EXTREMAL PROBLEMS FOR FUNCTIONS STARLIKE IN THE EXTERIOR OF THE UNIT CIRCLE

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1. Introduction. Let $\sum$ represent the class of analytic functions

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}, f(z) \neq 0, \tag{1}$$

which are regular, except for a simple pole at infinity, and univalent in $|z| > 1$ and map $|z| > 1$ onto a domain whose complement is starlike with respect to the origin. Further let $\sum^{-1}$ be the class of inverse functions of $\sum$ which at $w = \infty$ have the expansion

$$\phi(w) = w + \sum_{n=0}^{\infty} b_n w^{-n}. \tag{2}$$

In this paper we develop variational formulas for functions of the classes $\sum$ and $\sum^{-1}$ and obtain certain properties of functions that extremalize some rather general functionals pertaining to these classes. In particular, we obtain precise upper bounds for $|b_2|$ and $|b_3|$. Precise upper bounds for $|b_1|$, $|b_2|$ and $|b_3|$ are given by Springer **(8)** for the general univalent case, provided $b_0 = 0$.

Various results in this paper are analogues of results obtained by Hummel **(4)** for the class S of functions starlike and univalent in $|z| < 1$ and by Springer **(8)** for the class T of the functions univalent in $|z| > 1$.

There are several ways of deriving a variational formula for functions belonging to $\sum$. We have chosen the Julia variational method, mainly to exhibit another method for handling variational procedures for subclasses of univalent functions which possess certain geometric properties. As a starting point we could have used a formula of Schiffer **(6)** or the variational formula developed by Hummel **(3)** for functions of S in which Schiffer's formula is employed.

It should also be noted that the coefficient problem along with some other functionals for $\sum$ has been studied using the variational methods by Zamorski **(9)**.

2. Variational formulae. Let D be a simply connected domain in the w-plane bounded by an analytic curve and containing the point at infinity. Denote by $f(z)$ the univalent function defined in the exterior of the unit circle, E: $|z| > 1$, and mapping E onto D in such a manner that the points at infinity correspond to one another. If $f(z)$ is of the form

$$w = f(z) = az + a_0 + a_1/z + \dots, \qquad a > 0, \tag{3}$$

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and if we vary the domain D by means of the mapping $w^* = w + \rho^2 \delta w$, obtaining a new domain D^* which we shall suppose is bounded by an analytic curve, then we may write

$$(4) \qquad w^* = f^*(z) = a^*z + a_0^* + a_1^*/z + \ldots, a^* > 0.$$

Let $\delta f(z) = f^*(z) - f(z)$. Using Julia's variational formula **(5)** (for another interesting derivation see Springer **(8)**) we have

$$(5) \qquad \delta f(z) = \rho^2 z f'(z) \frac{1}{2\pi} \int_\gamma \frac{\zeta + z}{\zeta - z} \frac{\delta \xi d\xi}{\zeta^2 f'(\zeta)^2} + o(\rho^2)$$

where γ is the boundary of E, z is a point in E, and the path of integration is taken clockwise around γ and where

$$\delta \xi = i \frac{d\xi}{ds} \delta n, \; \xi = f(\zeta),$$

is the normal component of the displacement δf, s being the arc length along Γ, the boundary of D.

In order to preserve the starshapedness of the varied domain D^*, we use a variation first used by Hummel **(3)**, $w^* = w + \rho^2 w S[\phi(w)]$ where $S[\phi(w)]$ is real and bounded on $|z| = 1$. For ρ sufficiently small $w = 0$ will be exterior to D^* and Γ^*, the boundary of D^*, will again be starshaped with respect to the origin. The function $S[\phi(w)]$ can be taken as

$$(6) \qquad S[\phi(w)] = e^{i\theta} \frac{1 - \overline{\phi(w_0)}\phi(w)}{\phi(w) - \phi(w_0)} + e^{-i\theta} \frac{\phi(w) - \phi(w_0)}{1 - \overline{\phi(w_0)}\phi(w)}$$

where $z_0 = \phi(w_0)$ is a point in E. Since δw is the normal component of the displacement of δf it is easy to compute δw in terms of δf to obtain

$$\delta w = i \frac{dw}{ds} \operatorname{Re}\left(\delta f / i \frac{dw}{ds} \right), \; w = f(z).$$

If the function $f(z)$ given by (3) is starlike with respect to the origin then $f(z)/a$ belongs to $\sum$, that is, the value of the derivative of $f(z)$ at infinity is made equal to one. However, application of the variation $w^* = w + \rho^2 w$ $S[\phi(w)]$ to a member of $\sum$ does not necessarily yield a member of $\sum$ since $f^*(z)$ may not have the proper normalization. But, using (5) we have

$$(7) \qquad f'(\infty) = 1 - \rho^2 \int_\gamma \frac{\delta \xi d\xi}{2\pi i \zeta^2 f'(\zeta)^2} + o(\rho^2)$$

so that

$$(8) \quad \delta f(z) = \rho^2 z f'(z) \int_\gamma \frac{\zeta + z}{\zeta - z} \frac{\delta \xi d\xi}{2\pi i \zeta^2 f'(\zeta)^2} + \rho^2 f(z) \int_\gamma \frac{\delta \xi d\xi}{2\pi i \zeta^2 f'(\zeta)^2} + o(\rho^2)$$

and now $f^*(z)$ has the proper normalization.

