NUTATION AND THE OPPOLZER'S TERMS

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Two fundamental frames of reference are used in the study of the rotation of the Earth: the nonrotating celestial coordinate system XYZ attached to the directions to stars and/or extragalactic sources, and the terrestrial coordinate system xyz attached in "a prescribed way" to several points (observatories) on the surface of the Earth or to the pencil of unit vectors drawn from an arbitrary origin parallel to the local verticals at these points.

We may write

$$(x,y,z)^T = \underset{\sim}{M}(t)\cdot(X,Y,Z)^T \quad , \tag{1}$$

where $\underset{\sim}{M}(t)$ is the transformation matrix. Numerical values of its elements should be known for any moment t for the transformation (1) to be possible. Strictly speaking, the motion of the axes xyz relative to the XYZ system is unpredictable because of perturbations of the Earth's rotation by geophysical phenomena. It is the responsibility of the time and latitude services to monitor this motion by means of regular observations which allow the elements of the matrix $\underset{\sim}{M}(t)$ to be calculated for any past moment.

It seems convenient to introduce an intermediate frame of reference which would meet the following requirement: its rotation should approximate as close as possible that of the Earth and at the same time be precisely predictable. There may be a number of such intermediate systems. We shall describe one which seems to be the best choice.

Let $\vec{E}$ be a unit vector normal to the plane of the ecliptic of date and $\vec{H}$ be the angular momentum of the Earth. Designate by $\vec{F}$ a unit vector normal to the vectors $\vec{E}$ and $\vec{H}$ and define the intermediate right-handed system $\xi\eta\zeta$ in the following way: the 0ζ axis is the direction of the angular momentum vector $\vec{H}$; the 0ξ axis rotates around $\vec{H}$. Designate by Φ the angle between the vector $\vec{F}$ and the axis 0ξ. The rate $d\phi/dt$ should be equal to the mean angular velocity of the Earth measured by observations during a certain interval of time. But to calculate

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values of Φ for the future some standard of time independent of the Earth's rotation is to be used. In Woolard's (1953) development of the theory of the Earth's rotation, the nutations $\Delta\psi$ and $\Delta\theta$ which locate the $O\zeta$ axis in the mean equatorial system XYZ are obtained as a result of integrating Poisson's equations. They may be taken from Woolard's Table 24 after removing the small terms in his equation (55).

Then we may write

$$(\xi,\eta,\zeta)^T = \underset{\sim}{N}\cdot(X,Y,Z)^T \quad , \tag{2}$$

where

$$\underset{\sim}{N} = \underset{\sim}{R}_z(\phi)\cdot\underset{\sim}{R}_x(-\theta)\cdot\underset{\sim}{R}_z(\Delta\psi) \quad . \tag{3}$$

So long as the matrix $\underset{\sim}{N}$ is known, the position of the axes $\xi\eta\zeta$ can be calculated for any moment t; predicted values of the angles $\Delta\psi$ and $\Delta\theta$ are published in the Astronomical Ephemeris. For this reason the system $\xi\eta\zeta$ may be called the terrestrial ephemeris system.

There are two ways of transforming from this system to the conventional terrestrial system xyz:

(i) To add the perturbations $\delta\psi$, $\delta\theta$, $\delta\Phi$ to the Euler angles ψ, θ, Φ;

(ii) To rotate the $\xi\eta\zeta$ axes through small angles u, v, w shown in Fig. 1 to make them coincide with the axes xyz.

In the first case we have

$$(x,y,z)^T = (\underset{\sim}{N}+\delta\underset{\sim}{N})\cdot(X,Y,Z)^T \quad , \tag{4}$$

where

$$\delta\underset{\sim}{N} = \frac{\partial\underset{\sim}{N}}{\partial\psi}\,\delta\psi + \frac{\partial\underset{\sim}{N}}{\partial\theta}\,\delta\theta + \frac{\partial\underset{\sim}{N}}{\partial\Phi}\,\delta\Phi \quad . \tag{5}$$

