$. Suppose one or more (but not all) of these new 1's of $\mathfrak{P}$ were changed back to 0's. Then using the fact that $\mathfrak{Q}$ satisfies (*) we see that this new zero pattern cannot satisfy (*). Hence $\mathfrak{P}$ covers $\mathfrak{Q}$.

Conversely, suppose $\mathfrak{P} \in L(R, S)$ and $\mathfrak{P}$ covers $\mathfrak{Q}$. If $\mathfrak{P}$ has a 1 where some indecomposable component of $\mathfrak{Q}$ has a 0, then the pattern $\mathfrak{P}'$ which differs from $\mathfrak{Q}$ only in this respect is in $L(R, S)$ by the above. Hence $\mathfrak{P} = \mathfrak{P}'$. Hence suppose $\mathfrak{P}$ agrees with $\mathfrak{Q}$ in the positions of the indecomposable components of $\mathfrak{Q}$. Consider the directed graph D with nodes $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k$ such that there is a directed arc from $\mathfrak{Q}_i$ to $\mathfrak{Q}_j$, $i \neq j$, if and only if $\mathfrak{P}$ has a 1 in one or more positions of the (i, j)-zero block of $\mathfrak{Q}$. Suppose D did not have a directed cycle. Then the strong components (**5**, pp. 53–57) of D are precisely the vertices of D. But now it is seen that since $\mathfrak{Q}(R, S)$ is non-empty $\mathfrak{P}$ cannot satisfy condition (*). Hence D must have at least one directed cycle. But then since $\mathfrak{P}$ covers $\mathfrak{Q}$, D must consist of precisely one directed cycle apart from isolated vertices. This completes the proof.

5. The class $\overline{\mathfrak{P}(R, S)}$. Consider the non-empty class $\mathfrak{P}(R, S)$, topologized with the Euclidean topology. Then, in general, $\mathfrak{P}(R, S)$ is not a closed set. In fact $\mathfrak{P}(R, S)$ is closed if and only if $\mathfrak{P}(R, S)$ contains a unique matrix, that is, if and only if $\mathfrak{P}$ is an atom of the p.o. set $\hat{L}(R, S)$. The following theorem determines its closure $\overline{\mathfrak{P}(R, S)}$.

THEOREM 5.1. *If $\mathfrak{P} \in L(R, S)$, then*

$$\overline{\mathfrak{P}(R, S)} = \bigcup_{0 < \mathfrak{Q} \leqslant \mathfrak{P}} \mathfrak{Q}(R, S)$$

where the union is taken over all zero patterns $\mathfrak{Q}$ in $L(R, S)$ with $0 < \mathfrak{Q} \leqslant \mathfrak{P}$.

Proof. First it is clear that

$$\overline{\mathfrak{P}(R, S)} \subset \bigcup_{0 < \mathfrak{Q} \leqslant \mathfrak{P}} \mathfrak{Q}(R, S).$$

To prove the opposite containment it is enough to prove that if $\mathfrak{P}$ covers $\mathfrak{Q}$, then

$$\mathfrak{Q}(R, S) \subset \overline{\mathfrak{P}(R, S)}.$$

Thus let $A \in \mathfrak{Q}(R, S)$ with $\mathfrak{P}$ covering $\mathfrak{Q}$. Hence $\mathfrak{P}$ and $\mathfrak{Q}$ satisfy conditions (1) or (2) of Theorem 4.2. First assume that $\mathfrak{P}$ is obtained from $\mathfrak{Q}$ by replacing a 0 in an indecomposable component of $\mathfrak{Q}$ by a 1. There is no loss of generality in assuming that $\mathfrak{Q}$ itself is indecomposable. Let the position in which $\mathfrak{P}$ and $\mathfrak{Q}$ disagree be the (i, j) position. Consider the graph of $\mathfrak{Q}$ to be embedded in the graph of $\mathfrak{P}$, whose vertices are $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then since the graph of $\mathfrak{Q}$ is connected, we may find a path from x_i to y_j and hence a cycle in the graph of $\mathfrak{P}$ containing the edge $\{x_i, y_j\}$ joining x_i to y_j. Let C be the 0, -1, 1 matrix corresponding to this cycle which has a 1 in the (i, j) position. Then for all sufficiently small positive ϵ,

$$A + \epsilon C \in \mathfrak{P}(R, S).$$

Hence $A \in \overline{\mathfrak{P}(R, S)}$.

