FACTORING A GROUP AS AN AMALGAMATED FREE PRODUCT

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Even if in a decomposition of a group

$$G = \Pi^*(\{A_i,\ i \in I\};B)$$

the A_i are completely indecomposable, there may be another decomposition

$$G = \Pi^*(\{C_j,\ j \in J\};D)$$

with each C_j properly contained in some A_i and D a proper subgroup of B. The example of Bryce ([1], p. 636) may be modified, at the cost of having one $A_i = B$, so that $I = J$ and $C_i < A_i$ for all i. It is our object to study this relationship between decompositions of a group.

In section 1 notation is introduced and an example of Stallings is expanded. In section 2 machinery motivated by the Van Kampen Theorem is constructed to show that the problems arising in section 1 may not be insurmountable. Section 3 contains an application of this machinery to extend a theorem of Holmes concerning lattices of subgroups.

1. Decompositions of a group

All results here apply to free products of arbitrarily many groups with a single amalgamated subgroup. To simplify notation, results are stated for a product with only two factors. The extension to more factors is immediate except for Theorems 5 and 6, which require slight rephrasing.

Let G be a group and suppose G is the free product of its subgroups A and B with the subgroup C amalgamated. Then we write $G = (A*B;C)$. The subgroup of G generated by sets $R,S,T,\cdots$ and elements $a,b,c,\cdots$ will be denoted $(R,S,T,\cdots, a,b,c,\cdots)$. The group generated by elements $x,y,\cdots$ with relations $u = v,\cdots$, will be denoted $(x,y,\cdots \mid u = v,\cdots)$. Whether (x,y) is the free group. or a subgroup of a group G, will be clear by the context.

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DEFINITION 1. Suppose $G = (A*B; C)$ and $G = (A'*B'; C')$, with $A \leqq A'$, $B \leqq B'$, $C \leqq C'$. Then the first decomposition is called *finer* than the second, and the second *coarser* than the first. Among the immediate questions are: When can a decomposition be made finer? Are there minimal decompositions?

LEMMA 1. *Let* $G = (A*B; C) = (A'*B'; C')$ *with* $A \leqq A', B \leqq B', C \leqq C'$. *If* $A \neq A'$ *or* $B \neq B'$, *then* $C \neq C'$, *and conversely.*

PROOF. Since $C = A \cap B$ and $C' = A' \cap B'$, it is clear that $C \neq C'$ implies $A \neq A'$ or $B \neq B'$. In the forward direction, suppose

$$A \cap B = C = C' = A' \cap B',$$

and suppose $x \in B' \setminus B$. Now $x \notin A'$ (otherwise $x \in C' = C \subset B$) so $x \notin A$. Thus x may be written as

$$x = cg_1 g_2 \ldots g_n, \; n \geqq 2$$

by the well known theorem of Schreier (cf. [3], p. 205), with $c \in C$ and g_i alternately in $A \leqq A'$ and $B \leqq B'$, no $g_i \in C$. But in $G = (A'*B'; C')$ this word is reducible to $x \in B'$; hence some $g_i \in C'$, contradicting $C' = C$.

COROLLARY 1. *Let* $G = (A*B; \{1\})$ *be a free product. Then there is no strictly finer factorization of* G.

COROLLARY 2. *Let* $G = (A*B; C)$ *and suppose* C *is the center of* G. *Then there is no strictly finer factorization of* G.

PROOF. If $G = (A'*B'; C')$ with $C' \leqq C$, we have $C \leqq C'$ since the amalgamated subgroup must contain the center.

Unfortunately, it is not true that $G = (A*B; \{1\})$ is finer than any other decomposition of G, or even that any decomposition of a free product can be refined to be a free decomposition. This fact was pointed out by Stallings in [5] with the following example.

EXAMPLE 1. (Stallings). Letting (x,y) denote the free group on x and y,

$$(x,y) = ((x, y^2x^2yx^{-2}y^{-2}) * (x^2, y^2); (x^2, y^2x^2y^2x^{-2}y^{-2})).$$

Further, no decomposition of (x,y) finer than this one is free.