In the second case we may write

$$(x,y,z)^T = (\underset{\sim}{I}+\underset{\sim}{\sigma})\cdot\underset{\sim}{N}\cdot(X,Y,Z)^T \tag{6}$$

where $\underset{\sim}{I}$ is the unit matrix and

$$\underset{\sim}{\sigma} = \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix} \quad . \tag{7}$$

Substituting (3) in (5) and equating the matrix so obtained with $\underset{\sim}{\sigma}\underset{\sim}{N}$ we get nine equations connecting the angles u, v, w with the Euler angles perturbations $\delta\theta$ and $\delta\psi$ sin θ; of these equations we shall have need of the following:

$$v\cos\phi + u\sin\phi = -\delta\psi\sin\theta \quad , \tag{8}$$

$$v\sin\phi - u\cos\phi = \delta\theta \quad . \tag{9}$$

The angles u,v may be replaced by the direction cosines of the angular momentum vector $\vec{H}$ in the system xyz or, in other words, by the coordinates x,y of the end of the unit vector

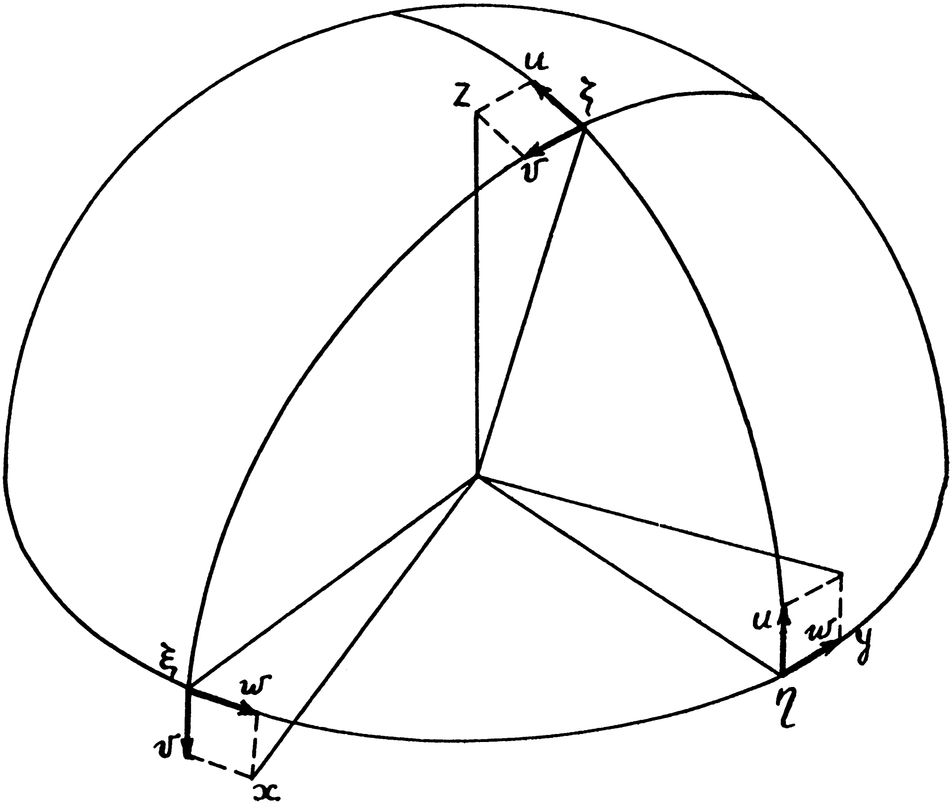

Figure 1

$$\vec{h} = \frac{\vec{H}}{|H|} \quad .$$

It is easy to see from Fig. 1 that

$$x = -v \quad , \quad y = u \quad . \tag{10}$$

Hence

$$-\delta\psi \sin \theta = -x \cos \phi + y \sin \phi \quad , \tag{11}$$

$$\delta\theta = -x \sin \phi - y \cos \phi \quad . \tag{12}$$

The motion of $\vec{h}$ in the system xyz may be broken into two parts. The first is an empirical one that is not predictable from theory and needs to be regularly monitored by observations. The second part is nearly diurnal motion due to the lunisolar torques. Oppolzer was the first to derive this second motion theoretically. Accordingly $\vec{h}$ may be represented as a sum of two vectors, $\vec{h}_1$ and $\vec{h}_2$. Then

$$x = x_1 + x_2 \quad , \quad y = y_1 + y_2 \quad , \quad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}_1 + \underset{\sim}{\sigma}_2 \quad , \tag{13}$$