Now assume that $\mathfrak{P}$ is obtained from $\mathfrak{Q}$ as indicated in (2). There is no loss of generality in assuming that $\mathfrak{Q}$ has the form

$$\begin{bmatrix} \mathfrak{Q}_1 & O & \ldots & O \\ O & \mathfrak{Q}_2 & \ldots & O \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ O & O & \ldots & \mathfrak{Q}_k \end{bmatrix}$$

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where the $\mathfrak{Q}_i$ are indecomposable and $\mathfrak{P}$ is obtained from $\mathfrak{Q}$ by replacing one 0 by a 1 in the zero blocks in positions $(1, 2,), \ldots, (k-1, k), (k, 1)$. Let the positions of these 0's be $(i_1, i_2), \ldots, (i_{k-1}, j_k), i_k, j_1)$. Then in the associated graph of $\mathfrak{Q}_1$ we may find a path from x_{i_1} to $y_{j_1}, \ldots$, in the associated graph of $\mathfrak{Q}_k$ we may find a path from x_{i_k} to y_{j_k}. Hence in the associated graph of $\mathfrak{P}$ (assuming the previous graphs are embedded in this) we may find a cycle containing the edges $\{x_{i_1}, y_{j_2}\}, \ldots, \{x_{i_{k-1}}, y_{j_k}\}, \{x_{i_k}, y_{j_1}\}$. Let C be the $0, -1, 1$ matrix associated with this cycle which has a 1 in the positions $(i_1, j_2), \ldots, (i_{k-1}, j_k), (i_k, j_1)$ (this is possible since each of the above paths consists of an odd number of edges). Then for all sufficiently small positive ϵ, $A + \epsilon C \in \mathfrak{P}(R, S)$ and hence $A = \overline{\mathfrak{P}(R, S)}$. This completes the proof of the theorem.

Let us extend a previous definition and define for $A, B \in \overline{\mathfrak{P}(R, S)}$ $A \sim B$ provided $A = B + \theta C$ where $\theta \geqslant 0$ and C is a $0, -1, 1$ matrix corresponding to a cycle of the graph $\mathfrak{P}$. The proof of the preceding theorem then shows that if $\mathfrak{P}$ covers $\mathfrak{Q}$, then there is a matrix $A \in \mathfrak{P}(R, S)$ and a matrix $B \in \mathfrak{Q}(R, S) \subset \overline{\mathfrak{P}(R, S)}$ such that $A \sim B$. Combining this with Theorem 3.1 we obtain the following theorem.

THEOREM 5.2. *Let $\mathfrak{P}$, $\mathfrak{Q} \in L(R, S)$ and suppose $\mathfrak{P}$ covers $\mathfrak{Q}$. Let $A \in \mathfrak{P}(R, S)$ and $B \in \mathfrak{Q}(R, S)$. Then there exist matrices*

$$A_1 = A, A_2, \ldots, A_i, A_{i+1}, \ldots, A_j = B$$

with $A_1, \ldots, A_i \in \mathfrak{P}(R, S)$ and $A_{i+1}, \ldots, A_j \in \mathfrak{Q}(R, S)$ such that

$$A = A_1 \sim A_2 \sim \ldots \sim A_i \sim A_{i+1} \sim \ldots \sim A_j = B.$$

COROLLARY 1. *Let $\mathfrak{P} \in L(R, S)$, $A \in \mathfrak{P}(R, S)$, and $B \in \overline{\mathfrak{P}(R, S)}$. Then there exist matrices $A_1 = A, A_2, \ldots, A_k = B$, all in $\overline{\mathfrak{P}(R, S)}$, such that*

$$A = A_1 \sim A_2 \sim \ldots \sim A_k = B.$$

That is, $\overline{\mathfrak{P}(R, S)}$ can be generated by finite sequences of "$\sim$" operations from any matrix $A \in \mathfrak{P}(R, S)$.