Stallings proves the second assertion by observing that if $(x,y) = (A*B; \{1\})$ with $A \leqq (x, y^2x^2yx^{-2}y^{-2})$ and $B \leqq (x^2, y^2)$, $A \cap B = \{1\}$ and the subgroup generated by $A \cup B$ cannot contain y. He produces an isomorphism to show that the amalgamated free product in question is actually (x,y); this may be done more routinely using Tietze transformations ([3], pp. 48ff). Denoting x, $y^2x^2yx^{-2}y^{-2}$, x^2, y^2 by x,b,c,d, the given decomposition may be written

$$\begin{aligned} G &= ((x,b) * (c,d); (x^2 = c, b^2 = dcdc^{-1}d^{-1})) \\ &= (x,b,c,d \mid x^2 = c, b^2 = dcdc^{-1}d^{-1}) \\ &= (x,b,c,d,y \mid c = x^2, b^2 = dx^2dx^{-2}d^{-1}, y = x^{-2}d^{-1}bdx^2) \\ &= (x,b,d,y \mid b^2 = dx^2dx^{-2}d^{-1}, b = dx^2yx^{-2}d^{-1}) \\ &= (x,b,d,y \mid b = dx^2yx^{-2}d^{-1}, dx^2dx^{-2}d^{-1} = dx^2y^2x^{-2}d^{-1}) \\ &= (x,d,y \mid d = y^2) \\ &= (x,y). \end{aligned}$$

This shows that (x,y) has a decomposition which fails in an essential way to be free. Can such a decomposition of a free group be minimal? For example, can any refinement of Stallings' decomposition be minimal? I am unable to settle this question, but the following example seems to argue for the negative:

Example 2. Let $A_1 = (x, y^2x^2yx^{-2}y^{-2})$, $B = (x^2, y^2)$, $C_1 = A_1 \cap B$. If $A_n = (a_n, b_n)$, let $A_{n+1} = (b_n^2a_nb_n^{-2},\ a_n^2b_na_n^{-2})$ and $C_{n+1} = A_{n+1} \cap B$. Then $(x,y) = (A_n * B; C_n)$ for all n, and $A_{n+1} < A_n$, $C_{n+1} < C_n$ for all $n \geqq 1$.

The fact that $A_{n+1} < A_n$ is immediate, by consideration of word length. $C_{n+1} \leqq C_n$ follows since $C_n = A_n \cap B$, and $C_n \neq C_{n+1}$ will follow from Lemma 1 once we have shown $(x,y) = (A_n * B; C_n)$ for all n. To do this, we apply an induction to the following statement:

Whenever a group G is the free product of free subgroups (a,b) and (c,d) with amalgamated subgroup $(a,b) \cap (c,d)$ generated by $a^2 = W(c,d)$ and $b^2 = V(c,d)$ (where W and V are words in c and d), then G is also the free product of its free subgroups (b^2ab^{-2}, a^2ba^{-2}) and (c,d) with amalgamated subgroup $(b^2ab^{-2}, a^2ba^{-2}) \cap (c,d)$ generated by $b^2a^2b^{-2} = VWV^{-1}$ and $a^2b^2a^{-2} = WVW^{-1}$.

That the intersection is correct may be proven by counting exponents. That the second amalgamated free product is in fact G may be proven by using Tietze transformations:

$$\begin{aligned} G &= (r,s,c,d \mid r^2 = VWV^{-1}, s^2 = WVW^{-1}) \qquad W = W(c,d),\ V = V(c,d) \\ &= (r,s,c,d,a,b \mid r^2 = VWV^{-1}, s^2 = WVW^{-1}, a = V^{-1}rV, b = W^{-1}sW) \\ &= (r,s,c,d,a,b \mid r = VaV^{-1}, s = WbW^{-1}, a^2 = W, b^2 = V) \\ &= (a,b,c,d \mid a^2 = W, b^2 = V) \\ &= ((a,b) * (c,d);\ (a^2 = W, b^2 = V)). \end{aligned}$$

This establishes the assertion of Example 2. It is worth noting that the intersection of all the A_n is $\{1\}$, and that B alone does not generate (x,y). Thus, we have constructed a descending chain with no lower bound in the set of decompositions of (x,y).