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$$\underset{\sim}{\sigma}_1 = \begin{pmatrix} 0 & w_1 & -v_1 \\ -w_1 & 0 & u_1 \\ v_1 & -u_1 & 0 \end{pmatrix} . \tag{14}$$

Up to the present the generally adopted practice has been to use transformation (6) to relate the terrestrial axes to the non-rotating frame of reference, because astronomers have believed that for their purposes the axis of figure is not directly of interest. The origin of this belief is in the assertion that astronomical latitudes and longitudes upon the Earth, which are the only geographic coordinates that are directly observable, depend upon the poles of rotation (Woolard, 1953, p. 17).

On the contrary, Jeffreys (1963) has pointed out that the instantaneous axis of rotation does not enter directly into any observation at all and that what the observations really do give is the axis of figure. This idea was supported and developed by Atkinson (1973,1975) and some other authors.

In principle we also adhere to the same point of view, but with the reservation that in the case of a deformable Earth the axis of figure shall be replaced by the Oz-axis of the conventional terrestrial system (in particular, it may be the axis connecting the geocenter with the CIO).

In terms of the present paper this means that the following matrix should be used to transform from the mean equatorial system of date to the conventional terrestrial system:

$$(\underset{\sim}{I}+\underset{\sim}{\sigma}_1)\cdot\underset{\sim}{R}_z(\phi+\delta\phi_2)\cdot\underset{\sim}{R}_x(-\Delta\theta-\delta\theta_2)\cdot\underset{\sim}{R}_z(\Delta\psi-\delta\psi_2) \quad . \tag{15}$$

To compute the Euler angle perturbations $\delta\theta_2$ and $\delta\psi_2$, x_2 and y_2 are to be substituted in (11) and (12) instead of x and y. Note that $\delta\psi_2$ is subtracted from $\Delta\psi$ because it is reckoned positive to the east while nutation is traditionally reckoned positive to the west.

The x_2, y_2-components of the unit vector $\underset{\sim}{h}_2$ are the sums of periodic terms called the Oppolzer terms. McClure's (1973) Table 8-1 contains coefficients of 135 terms in both of these sums. Terms are called symmetrical when their arguments may be represented as $\phi + \pi + \beta$ and $\phi + \pi - \beta$ where β denotes a linear combination of Brown's fundamental arguments. Two symmetrical terms are combined to form

$$\begin{aligned} x_2 &= A_+ \sin(\phi+\beta) + A_- \sin(\phi-\beta) \quad , \\ y_2 &= A_+ \cos(\phi+\beta) + A_- \cos(\phi-\beta) \quad . \end{aligned} \tag{16}$$

Hence

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$$-\delta\psi_2 \sin\theta = (-A_+ + A_-)\sin\beta \quad ,$$
$$\delta\theta_2 = (-A_+ - A_-)\cos\beta \quad . \tag{17}$$

Further we have

$$\delta\phi = x_2 \cos\lambda + y_2 \sin\lambda \quad ,$$

$$\delta t = (x_2 \sin\lambda - y_2 \cos\lambda)\tan\phi$$

where λ is the longitude reckoned positive eastward. Thus the transition from the angular momentum $\vec{H}$ to the Oz axis of the conventional terrestrial system results in the following variations of the latitude and time obtained from observations:

$$\delta\phi = \delta\psi_2 \sin\theta \cos S - \delta\theta_2 \sin S \quad , \tag{18}$$

$$\delta t = (\delta\psi_2 \sin\theta \sin S + \delta\theta_2 \cos S)\tan\phi \quad , \tag{19}$$

