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A FUNCTION THEORETIC PROOF OF AXLER'S ZERO MULTIPLIER THEOREM

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ABSTRACT. A function theoretic proof of Axler's zero multiplier theorem of Bergman spaces is given.

Let G be an open, connected, nonempty subset of C^N. Let dA be the normalized Lebesgue measure on C^N and w be a positive continuous function on G. For $0 < p \leqq \infty$, we denote by $L^p(G, wdA)$ the usual Lebesgue space. The Bergman space $L^p_a(G, wdA)$ is defined by

$$L^p_a(G, wdA) = \{g \in L^p(G, wdA); g \text{ is analytic in } G\}.$$

We note that $L^\infty_a(G, wdA)$ coincides with the space of bounded analytic functions on G. For $f \in L^p_a(G, wdA)$, put

$$\|f\|_p = \begin{cases} \sup\{|f(z)|; z \in G\} & \text{if } p = \infty \\ \left(\int_G |f|^p wdA\right)^{1/p} & \text{if } 1 \leqq p < \infty \\ \int_G |f|^p wdA & \text{if } 0 < p < 1. \end{cases}$$

Then $L^p_a(G, wdA)$ becomes a complete metric space with the metric defined by $d(f, g) = \|f - g\|_p$ for $f, g \in L^p_a(G, wdA)$.

In [1], Axler showed the following zero multiplier theorem. His paper [1] gives good references for multiplier theorems on Bergman spaces.

THEOREM 1. *Suppose that $L^t_a(G, wdA)$ has dimension greater than* 1 *for each* $0 < t < \infty$. *Let* $0 < p < s \leqq \infty$, *and let g be an analytic function on G such that*

$$gL^p_a(G, wdA) \subset L^s_a(G, wdA).$$

Then $g = 0$.

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To prove this theorem, Axler used the Fredholm alternative from operator theory as a major tool. In this paper, we shall prove the above theorem without using operator theory, giving a purely function theoretic proof. The following is our main theorem. As a corollary we can get Theorem 1.

THEOREM 2. *Let* $0 < p < \infty$. *Suppose that* $L_a^p(G, wdA)$ *has dimension greater than* 1. *Let g be an analytic function on G such that*

$$gL_a^p(G, wdA) \subset L_a^\infty(G, wdA).$$

Then $g = 0$.

PROOF. To show $g = 0$, suppose not. We shall get a contradiction. Since $\dim L_a^p(G, wdA) \geqq 2$, there exists a function h in $L_a^p(G, wdA)$ such that gh is nonconstant. Since $gh \in L_a^\infty(G, wdA)$, we may assume

$$(1) \qquad \|gh\|_\infty = 1.$$

Hence there is a sequence $\{\lambda_n\}_{n=0}^\infty$ in G such that

$$(2) \qquad |(gh)(\lambda_n)| \to 1 \ (n \to \infty).$$

We shall show the existence of increasing positive integers $\{k_n\}_{n=1}^\infty$ such that

$$(3) \qquad \sum_{n=1}^\infty n2^n (gh)^{k_n} h \in L_a^p(G, wdA)$$

and

$$(4) \qquad g\left(\sum_{n=1}^\infty n2^n (gh)^{k_n} h\right) \notin L_a^\infty(G, wdA).$$

Then these contradict our assumption.

To show the existence of $\{k_n\}$ satisfying (3) and (4), first we show by induction that there are increasing sequences of positive integers $\{k_n\}_{n=1}^\infty$ and $\{i_n\}_{n=1}^\infty$ such that

$$(5, n) \qquad \|(gh)^{k_n} h\|_p < (1/3)^n,$$

$$(6, n) \qquad |(gh)^{k_n}(\lambda_j)| < (1/3)^n$$

for every j with $0 \leqq j \leqq i_{n-1}$,

$$(7, n) \qquad |(gh)^{k_n}(\lambda_{i_n})| > 1 - 1/n2^n.$$

For convenience, we put $i_0 = 0$. We only prove the general step. We can get the first step by the same way. Suppose that there exist k_n and i_n satisfying (5, n), (6, n) and (7, n). Since gh is a nonconstant analytic function with $\|gh\|_\infty = 1$, $(gh)^n$ converges 0 uniformly on each compact subset of G. Since $h \in L_a^p(G, wdA)$, by the dominated convergence theorem, we can take a sufficiently large positive integer k_{n+1} satisfying (5, $n + 1$) and (6, $n + 1$). Next, by (2), we can take i_{n+1} satisfying (7, $n + 1$). This completes the induction.