From the definition of δw we have

$$(9)\quad \delta f(z) = \frac{\rho^2}{2\pi i}\int_\gamma \left[zf'(z)\frac{\zeta+z}{\zeta-z}+f(z)\right]\mathrm{Re}\left\{\frac{f(\zeta)S[\phi(\xi)]}{i\dfrac{d\xi}{ds}}\right\}\cdot i\frac{d\xi}{ds}\frac{d\xi}{\zeta^2 f'(\zeta)^2}+o(\rho^2)$$

Since

$$\frac{d\xi}{ds}\frac{d\xi}{\zeta^2 f'(\zeta)^2}$$

is real on γ and

$$z = \frac{1}{\bar{z}}$$

on γ, (9) may be written as

$$(10)\quad \delta f(z) = \frac{\rho^2}{4\pi}\int_\gamma \left[zf'(z)\frac{\zeta+z}{\zeta-z}+f(z)\right]S[\phi(\xi)]f(\zeta)\frac{d\zeta}{i\zeta^2 f'(\zeta)}$$
$$\overline{+\frac{\rho^2}{4\pi}\int_\gamma \left[\overline{\bar{z}f'(\bar{z})}\frac{1+\bar{z}\zeta}{1-\bar{z}\zeta}+\overline{f(\bar{z})}\right]S[\phi(\xi)]f(\zeta)\frac{d\zeta}{i\zeta^2 f'(\zeta)}}+o(\rho^2).$$

Evaluating these integrals by the residue theorem we have

$$\begin{aligned}(11)\quad \delta f(z) = \frac{\rho^2}{2}\Bigg\{&2f(z)\left(e^{i\theta}\frac{1-\bar{\zeta}_0 z}{z-\zeta_0}+e^{-i\theta}\frac{z-\zeta_0}{1-\bar{\zeta}_0 z}\right)\\ &+\left(zf'(z)\frac{\zeta_0+z}{\zeta_0-z}+f(z)\right)\left(\frac{f(\zeta_0)}{\zeta_0^2 f'(\zeta_0)}\right)e^{i\theta}(1-|\zeta_0|^2)\\ &+(zf'(z)+f(z))\left(e^{i\theta}\bar{\zeta}_0+\frac{e^{-i\theta}}{\zeta_0}\right)\\ &+\left(zf'(z)\frac{1+\bar{\zeta}_0 z}{1-\bar{\zeta}_0 z}+f(z)\right)\left(\overline{\frac{f(\zeta_0)}{\zeta_0^2 f'(\zeta_0)}}\right)e^{-i\theta}(1-|\zeta_0|^2)\\ &-(zf'(z)-f(z))\left(e^{-i\theta}\zeta_0+\frac{e^{i\theta}}{\zeta_0}\right)\Bigg\}+o(\rho^2)\end{aligned}$$

which upon collecting terms and simplifying becomes

$$\begin{aligned}(12)\quad \delta f(z) = \frac{\rho^2}{2}(1-|\zeta_0|^2)\Bigg\{&\frac{2f(z)e^{i\theta}}{z-\zeta_0}+\frac{2f(z)e^{-i\theta}}{\bar{\zeta}_0(1-\bar{\zeta}_0 z)}\\ &+e^{i\theta}\left(\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}\right)\frac{2zf'(z)}{\zeta_0-z}+e^{-i\theta}\left(\overline{\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}}\right)\left(\frac{2z^2 f'(z)}{1-\bar{\zeta}_0 z}+\frac{2f(z)}{\bar{\zeta}_0}\right)\\ &+(zf'(z)-f(z))\Bigg[-\frac{e^{i\theta}}{\zeta_0}\left(\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}\right)+\frac{e^{-i\theta}}{\bar{\zeta}_0}\left(\overline{\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}}\right)\\ &-\frac{e^{i\theta}}{\zeta_0}+\frac{e^{-i\theta}}{\bar{\zeta}_0}\Bigg]\Bigg\}+o(\rho^2)\end{aligned}$$

A rotational variation of the form $e^{-i\epsilon}f(ze^{i\epsilon})$ will remove the terms involving $zf'(z) - f(z)$. Also, noting that $(\zeta_0 - z)^{-1} = \frac{1}{2}\zeta_0^{-1}[(\zeta_0 + z)(\zeta_0 - z)^{-1} + 1]$,

$$(1 - \bar{\zeta}_0 z)^{-1} = \tfrac{1}{2}[(1 + \bar{\zeta}_0 z)(1 - \bar{\zeta}_0 z)^{-1} + 1]$$

and

$$\bar{\zeta}_0 z(1 - \bar{\zeta}_0 z)^{-1} = \tfrac{1}{2}[(1 + \bar{\zeta}_0 z)(1 - \bar{\zeta}_0 z)^{-1} - 1]$$