Regarded as a subset of Euclidean mn-space, $\mathfrak{P}(R, S)$ is a convex set. Likewise its closure $\overline{\mathfrak{P}(R, S)}$ is convex. In **(7)** Jurkat and Ryser and in **(4)** Fulkerson determine the extreme points of the convex set $\overline{\mathfrak{F}(R, S)}$ (in the notation of **(7)**, this is $\mathfrak{A}(R, S)$). The following theorem characterizes the extreme points for $\overline{\mathfrak{P}(R, S)}$.

THEOREM 5.3. *If $\mathfrak{P} \in L(R, S)$, then the extreme points of the convex set $\overline{\mathfrak{P}(R, S)}$ consist of the unique matrices in the classes $\mathfrak{T}(R, S)$ where $\mathfrak{T}$ ranges over all atoms of the p.o. set $\hat{L}(R, S)$ with $\mathfrak{T} \leqslant \mathfrak{P}$.*

Proof. Suppose $A \in \overline{\mathfrak{P}(R, S)}$. Then by Theorem 5.1, $A \in \mathfrak{Q}(R, S)$ for some $\mathfrak{Q} \in L(R, S)$ with $\mathfrak{Q} \leqslant \mathfrak{P}$. If $\mathfrak{Q}$ is not an atom of the p.o. set $\hat{L}(R, S)$, then the associated graph of $\mathfrak{Q}$ has a cycle. But then for all sufficiently small positive ϵ, $A \pm \epsilon C \in \mathfrak{Q}(R, S) \subset \overline{\mathfrak{P}(R, S)}$ where ϵC are cycle matrices corresponding to a cycle of the graph. But then

$$A = \tfrac{1}{2}(A + \epsilon C) + \tfrac{1}{2}(A - \epsilon C)$$

is a proper convex decomposition of A and hence A is not an extreme point of $\overline{\mathfrak{P}(R, S)}$.

Conversely, suppose A is the unique matrix in a class $\mathfrak{T}(R, S)$ where $\mathfrak{T}$ is an atom and $\mathfrak{T} \leqslant \mathfrak{P}$. Consider the possibility of a convex decomposition of A:

$$A = \alpha A_1 + (1 - \alpha)A_2, \qquad 0 < \alpha < 1,$$

where $A_1, A_2 \in \overline{\mathfrak{P}(R, S)}$. But if $A_1 \in \mathfrak{Q}(R, S)$ with $\mathfrak{Q} \leqslant \mathfrak{P}$, then clearly $\mathfrak{Q} \leqslant \mathfrak{T}$. Since $\mathfrak{T}$ is an atom, $\mathfrak{Q} = \mathfrak{T}$. Thus $A_1 \in \mathfrak{T}(R, S)$ and similarly $A_2 \in \mathfrak{T}(R, S)$. But then $A_1 = A_2 = A$. Hence A is an extreme point of $\overline{\mathfrak{P}(R, S)}$. This completes the proof of the theorem.

As already mentioned, Jurkat and Ryser determine the extreme points of $\overline{\mathfrak{J}(R, S)}$ and do so by a construction process. They then go on to derive characterizing properties of these matrices. They, for instance, show that the extreme points of $\overline{\mathfrak{J}(R, S)}$ are uniquely determined by the positions of their positive elements. Our approach, besides being more general, relates the extreme points of the convex set $\overline{\mathfrak{P}(R, S)}$ with the atoms (which might be regarded as extreme) of the p.o. set $\hat{L}(R, S)$. Everything we have done ultimately rests on the existence theorem of § 2.

To conclude this section we want to show how Theorem 2.1 may be used to give a simpler proof of Jurkat and Ryser's formula for the minimal term rank of the matrices in $\overline{\mathfrak{J}(R, S)}$. The *term rank* $\rho(A)$ of a non-negative matrix A is defined as the maximal number of positive elements of A with no two on a line (row or column). According to the König–Egervary theorem the term rank of A is also equal to the minimum number of lines of A that contain all the positive elements of A. Since the term rank of A is invariant under arbitrary permutations of rows and columns, there is no loss of generality in assuming that in $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ we have $r_1 \leqslant \ldots \leqslant r_m$ and $s_1 \geqslant \ldots \geqslant s_n$.