where S is the local sidereal time. The expressions for $\delta\theta_2$ and $\delta\psi_2 \sin\theta$ from (17) are substituted in (18) and (19) to obtain

$$\delta\phi = A_+ \sin(S+\beta) + A_- \sin(S-\beta) \quad , \tag{20}$$

$$\delta t = -[A_+ \cos(S+\beta) + A_- \cos(S-\beta)]\tan\phi \quad . \tag{21}$$

The lengthening of the period of the free nutation is known to be the most noticeable manifestation of the effect of the Earth's deformation on its rotation. The influence of this same deformation on forced nutation was considered by Schweydar (1916), Fedorov (1963), and McClure (1973). The elastic deformation has been shown not to affect the motion in space of the angular momentum vector $\vec{H}$ of the whole Earth. Moreover the equations of its motion remain practically unchanged for any assumption about the interior of the Earth. The reason is that the tide-producing body stretches the Earth along the line connecting its center with that of the Earth. Such a deformation does not change the moment of the couple exerted on the Earth.

Thus the introduction of the system related to the angular momentum vector $\vec{H}$ as an intermediate ephemeris system has an advantage in the exposition of the theory of the rotation of the deformable Earth. Concerning this matter Jeffreys (1963) writes: "Fedorov is certainly right in maintaining that if we want an intermediate standard of reference independent of all properties of the Earth other than its moments of inertia, that standard must be the axis of angular momentum."

From what has been said it is clear that for computing nutation of the angular momentum vector we may continue to use the theory of the rotation of the rigid Earth. At the same time tidal deformation results in considerable changes in the relative positions of the axis Oz and the axis of the greatest moment of inertia often called the axis of figure: the amplitude of the variation in the angle between these axes reaches two seconds of arc. That's why the axes of inertia are inappropriate as a terrestrial frame of reference. The motion of these axes in the xyz system affects the motion of the vector $\vec{H}$ (but not of the axis of rotation) in the same system. The elastic deformation of the Earth leads to nearly the same relative diminution of the coefficients of all the terms induced by the tide-generating force, in as much as the factor $(1 - K/K_s) \approx 2/3$ appears throughout. Here K and K_s are the tidal-effective and secular Love numbers.

At the same time the suggestion to use the system xyz as an intermediate system has met with some criticism, mainly because "there is no way at all of observing the angular momentum" (Jeffreys, 1963). However, the mean of two positions of the axis of rotation taken 12 hours apart approximates the position of the angular momentum $\vec{H}$. Thus observations of the same star when it crosses the meridian above and below the pole are capable of giving the angle between the direction of $\vec{H}$ and the direction to the observed star (Fedorov *et al.*, 1972).

Fundamental declinations and latitudes obtained from upper and lower transit observations pertain to the angular momentum axis rather than to the instantaneous axis of rotation. Time and latitude services observe the motion of stars relative to the terrestrial frame of reference xyz, which enables the motion of the vector $\vec{H}$ to be obtained in the same frame (Fedorov, 1975). Several attempts were made to derive from observations the coefficients of the principal, semiannual and fortnightly terms of nutation. In the remainder of this paper, we shall be concerned with these terms only.

Table 1 comprises the coefficients of circular motions of the rotational axis, the axis of figure and the angular momentum in the system xyz for both the rigid and deformable (elastic) Earth. The coefficients are taken from McClure's Tables 9-1, 9-2. Note that the coefficients for the rigid Earth differ slightly from those for a deformable Earth. In Table 2 the perturbations $\delta\theta$ and $\delta\psi \cdot \sin\theta$ are given in the sense "the Euler angles of the angular momentum vector minus those of the z-axis of the conventional terrestrial system." Ultimately this system is to be related to the nonrotating axes XYZ, and in particular to the mean equatorial axes of date. This may be achieved by adding the values of $\delta\theta$ and $\delta\psi \cdot \sin\theta$ to the adopted nutations of the angular momentum, which are independent of the internal structure of the Earth, and by subsequent rotation through the small angles u,v,w.