Now we get

$$\left\| \sum_{n=1}^{\infty} n2^n (gh)^{k_n} h \right\|_p \leqq \sum_{n=1}^{\infty} n2^n \| (gh)^{k_n} h \|_p$$

$$\leqq \sum_{n=1}^{\infty} n(2/3)^n$$

by (5, n)

$$< \infty.$$

The first inequality is easy to see for $1 \leqq p < \infty$. If $0 < p < 1$, it follows from

$$\left\| \sum_{n=1}^{\infty} n2^n (gh)^{k_n} h \right\|_p \leqq \sum_{n=1}^{\infty} \| n2^n (gh)^{k_n} h \|_p$$

$$= \sum_{n=1}^{\infty} (n2^n)^p \int | (gh)^{k_n} h|^p w dA$$

by the definition

$$\leqq \sum_{n=1}^{\infty} n2^n \| (gh)^{k_n} h \|_p \text{ because } n2^n \geqq 1.$$

Hence we get (3).

Also we have the following inequalities for sufficiently large j.

$$\left| g(\lambda_{i_j}) \left(\sum_{n=1}^{\infty} n2^n (gh)^{k_n} h \right) (\lambda_{i_j}) \right|$$

$$\geqq | (gh)(\lambda_{i_j}) | \Bigg\{ j2^j | (gh)^{k_j} (\lambda_{i_j}) |$$

$$- \sum_{n=1}^{j-1} n2^n | (gh)^{k_n} (\lambda_{i_j}) | - \sum_{n=j+1}^{\infty} n2^n | (gh)^{k_n} (\lambda_{i_j}) | \Bigg\}$$

$$\geqq | (gh)(\lambda_{i_j}) | \left\{ j2^j (1 - 1/j2^j) - \sum_{n=1}^{j-1} n2^n - \sum_{n=j+1}^{\infty} n(2/3)^n \right\}$$

by (7, j), (1) and (6, n)

$$\geqq | (gh)(\lambda_{i_j}) | \{ j2^j - 1 - (j2^j - j) - 1 \}$$

$$= | (gh)(\lambda_{i_j}) | (j - 2).$$

The last inequality follows from

$$\sum_{n=j+1}^{\infty} n(2/3)^n < 1$$

for sufficient large j, and

$$\sum_{n=1}^{j-1} n2^n \leqq (j-1)\sum_{n=1}^{j-1} 2^n = (j-1)(2^j-1)$$

$$= j2^j - j - 2^j + 1 < j2^j - j.$$

Hence, by (2), we get

$$\left| g(\lambda_{i_j})\left(\sum_{n=1}^{\infty} n2^n (gh)^{k_n} h\right)(\lambda_{i_j})\right| \to \infty \ (j \to \infty).$$

Thus we get (4). This completes the proof.

PROOF OF THEOREM 1. Let t be a positive number such that $1/s + 1/t = 1/p$. For each $f \in L_a^s(G, wdA)$ and $h \in L_a^t(G, wdA)$, we have $fh \in L_a^p(G, wdA)$ by the generalized Hölder's inequality. For each $k \in L_a^p(G, wdA)$, by our assumption, $gk \in L_a^s(G, wdA)$. Hence

$$(gh)k = (gk)h \in L_a^p(G, wdA).$$

Thus

$$(gh)L_a^p(G, wdA) \subset L_a^p(G, wdA).$$

By Lemma 11 of [2], $gh \in L_a^\infty(G, wdA)$. Hence $gL_a^t(G, wdA) \subset L_a^\infty(G, wdA)$. By Theorem 2, $g = 0$.

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