THEOREM 5.4 (Jurkat and Ryser). *If the components of $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ are assumed in non-decreasing order and non-increasing order respectively, then the minimal term rank $\tilde{\rho}$ of the matrices in $\overline{\mathfrak{J}(R, S)}$ is given by*

$$\tilde{\rho} = \min\{m - e + f\} \tag{5.1}$$

where the minimum is taken over all pairs (e, f) with $0 \leqslant e \leqslant m$, $0 \leqslant f \leqslant n$, and

$$s_1 + \ldots + s_f \geqslant r_1 + \ldots + r_e. \tag{5.2}$$

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Proof. Consider such an e and f satisfying (5.2). Then the $m \times n$ zero pattern

$$\mathfrak{P} = \begin{bmatrix} \mathfrak{J}_1 & O \\ X & \mathfrak{J}_2 \end{bmatrix},$$

where $\mathfrak{J}_1$ and $\mathfrak{J}_2$ consist of all 1's, $\mathfrak{J}_1$ being $e \times f$, and where X is either all 1's or all 0's depending on whether inequality or equality occurs in (5.2) for this e and f, satisfies condition (*) of Theorem 2.1 and thus $\mathfrak{P}(R, S)$ is non-empty. But if $A \in \mathfrak{P}(R, S)$, then the term rank of A is less than or equal to $m - e + f$ by the König–Egervary theorem. Hence $\tilde{\rho} \leqslant \min \{m - e + f\}$. Conversely, let $A \in \overline{\mathfrak{J}(R, S)}$ and let rows $i_1, \ldots, i_{m-e}$ and columns $j_1, \ldots, j_f$ contain all the positive elements of A with $m - e + f$ equal to the term rank of A. Then permuting rows and columns of A so that rows $i_1, \ldots, i_{m-e}$ are among the last $m - e$ rows and columns $j_1, \ldots, j_f$ are among the first f columns, we obtain

$$\begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix},$$

where A_1 is an $e \times f$ matrix. But then

$$s_{j_1} + \ldots + s_{j_f} \geqslant r_{i_1} + \ldots + r_{i_e}.$$

But by the assumed monotonicities, $s_1 + \ldots + s_f \geqslant s_{j_1} + \ldots + s_{j_f}$ and $r_{i_1} + \ldots + r_{i_e} \geqslant r_1 + \ldots + r_e$. Hence $s_1 + \ldots + s_f \geqslant r_1 + \ldots + r_e$ and

$$\tilde{\rho} = \min \{m - e + f\},$$

and this proves (5.1).

We remark that our expression for $\tilde{\rho}$, although a little different in appearance, is entirely equivalent to that obtained in **(7)**.

As remarked in **(7)** the maximal term rank $\bar{\rho}$ of matrices in $\overline{\mathfrak{J}(R, S)}$ is always given by $\bar{\rho} = \min \{m, n\}$ and, of course, is attained for all matrices in $\mathfrak{J}(R, S)$. Let ρ be an integer with $\tilde{\rho} \leqslant \rho \leqslant \bar{\rho}$. Then there exists a matrix in $\overline{\mathfrak{J}(R, S)}$ whose term rank is equal to ρ. For if e and f are integers for which the minimum occurs in (5.1), then from $s_1 + \ldots + s_f \geqslant r_1 + \ldots + r_e$ we conclude that $s_1 + \ldots + s_f \geqslant r_1 + \ldots + r_{e-1}$. Hence by Theorem 2.1, the zero pattern

$$\mathfrak{Q} = \begin{bmatrix} \mathfrak{J}_1 & O \\ X & \mathfrak{J}_2 \end{bmatrix}$$

where $\mathfrak{J}_1$ and $\mathfrak{J}_2$ consist of all 1's, $\mathfrak{J}_1$ being $(e - 1) \times f$ is such that $\mathfrak{Q}(R, S)$ is non-empty. If $A \in \mathfrak{Q}(R, S)$, then $\rho(A) = m - e + 1 + f = \tilde{\rho} + 1$. Note that if $e = 0$, then $\bar{\rho} = \tilde{\rho}$ and there is nothing to prove. By repeating the above argument the desired result follows.

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