If the Earth were an elastic body from its center to the surface the analysis of latitude and time observations would detect the following leading short-periodic terms:

$$\delta\phi = -0''.0020 \cdot \sin(S-2L') - 0''.0046 \sin(S-2L) \quad ,$$

$$\delta t = [0''.0020 \cos(S-2L') + 0''.0046 \cos(S-2L)] \tan\phi \quad .$$

Table 1.

Term	Arg.	Rotation axis		Axis of figure	Angular momentum	
		Rigid Earth	Deform. Earth	Deform. Earth	Rigid Earth	Deform. Earth
Principal	Ω	0.″00118 -0.00017	0.″00118 -0.00017	0.″11100 0.01623	0.″00117 -0.00017	0.″00081 -0.00012
Semi-annual	2L'	-0.00290 0.00012	-0.00290 0.00012	0.27186 -0.01171	-0.00290 0.00012	-0.00200 0.00008
Fortnightly	2L	-0.00667 0.00025	-0.00652 0.00025	0.58241 -0.00251	-0.00665 0.00025	-0.00459 0.00017

Ω = longitude of the mean ascending node of the lunar orbit on the ecliptic.

L' = solar mean longitude.

L = lunar mean longitude.

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Table 2.

Term	Period days	Tidal code number	Arg.	$\delta\theta$ Rigid Earth	$\delta\theta$ Deform. Earth	$\delta\psi \cdot \sin\theta$ Rigid Earth	$\delta\psi \cdot \sin\theta$ Deform. Earth
Principal	6798	165.565 165.545	Ω	$-0\overset{''}{.}00100$	$-0\overset{''}{.}00069$	$-0\overset{''}{.}00135$*	$-0\overset{''}{.}00093$
Semi-annual	183	163.555 167.555	2L'	0.00277	0.00193	0.00302	0.00208
Fortnightly	13.7	145.555 185.555	2L	0.00640	0.00442	0.00690	0.00476

*These terms in McClure's Tables 8-3, 8-4, 9-3 and 9-4 are given erroneously with the positive sign.

Table 3.

	Nutation in obliquity			Nutation in longitude		
	Ω	2L'	2L	Ω	2L'	2L
Nutations for the angular momentum axis:	9″.2272	0″.5522	0″.0885	-6″.8733	-0″.5067	-0″.0812
Perturbations for Oz axis:						
for rigid Earth	-0.0010	0.0028	0.0064	0.0014	-0.0030	-0.0069
for elastic Earth	-0.0007	0.0019	0.0044	0.0009	-0.0028	-0.0048
Observed:	-0.0298	0.0260	0.0080	0.0296	-0.0263	-0.0122

These equations result from (18), (19) and the data of Table 3. Formerly we obtained (Fedorov, 1963)

$$\delta\phi = -0''.0022 \sin(S-2L') - 0''.0051 \sin(S-2L) \quad .$$

Now we shall turn to the case of the Earth composed of a mantle and a liquid core. In this case the conception of the rotational axis of the whole Earth loses its sense but that of the angular momentum vector remains. Nutation of this vector will still be the same as in the case of a rigid Earth, and for descriptive purposes the conventional terrestrial system xyz may be considered as attached to the mantle.

The effect of the liquid core manifests itself in the Euler angle perturbations $\delta\theta_2$ and $\delta\psi_2 \cdot \sin\theta$ which may be deduced theoretically. This already has been done for several models of the Earth, but the discussion of the results so obtained is out of the scope of the present paper.

However for estimating the perturbations $\delta\theta_2$ and $\delta\psi_2 \cdot \sin\theta$ we may use another way, namely the comparison of nutations of the angular momentum vector $\vec{H}$ obtained in the theory of the rotation of the rigid Earth with nutations of the z-axis derived from the analysis of observations. For this comparison we use our previous results (Fedorov, 1963), though it is hoped that after discussing some new determinations more reliable and precise values will be obtained.

The perturbations $\delta\theta_2$ and $\delta\psi_2 \cdot \sin\theta$ given in Table 3 should be added to the coefficients of nutation of the angular momentum vector in obliquity and longitude to obtain coefficients of nutation of the axis Oz.

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