2. General theorems

The following theorem, motivated by the Van Kampen Theorem by way of [5], and proven in [4] (Theorem 3.2.2), gives a way of going from a finer to a coarser decomposition.

THEOREM 1. *Let* $G = (A * B; C)$. *Let* S *be a well ordered subset of* G *such that if* $s \in S$,

$$s \in (A, \{r \in S \mid r < s\}) \cup (B, \{r \in S \mid r < s\}).$$

In particular, the first element of S *lies in* $A \cup B$. *Then* $G = ((A,S) * (B,S); (C,S))$.

The basic result of this section is a converse of Theorem 1; namely that *any* decomposition coarser than a given one may be obtained by adjoining such a set S to each factor.

LEMMA 2. *Let* $G = (A * B; C) = (A' * B', C'), A \leqq A', B \leqq B', C < C'$. *Then either* $A \cap (C' \setminus C)$ *or* $B \cap (C' \setminus C)$ *is nonempty.*

PROOF. Let $h \in C' \setminus C$, so that $h = cg_1 \cdots g_n$ with the g_1 alternately in $A \setminus C$ and $B \setminus C$. The g_1 are alternately in A' and B', and $h \in C'$ so some $g_i \in C'$. Since this g_i is in $A \setminus C$ or in $B \setminus C$, we are done.

THEOREM 2. *Let* $G = (A * B; C) = (A' * B'; C')$, $A \leqq A'$, $B \leqq B'$, $C < C'$. *Then there is a well-ordered subset* S *of* C' *such that if* $s \in S$,

$$s \in (A, \{r \in S \mid r < s\}) \cup (B, \{r \in S \mid < s\}),$$

and such that A' *is generated by* $A \cup S$, B' *by* $B \cup S$ *and* C' *by* $C \cup S$.

PROOF. Lemma 2 yields a first element s_1 for S. Using it, write

$$G = ((A,s_1) * (B,s_1);\ (C,s_1))$$

by Theorem 1. Apply Lemma 2 to this decomposition to find s_2. Continue by transfinite induction to build up $S = \{s_1, s_2, \cdots\}$ (The limit steps are routine, since each intermediate S satisfies the hypotheses for Theorem 1) until $(C,S) = C'$, which happens after a number of steps not exceeding the cardinality of $C' \setminus C$. Now by Theorem 1,

$$G = ((A,S) * (B,S);\ (C,S)).$$

Since $A \leqq A'$ and $S \leqq C' \leqq A'$, $(A,S) \leqq A'$; similarly $(B,S) \leqq B'$. Hence by Lemma 1, $(A,S) = A'$ and $(B,S) = B'$.

As previously noted, the structure of the set of decompositions of G is unpleasant; there are descending chains with no lower bounds, and pairs of decompositions with no common refinement. If we stay away from the bottom, however, we can find the following structure;

THEOREM 3. *Suppose* $G = (G_1 * G_2; G_0)$, *and consider the set of all decompositions of* G *coarser than this one. Any two elements of this set have a greatest lower bound and a least upper bound in the set.*

PROOF. Suppose $G = (G_1^1 * G_2^1;\ G_0^1)$ and $(G_1^2 * G_2^2;\ G_0^2)$ are the decompositions. We first construct a least upper bound. By Theorem 2, there is $S \subset G_0^1$

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with $G_i^1 = (G_i, S)$, $i=0,1,2$. Now $G = (G_1^2 * G_2^2 \colon G_0^2)$ and S satisfy the hypotheses of Theorem 1, so

$$G = ((G_1^2,S) * (G_2^2,S);\ (G_0^2,S)).$$

That this is the desired least upper bound is clear since

$$(G_i^2,S) = (G_i^2,G_i,S) = (G_i^2,G_i^1)$$

is the smallest subgroup of G containing $G_i^2 \cup G_i^1$.