we have that

$$(13)\quad \delta f(z) = \rho^2(1 - |\zeta_0|^2)\left\{ \frac{e^{i\theta}}{\zeta_0}\left[- f(z)\,\frac{\zeta_0 + z}{\zeta_0 - z} - f(z) \right.\right.$$
$$\left. + \left(\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}\right)\left(zf'(z)\,\frac{\zeta_0 + z}{\zeta_0 - z} + zf'(z)\right)\right] + \frac{e^{-i\theta}}{\bar{\zeta}_0}\left[f(z)\,\frac{1 + \bar{\zeta}_0 z}{1 - \bar{\zeta}_0 z}\right.$$
$$\left.\left. + f(z) + \left(\frac{\overline{f(\zeta_0)}}{\overline{\zeta_0 f'(\zeta_0)}}\right)\left(zf'(z)\,\frac{1 + \bar{\zeta}_0 z}{1 - \bar{\zeta}_0 z} - zf'(z) + 2f(z)\right)\right]\right\} + o(\rho^2)$$

which yields a variational formula for the class $\sum$.

Let $\zeta = \phi(\xi)$, the inverse of $\xi = f(\zeta)$, belong to $\sum^{-1}$. Let $\phi^*(\xi)$ be the inverse of $f^*(\zeta)$. Set $w = f(z)$ and $w^* = f^*(z)$. If we expand $\phi(w)$ by Taylor's formula and note that $z = \phi^*(w^*) = \phi(w)$ we get

$$(14)\qquad \delta\phi(w) = -\phi'(w)\delta f(z) + o(\rho^2), \qquad |w^* - w| = O(\rho^2).$$

Consequently, the variational formula for the inverse function $z = \phi(w)$ is obtained by substituting (13) into (14) giving

$$(15)\quad \delta\phi(w) = \rho^2(1 - |\phi(\xi_0)|^2)\left\{\frac{e^{i\theta}}{\phi(\xi_0)}\left[- w\phi'(w)\,\frac{\phi(\xi_0) + \phi(w)}{\phi(\xi_0) - \phi(w)}\right.\right.$$
$$\left. - w\phi'(w) + \left(\frac{\xi_0\phi'(\xi_0)}{\phi(\xi_0)}\right)\left(\phi(w)\,\frac{\phi(\xi_0) + \phi(w)}{\phi(\xi_0) - \phi(w)} + \phi(w)\right)\right]$$
$$+ \frac{e^{-i\theta}}{\overline{\phi(\xi_0)}}\left[w\phi'(w)\,\frac{1 + \overline{\phi(\xi_0)}\phi(w)}{1 - \overline{\phi(\xi_0)}\phi(w)} + w\phi'(w) + \left(\overline{\frac{\xi_0\phi'(\xi_0)}{\phi(\xi_0)}}\right)\right.$$
$$\left.\left.\cdot\left(\phi(w)\,\frac{1 + \overline{\phi(\xi_0)}\phi(w)}{1 - \overline{\phi(\xi_0)}\phi(w)} - \phi(w) + 2w\phi'(w)\right)\right]\right\} + o(\rho^2).$$

Although contour integrals were used to obtain these formulae, the variational formulae (13) and (15) do not involve contour integrals. These contour integrations have been made upon the assumption that the curves were analytic. An arbitrary boundary curve may be considered as the limit of a sequence of approximately analytic curves and since formulae (13) and (15) involve only the mapping functions and their derivatives at an interior point the convergence of the formulae to the formulae for arbitrary domains is uniform. To show that the terms of higher order in the formulae also converge uniformly under the above limiting operation we may use the forms of the formulae to construct a direct proof of the formula which does allow

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us to make a better estimate on terms of higher order. The procedure we shall use is due to Garabedian and Schiffer **(1)**. Let

$$\phi^*(w) = b_{-1}^*w + b_0^* + \frac{b_1^*}{w} + \ldots, b_{-1}^* > 0, \tag{16}$$

map D^* onto E. Then by Taylor's formula we have

$$\log|\phi(w)| = \log|\phi^*(w^*)| = \log|\phi^*(w)| + \mathrm{Re}\frac{\phi'(w)}{\phi(w)}\delta w + o(\rho^2) \tag{17}$$

where $\phi^{*\prime}(w)/\phi(w)$ has been replaced by $\phi'(w)/\phi(w)$ and $\delta w = wS[\phi(w)]$. Since $\log|\phi^*(w^*)|$ vanishes for w on Γ and the first two terms of the right-hand side of (17) constitute an harmonic function of w, except at $w = \xi_0$ where it has a simple pole and at infinity where it has a logarithmic singularity, this harmonic function has small values of magnitude $o(\rho^2)$ when w is on Γ. From the analytic completion of this harmonic function we subtract an analytic function which has zero boundary values on Γ, a simple pole at $w = \xi_0$ and has at $w = \xi_0$ the same residue as the original function. This new function

$$\begin{aligned} H(w, \xi_0) = \log \phi^*(w) + \rho^2\Bigg\{&\frac{\phi'(w)}{\phi(w)}\left(e^{i\theta}\frac{1 - \overline{\phi(\xi_0)}\phi(w)}{\phi(w) - \phi(\xi_0)}\right. \\ &\left.+ e^{-i\theta}\frac{\phi(w) - \phi(\xi_0)}{1 - \overline{\phi(\xi_0)}\phi(w)}\right) - \xi_0\frac{\phi'(\xi_0)}{\phi(\xi_0)}\frac{1 - |\phi(\xi_0)|^2}{\phi(w) - \phi(\xi_0)}e^{i\theta} \\ &- \left(\overline{\frac{\xi_0\phi'(\xi_0)}{\phi(\xi_0)}}\right)\frac{1 - |\phi(\xi_0)|^2}{1 - \overline{\phi(\xi_0)}\phi(w)}e^{-i\theta}\phi(w)\Bigg\} \end{aligned} \tag{18}$$

is a function with a real part that has a logarithmic singularity at infinity and on Γ has the order of magnitude $o(\rho^2)$. It, therefore, differs from the Green's function, $\log|\phi(w)|$ by terms of order $o(\rho^2)$, that is, $\log|\phi(w)| = \mathrm{Re}\{H(w, \xi_0)\} + o(\rho^2)$. Completing this harmonic function to an analytic function we get

$$\log\frac{\phi^*(w)}{\phi(w)} = -\rho^2\Phi(w, \xi_0) + ic\rho^2 + o(\rho^2) \tag{19}$$

where $\Phi(w, \xi_0)$ is defined by the relation $H(w, \xi_0) = \log\phi^*(w) + \rho^2\Phi(w, \xi_0)$ and where c is a real function of ξ_0. To determine c let $w \to \infty$. Then since $\log b^*_{-1}$ is real we get

$$c = \frac{1}{2i}(1 - |\phi(\xi_0)|^2)\left[\left(\frac{\xi_0\phi'(\xi_0)}{\phi(\xi_0)}\right)e^{i\theta} - \left(\overline{\frac{\xi_0\phi'(\xi_0)}{\phi(\xi_0)}}\right)e^{-i\theta} + \frac{e^{i\theta}}{\phi(\xi_0)} - \frac{e^{-i\theta}}{\overline{\phi(\xi_0)}}\right].$$

We now obtain the variational formula (15) by applying the exponential function to both sides of (19) where the above value of c has been used. In this second proof of the formula if Γ is not an analytic curve we can again approximate Γ uniformly by a sequence of analytic curves Γ_n for which the formulae hold as a first variation. The corrective terms of order $o(\rho^2)$ are composed of the mapping function and its derivatives which converge uniformly to the mapping function of D and its derivatives, and hence are of order $o(\rho^2)$. Hence we may use formulae (13) and (15) for arbitrary domains.

3. Extremal functions. It is convenient to introduce a complex functional in order to handle rather general extremal problems pertaining to $\sum$. Following Hummel **(4)**, a complex function $J[f]$ is called linear in the small if for any f and $f + \epsilon g$ belonging to $\sum$ then $J[f + \epsilon g] = J[f] + \epsilon J_1[f; g] + o(\epsilon)$, where $J_1[f; g]$ is a complex valued functional linear in g.

Given a function $f \in \sum$ and $|\zeta_0| > 1$, define

$$K(\zeta_0) = J_1\left[f;f(z)\frac{\zeta_0 + z}{\zeta_0 - z}\right], \quad J_1\left[f;zf'(z)\frac{\zeta_0 + z}{\zeta_0 - z}\right] = L(\zeta_0) \tag{20}$$

$$M = J_1[f;f(z)] \qquad , N = J_1[f;zf'(z)].$$

Denoting $J[f^*] - J[f]$ by $\delta J[f]$ we easily see from (13) that

$$\begin{aligned} J[f] &= \rho^2 J_1[f;g] + o(\rho^2) \\ &= \frac{\rho^2}{2}(1 - |\zeta_0|^2)\left\{\frac{e^{i\theta}}{\zeta_0}\left[-K(\zeta_0) - M + \left(\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}\right)(L(\zeta_0) + N)\right]\right. \\ &\quad \left. + \frac{e^{-i\theta}}{\bar{\zeta}_0}\left[K\left(\frac{1}{\bar{\zeta}_0}\right) + M + \left(\overline{\frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}}\right)\left(L\left(\frac{1}{\bar{\zeta}_0}\right) - N + 2M\right)\right]\right. + o(\rho^2). \end{aligned} \tag{21}$$

The results we shall now prove for the class $\sum$ correspond to already known results for the class S and to results for the class $\sum^{-1}$ which are given in § 4.

We should note for later reference that if f is extremal for $\mathrm{Re}\{J[f]\} =$ maximum then $M - N$ is real. For, employing a rotational variation $e^{-i\epsilon}f(e^{i\epsilon}z)$, ϵ real, we get

$$\delta J[f] = i\epsilon J_1[f; zf'(z) - f(z)] + o(\epsilon) = - i\epsilon(M - N) + o(\epsilon). \tag{22}$$

Since $\mathrm{Re}\{J[f]\}$ is to be a maximum and noting that ϵ is arbitrary the result immediately follows.