We now construct a greatest lower bound. Denote $G_0^1 \cap G_0^2$ by B. We now construct a set S by transfinite induction. Let

$$S_1 = (G_1 \cup G_2) \cap B$$

and well-order it arbitrarily.

Let

$$S_{\lambda+1} = ((G_1,S_\lambda) \cup (G_2,S_\lambda)) \cap B \text{ for } \lambda \geqq 1;$$

retain the order on $S_\lambda \subset S_{\lambda+1}$ and well-order the new elements arbitrarily to follow them. When this process terminates, denote the final S_λ (which is the union of all all the S_λ) by S_0. Now

$$G = ((G_1,S_0) * (G_2,S_0);\ (G_0,S_0))$$

is a lower bound for the original factorizations since $G_i \leqq G_i^j$ and $S_0 \subset B \leqq G_0^j \leqq G_i^j$. To show this is the greatest lower bound, suppose $G = (H_1 * H_2; H_0)$ is any other lower bound coarser than $G = (G_1 * G_2, G_0)$. By Theorem 2, $H_0 = (G_0,S)$ and

$$S = S^1 \cup S^2 \cup \cdots \cup S^\lambda \cup \cdots,$$

where

$$S^1 \subset (G_1 \cup G_2) \cap B = S_1$$
$$S_2 \subset ((G_1,S^1) \cup (G_2,S^1)) \cap B \subset ((G_1,S_1) \cup (G_2,S_1)) \cap B = S_2$$

and by transfinite induction $S^\lambda \subset S_\lambda$, so $S \subset S_0$. Hence $H_i = (G_i,S) \subset (G_i,S^0)$ and our lower bound is coarser than any other.

3. Products with normal amalgamation

This section is due in large part to conversations with Professor Charles S. Holmes, in some of which Professor L. M. Sonneborn participated.

Let $G = (A * B;C)$. Clearly C is normal in G if and only if it is normal in A and in B. Now suppose C is not normal in G. Then C^A (the normal closure of C in A) or C^B is not C. Denote $C^A \cup C^B$ by S_1 and well-order it arbitrarily. By Theorem 1,

$$G = ((A,S_1) * (B,S_1);(C,S_1)).$$

If (C,S_1) is not normal in G, repeat the process; $S_2 = (C,S_1)^{A'} \cup (C,S_1)^{B'}$, where $A' = (A,S_1)$ and $B' = (B,S_1)$. $S_1 \subset S_2$, so we can well-order S_2 so that elements of $S_2 \setminus S_1$ follow elements of S_1. Define S_λ by transfinite induction, and let S be the (suitably ordered) union of the S_λ. Now

$$G = ((A,S * (B,S);(C,S))$$

where (C,S) is simply C^G, the normal closure of C in G. Since $(A,S) = (A,C,S) = (A,C^G)$, we have proven:

THEOREM 4. *Let* $G = (A * B;C)$. *Then*

$$G = ((A,C^G) * (B,C^G);C^G).$$

Since C^G is normal in (A,C^G), we may compute its index.

LEMMA 3. $[(A,C^G):C^G] = [A;C^A]$, *and similarly for* B.

PROOF. This may be proven by using the word problem, or by first looking at the isomorphism

$$\frac{(A * B;C)}{C^G} \cong \frac{A}{C^A} * \frac{B}{C^B}$$

from which it is clear that $C^A = A \cap C^G$. Then since C^G is normal in G,

$$\frac{A}{C^A} = \frac{A}{A \cap C^G} \cong \frac{(A,C^G)}{C^G}.$$

We are now able to extend somewhat the following theorem of Holmes [2].

THEOREM 5. *Let* $G = (A * B;C)$, C *normal in* $G, A \neq C \neq B$, $[A:C] > 2$ *or* $[B:C] > 2$. *Then* G *is determined by its lattice of subgroups.*
Our extension is:

THEOREM 6. *Let* $G = (A * B;C)$, $[A;C^A] > 2$, $[B;C^B] \geqq 2$. *Then* G *is determined by its lattice of subgroups.*

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