THEOREM 3.1. *Let $f(\zeta) \in \sum$ be a solution of the extremal problem* $\mathrm{Re}\{J[f]\} =$ *maximum (minimum). Then $f(\zeta)$ must satisfy a differential equation of the form $\zeta f'(\zeta)/f(\zeta) = P(\zeta)/Q(\zeta)$ for all $|\zeta| > 1$.*

Proof. If $\mathrm{Re}\{J[f]\}$ is to be a maximum then $\mathrm{Re}\{\delta J[f]\} \leqslant 0$. Hence taking real parts in (12) and noting that θ is arbitrary we get

$$\begin{aligned} K(\zeta_0) + M - \overline{K\left(\frac{1}{\bar{\zeta}_0}\right)} - \overline{M} \\ = \frac{f(\zeta_0)}{\zeta_0 f'(\zeta_0)}\left(L(\zeta_0) + N + \overline{L\left(\frac{1}{\bar{\zeta}_0}\right)} - \overline{N} + \overline{2M}\right). \end{aligned} \tag{23}$$

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Replacing ζ_0 by ζ we have

$$(24) \qquad \frac{\zeta f'(\zeta)}{f(\zeta)} = \frac{L(\zeta) + L\left(\frac{1}{\bar{\zeta}_0}\right) + N - \bar{N} + 2\bar{M}}{K(\zeta) - K\left(\frac{1}{\bar{\zeta}}\right) + M - \bar{M}} = \frac{P(\zeta)}{Q(\zeta)}$$

for $|\zeta| > 1$, which is the desired result.

Now on $|\zeta| = 1$ the relation $\zeta = \bar{\zeta}^{-1}$ holds so that $Q(\zeta)$ is purely imaginary on $|\zeta| = 1$ and since $\bar{M} - \bar{N}$ is real $P(\zeta)$ is real on $|\zeta| = 1$. Hence on the unit circle $\zeta f'(\zeta)/f(\zeta)$ is purely imaginary wherever it is regular. On the unit circle $d\zeta/\zeta = id\phi$, $\zeta = e^{i\phi}$, so that $f'(\zeta)/f(\zeta)$ is real on $|\zeta| = 1$ and hence wherever $P(\zeta)$ and $Q(\zeta)$ are finite, continuous and non-zero we have $\mathcal{I}m\{\log f(z)\} =$ constant. Therefore $f(\zeta)$ maps arcs on which $P(\zeta)$ and $Q(\zeta)$ have the above-named properties onto radial line segments.

In a large number of extremal problems $P(\zeta)$ and $Q(\zeta)$ turn out to be rational functions. For this class of extremal problems we have the following

THEOREM 3.2. *Suppose $K(\zeta)$ and $L(\zeta)$ are rational functions of degree n and are regular on $|\zeta| = 1$. Let $Q(\zeta)$ (defined in (24)) have $2k$ zeros not on $|\zeta| = 1$. Then the extremal function $f(\zeta)$ maps $|\zeta| > 1$ onto the plane cut by $n - k$ or fewer radial slits and $f(\zeta)$ has the form*

$$(25) \qquad f(\zeta) = \prod_{p=1}^{m} \left(1 - \frac{\beta_p}{\zeta}\right)^{\alpha p}, \; m \leqslant n - k,$$

where

$$\alpha_p > 0, \; \sum_{p=1}^{m} \alpha_p = 2 \text{ and } |\beta_p| = 1.$$

Proof. The proof of this theorem follows the pattern set forth by the proof of the corresponding theorem in the class S. However, for completeness we shall give the proof.

Since $K(\zeta)$ and $L(\zeta)$ are regular on $|\zeta| = 1$ and of degree n, then $P(\zeta)$ and $Q(\zeta)$ are also regular and are rational of degree $2n$. Hence, $P(\zeta)/Q(\zeta)$ is regular except at the $2k$ zeros of $Q(\zeta)$. Also $Q(\zeta) = -Q(\bar{\zeta}^{-1})$ on $|\zeta| = 1$ so that the zeros of $Q(\zeta)$ are inverse points with respect to the unit circle. Since $\zeta f'/f$ is purely imaginary on $|\zeta| = 1$, regular (except at a finite number of poles) and not zero in $|\zeta| > 1$ we have then by the symmetry principle that $\zeta f'/f$ satisfies the same conditions in $|\zeta| < 1$. Hence, all poles and zeros of $\zeta f'/f$ must lie on $|\zeta| = 1$. Now $P(\zeta)$ and $Q(\zeta)$ are regular on $|\zeta| = 1$, therefore all poles and zeros of $\zeta f'/f$ come from zeros of $P(\zeta)$ and $Q(\zeta)$. Note that $P(\zeta)/Q(\zeta)$ is rational of degree $2(n - k)$. By the remark preceding the theorem we see that the boundary of the unit circle maps onto a set of radial slits. The zeros of $P(\zeta)$ correspond to the tips of the slits and the origin corresponds to a simple pole of $\zeta f'/f$, hence there are at most $2(n - k)$ slits. Expanding $\zeta f'/f$ in terms of its poles and integrating we get formula (25) with $m \leqslant 2(n - k)$.

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Let us show that $m \leqslant n - k$. Using Julia's formula on $\delta J[f]$ and noting that

$$(\zeta + z)(\zeta - z)^{-1} = \tfrac{1}{2}[(\zeta + z)(\zeta - z)^{-1} + (1 + \bar{\zeta}z)(1 - \bar{\zeta}z)^{-1}]$$

we get

$$\delta J[f] = \frac{\rho^2}{4\pi} \int_\gamma \left[L(\zeta) + L\left(\frac{1}{\bar{\zeta}}\right) + 2M\right] \frac{\delta \xi d\xi}{i\zeta^2 f'(\zeta)^2} + o(\rho^2). \tag{26}$$

Hence, since $[\delta\xi d\xi / i\zeta^2 f'(\zeta)^2]$ is real,

$$\begin{aligned}
\text{(27)}\quad & \operatorname{Re}\{\delta J[f]\} \\
&= \frac{\rho^2}{4\pi} \int_\gamma \operatorname{Re}\left\{L(\zeta) + L\left(\frac{1}{\bar{\zeta}}\right) + 2\overline{M} + N - \overline{N}\right\} \frac{\delta\xi d\xi}{i\zeta^2 f'(\zeta)^2} + o(\rho^2) \\
&= \frac{\rho^2}{4\pi} \int_\gamma \operatorname{Re} P(\zeta) \frac{\delta\xi d\xi}{i\zeta^2 f'(\zeta)^2} + o(\rho^2) \\
&= \frac{\rho^2}{4\pi} \int_\gamma P(\zeta)\left(-\delta n \frac{d\phi}{ds}\right) d\phi + o(\rho^2), \qquad \zeta = e^{i\phi},
\end{aligned}$$

where we have used the fact that $P(\zeta)$ is real on γ and where $((d\phi)/(ds))d\phi > 0$.

Since we know that the boundary Γ is made up of radial slits, choose one of the slits and separate the two edges so that one edge stays fixed while a part of the other edge is shifted into the interior. In order that we remain in the class $\sum$ the shift will have to go all the way into the origin and be of such a nature that the new domain is starlike. With this type of shift we can conclude that $P(\zeta)$ must have at least two zeros corresponding to each slit. Indeed, $P(\zeta)$ is continuous on $|\zeta| = 1$ and we can take the shift sufficiently close to the origin that $P(\zeta)$ will be of constant sign along the varied portion. This shift defines a positive δn which implies by (27) that in the extremal case $P(\zeta)$ must be positive in a neighbourhood of a point ζ which maps onto the origin. Since one zero of $P(\zeta)$ corresponds to the tip of the slit we see that between two such points which map onto the origin $P(\zeta)$ has at least two zeros, and hence an even number of zeros. The function $P(\zeta)$ has at most $2(n - k)$ zeros on γ, therefore, there are at most $n - k$ slits. This proves the theorem.

Various applications to the above theorem can be made. Many of them have already been done by elementary methods, for example, the region of values of $f(z_0)$ and $\log(f(z_0))/(z_0))$ for z_0 fixed in $|z| > 1$ and where f ranges over $\sum$, and others. In determining the set of values of $\log f'(z_0)$ for fixed z_0, $|z_0| > 1$, and all $f \in \sum$ we find upon applying Theorem 3.2 that $K(z)$ and $L(z)$ are of degree 2. Hence, the extremal functions have at most two slits. Goodman **(2)** showed that the function maximizing $|\mathscr{I}_m\{\log f'(z)\}|$ cannot be a function with only one slit so by the above remark we see that the function maximizing $|\arg f'(z)|$ has exactly two slits.

4. The class of inverse functions. The variational formula for a member of the class $\sum^{-1}$ is given by (15). If we consider functionals, linear in the small, for this class and introduce the notation

$$k(\zeta_0) = k(\phi(\xi_0)) = J_1\left[\phi;\, \phi(w)\, \frac{\phi(\xi_0) + \phi(w)}{\phi(\xi_0) - \phi(w)}\right]$$

$$(28) \qquad l(\zeta_0) = l(\phi(\xi_0)) = J_1\left[\phi;\, w\phi'(w)\, \frac{\phi(\xi_0) + \phi(w)}{\phi(\xi_0) - \phi(w)}\right]$$

$$m = J_1[\phi;\, \phi(w)], \qquad n = J_1[\phi;\, w\phi'(w)].$$

Then we have, applying J to (15),

$$(29) \qquad \delta J[\phi] = -\rho^2(1 - |\phi(\zeta_0)|^2)\left\{\frac{e^{i\theta}}{\phi(\xi_0)}\left[- l(\zeta_0) - n + \left(\frac{\xi_0\phi'(\xi_0)}{\phi(\xi_0)}\right)(k(\zeta_0) + m)\right] + \frac{e^{-i\theta}}{\overline{\phi(\xi_0)}}\left[l\left(\frac{1}{\bar{\zeta}_0}\right) + n + \left(\overline{\frac{\xi_0\phi'(\xi_0)}{\phi(\xi_0)}}\right)\left(k\left(\frac{1}{\bar{\zeta}_0}\right) - m + 2n\right)\right]\right\} + o(\rho^2).$$

Using (14) and (22) we easily see that $n - m$ is purely real. Then proceeding as in the proof of Theorem 3.1 we can prove the following

THEOREM 4.1. *Let $\phi(w) \in \sum^{-1}$ be a solution of the extremal problem* $\text{Re}\{J[\phi]\}$ $=$ *maximum for $w \in D$, then $\phi(w)$ must satisfy a differential equation of the form*

$$(30) \qquad \frac{w\phi'(w)}{\phi(w)} = \frac{l(\phi(w)) - \overline{l\left(\dfrac{1}{\overline{\phi(w)}}\right)} + n - \bar{n}}{k(\phi(w)) + \overline{k\left(\dfrac{1}{\overline{\phi(w)}}\right)} + m - \bar{m} + 2n} = \frac{q(\phi(w))}{p(\phi(w))}$$

for all w in D.

Equation (30) may also be written as

$$[zf'(z)/f(z)] = p(z)/q(z) \qquad \text{for} \qquad |z| > 1.$$

Another property of the extremal function is obtained upon noting that if $p(\phi(w))$ and $q(\phi(w))$ are finite, non-zero, and continuous at a point w_0 on Γ then $f(z)$, the inverse of the extremal function $\phi(w)$, maps an arc of $|z| = 1$ containing the map of w_0 onto a radial line segment.

In a manner analogous to Theorem 3.2 we can also prove

THEOREM 4.2. *Let $k(z)$ and $l(z)$, $z = \phi(w)$, be rational functions in z of degree n and be regular on $|z| = 1$. Let $q(z)$ (defined in (30)) have $2k$ zeros not on $|z| = 1$. Then $f(z)$, the inverse of the extremal function $\phi(w)$, maps $|z| > 1$ onto the w plane cut by $n - k$ or fewer radial slits and $f(z)$ has the form*

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(31) $$f(z) = z \prod_{p=1}^{m} \left(1 - \frac{\beta_p}{z}\right)^{\alpha p}, \; m \leqslant n - k$$

where

$$\alpha_p > 0, \; \sum_{p=1}^{m} \alpha_p = 2, \text{ and } |\beta_p| = 1.$$

An interesting application of Theorem 4.2 is the coefficient problem for the class $\sum^{-1}$. In order to compute $k(z)$ and $l(z)$ it is necessary to rewrite the second arguments appearing in the J_1 functionals, namely,

(32) $$\phi(w) \frac{\phi(\xi_0) + \phi(w)}{\phi(\xi_0) - \phi(w)} = -\left(\frac{2\phi(\xi_0)^2}{\phi(w) - \phi(\xi_0)} + 2\phi(\xi_0) + \phi(w)\right)$$

$$w\phi'(w) \frac{\phi(\xi_0) + \phi(w)}{\phi(\xi_0) - \phi(w)} = -\left(w\phi'(w) + 2\phi(\xi_0) \frac{w\phi'(w)}{\phi(w) - \phi(\xi_0)}\right).$$

It is well known **(7)** that the function

(33) $$\log \frac{\phi(w) - t}{w} = -\sum_{m=1}^{\infty} \frac{1}{m} F'_m(t) w^{-m},$$

the development being valid in the neighbourhood of $w = \infty$, generates the Faber polynomials $F_m(t)$ of degree m. Differentiating (33) with respect to t yields

(34) $$\frac{1}{\phi(w) - t} = \sum_{m=1}^{\infty} \frac{1}{m} F'_m(t) w^{-m}$$

which generates the derivatives of the Faber polynomials. Also, differentiating (33) with respect to w gives

(35) $$\frac{w\phi'(w)}{\phi(w) - t} = 1 + \sum_{m=1}^{\infty} F_m(t) w^{-m}.$$

Using (34), (35), and $J[\phi] = b_\nu$ we can now write

(36) $$k(\zeta_0) = \left[2\phi(\xi_0)^2 \frac{1}{\nu} F'_\nu(\phi(\xi_0)) + b_\nu\right]$$

$$l(\zeta_0) = -[-\nu b_\nu + 2\phi(\xi_0) F_\nu(\phi(\xi_0))]$$

$$m = b_\nu$$

$$n = -\nu b_\nu.$$

Our problem is to find the functions of the class $\sum^{-1}$ for which $|b_\nu|$ is a maximum. Since $\sum^{-1}$ is a normal family we know such functions exist. We may suppose that the extremal b_ν is real and hence seek to maximize $\mathrm{Re}\{b_\nu\}$. Thus Theorem 4.1 applies and we have upon substituting (36) into (30) that the extremal functions $f(z)$ must satisfy

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$$(37)\qquad \frac{zf'(z)}{f(z)} = \frac{\dfrac{z^2}{\nu}F'_\nu(z) + (\nu+1)b_\nu + \dfrac{1}{\nu z^2}\overline{F'_\nu\left(\dfrac{1}{\bar z}\right)}}{zF_\nu(z) - \dfrac{1}{z}\overline{F_\nu\left(\dfrac{1}{\bar z}\right)}}, \nu \geqslant 1.$$

It is well known that $|b_0| = |a_0| \leqslant 2$ and $|b_1| = |a_1| \leqslant 1$. For $\nu = 2$ (37) becomes

$$(38)\qquad \frac{zf'(z)}{f(z)} = \frac{z^3 - b_0z^2 + 3b_2 - \overline{b_0}/z^2 + 1/z^3}{z^3 - 2b_0z^2 + (b_0^2 - 2b_1)z - (\bar b_0^2 - 2\bar b_1)/z + 2\bar b_0/z^2 - 1/z^3} = \frac{p(z)}{q(z)}.$$

We know that $p(z)$ has at least one zero on the unit circle, say at $z = z_0$, so that $p(z_0) = 0$. If $p(z_0) = 0$, then $3b_2 = b_0z_0^2 + \bar b_0/z_0^2 - z_0^3 - 1/z_0^3$ and hence $3|b_2| \leqslant 2|b_0| + 2$. If $b_0 = 0$ then $|b_2| \leqslant 2/3$ and if $b_0 \neq 0$ then $|b_0| \leqslant 2$, $|b_2| \leqslant 2$. Upon computing the derivative of the third Faber polynomial we get $F_3'(z) = 3(z^2 - 2b_0z + b_0^2 - b_1)$ so that for $\nu = 3$, $p(z) = z^2 - 2b_0z + b_0^2 - b_1 + 4b_3 + \bar b_0^2 - \bar b_1 - 2\bar b_0/z + 1/z^2$. As we reasoned before, $p(z)$ has at least one root $z = z_0$ and we obtain $4|b_3| \leqslant 2|b_1| + 2|b_0|^2 + 4|b_0| + 2$ which implies, $b_0 \neq 0$, that $|b_3| \leqslant 5$ and, $b_0 = 0$, that $|b_3| \leqslant 1$. We have now the following

THEOREM 4.3. *If $\phi(w) = w + b_0 + b_1/w + \ldots$ belongs to the class $\sum^{-1}$ then*

$$|b_\nu| \leqslant \frac{1}{\nu}\binom{2\nu}{\nu+1}, \nu = 1, 2, 3, \text{ for } b_0 \neq 0$$

and

$$|b_1| \leqslant 1, \quad |b_2| \leqslant 2/3, \quad |b_3| \leqslant 1, \quad \text{for} \quad b_0 = 0.$$

All these inequalities are sharp.

Unfortunately the above trend does not continue for $\nu > 3$, that is, the sharp bounds are not obtained by solving $p(z_0) = 0$ for b_ν and substituting in the maximum values for the other coefficients.

Springer **(8)** has shown for the more general class of univalent functions with $b_0 = 0$ that $|b_3| \leqslant 1$ and has conjectured that

$$|b_\nu| \leqslant \frac{1}{\nu}\binom{2\frac{\nu}{n}}{k},$$

$n \geqslant 2$, $nk = \nu + 1$, k an integer, where n is the least prime divisor of $\nu + 1$.

We can show, as did Springer for his class, the following: Let D_n be the domain consisting of the whole w plane except for $n(n \geqslant 2)$ symmetric radial slits, each of length $(2)^{2/n}$, originating at the origin. Let $\phi_n(w) = w + c_1^{(n)}/w + c_2^{(n)}/w + \ldots$ be the function mapping D_n conformally on $|z| > 1$. The inverse of $\phi_n(w)$ is essentially $w = f_n(z) = z(1+z^{-n})^{2/n}$. If n is a divisor of $\nu+1$ then $f_n(z)$ satisfies (37). As Springer shows, the $c_\nu^{(n)}$ are the largest when n is the least prime divisor of $\nu+1$. If $b_0 \neq 0$ then $w = f_1(z) = z(1+z^{-1})^2$ satisfies (37). As we have found, this function is extremal for $|b_2|$ and $|b_3|$. Thus one might suspect in the general case with $b_0 \neq 0$ that

$$|b_\nu| \leqslant \frac{1}{\nu}\binom{2\nu}{\nu+1}$$

which is also the bound given by Loewner for the coefficients of the inverse schlicht functions in the unit circle.

References

1. P. R. Garabedian and M. Schiffer, *Identities in the theory of conformal mapping*, Trans. Amer. Math. Soc., *65* (1949), 187–238.
2. A. W. Goodman, *The rotation theorem for starlike univalent functions*, Proc. Amer. Math. Soc., *4* (1953), 278–286.
3. J. A. Hummel, *A variational method for starlike functions*, Proc. Amer. Math. Soc., *9* (1958), 82–87.
4. ——— *Extremal problems in the class of starlike functions*, Proc. Amer. Math. Soc., *11* (1960), 741–749.
5. G. Julia, *Sur une équation aux derivées fonctionelles*, Ann. Ecole Norm., *39* (1922), 1–28.
6. M. Schiffer, *Applications of variational methods in the theory of conformal mapping*, Proceedings of the Symposia in Applied Mathematics, 1956.
7. ——— *Faber polynomials in the theory of univalent function*, Bull. Amer. Math. Soc., *54* (1948), 503–517.
8. G. Springer, *The coefficient problem for schlicht mappings of the exterior of the unit circle*, Trans. Amer. Math. Soc., *70* (1951), 421–450.
9. J. Zamorski, *Differential equations for the extremal starlike functions*, Ann. Pol. Math., *7* (1960), 279–283.

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