ANGULAR AND TANGENTIAL LIMITS OF BLASCHKE PRODUCTS AND THEIR SUCCESSIVE DERIVATIVES

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1. Introduction. In this paper, we shall be concerned with bounded, holomorphic functions of the form

$$B(z; \{a_n\}) = \prod_{n=1}^{\infty} b(z; a_n)$$

where

$$b(z;a) = \frac{|a|}{a}\frac{a-z}{1-\bar{a}z}, \tag{1}$$

$$0 < |a_n| < 1 \qquad (n = 1, 2, \ldots), \tag{2}$$

and

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty. \tag{3}$$

$B(z; \{a_n\})$ is called a *Blaschke product*, and any sequence $\{a_n\}$ which satisfies (2) and (3) is called a *Blaschke sequence*. For a general discussion of the properties of Blaschke products, see **(18**, pp. 271–285**)** or **(14**, pp. 49–52**)**.

According to a theorem due to Riesz **(15)**, a Blaschke product has radial limits of modulus one almost everywhere on $C = \{z: |z| = 1\}$. Moreover, it is common knowledge that, if a Blaschke product has a radial limit at a point, then it also has an angular limit at the point **(**see **14**, p. 19 and **6**, p. 457**)**. For this reason, it seems natural to go one step farther and investigate the existence of tangential limits.

First, let us carefully define the notion of a tangential limit. Let

$$R(m, \theta, \gamma) = \{z: 1 - |z| \geqslant m\,|\arg z - \theta|^{\gamma}; 0 < |z| < 1\}$$

where by $|\arg z - \theta|$ we mean the length of the shorter one of the two arcs on C joining $z/|z|$ and $e^{i\theta}$. If $f(z)$ is a function defined on $D = \{z: |z| < 1\}$, we say that $f(z)$ *has a T_γ-limit at $e^{i\theta}$ provided there exists a number L such that, for each $m (m > 0)$, $f(z) \to L$ as $z \to e^{i\theta}$, z being confined to $R(m, \theta, \gamma)$.*

We observe that a T_1-limit exists if and only if the classical angular limit exists. For this reason, a Blaschke product has a T_1-limit almost everywhere. However, when $\gamma > 1$, a different situation prevails. Indeed, Lohwater and Piranian **(11)** have shown that, corresponding to each $\gamma (\gamma > 1)$, there exists a Blaschke product which has no T_γ-limit whatsoever on C. Also, in this connection, see **(17)** or **(18**, p. 280**)**.

Received January 3, 1961.

https://doi.org/10.4153/CJM-1962-026-2 Published online by Cambridge University Press

One is thus led to seek sufficient conditions for the existence of T_γ-limits. Frostman **(8)** has proved a global theorem dealing with radial limits which serves as a prototype: *If* $\{a_n\}$ *is a Blaschke sequence and*

$$\sum_{n=1}^{\infty} (1 - |a_n|)^{\alpha} < \infty \quad (0 < \alpha < 1) \tag{4}$$

for some fixed α, *then there is a set* E_1 *whose capacity of order* α *is zero such that* $B(z; \{a_n\})$ *and all its subproducts have radial limits of modulus one everywhere on* $C - E_1$. The function

$$B(z; \{a_{nk}\})$$

is called a *subproduct* of $B(z; \{a_n\})$ if

$$\{a_{nk}\}$$

is a subsequence of $\{a_n\}$. (The existence of radial limits for *all* subproducts of a Blaschke product at a point on C entails much more than the mere existence of a radial limit for the Blaschke product. For example, it implies that the radial limits are all of modulus one and that the radius terminating at the point is carried onto curves of finite length by the Blaschke product and all its subproducts **(**see **2)**.)

Recently Kinney **(9)** has shown that, *if* (4) *holds, then* $B(z; \{a_n\})$ *has* T_γ*-limits* $(\gamma > 1)$ *of modulus one everywhere off a set* I_γ *whose capacity of order* $\alpha\gamma/(1 - \alpha)$ *is zero.* (Actually, Kinney proved that the Hausdorff–Besicovitch dimension of I_γ does not exceed $\alpha\gamma/(1 - \alpha)$, which is a slightly weaker conclusion. Also, his definition of tangential limit being somewhat different, he only established convergence on $R(m, \theta, \gamma)$ for $m = 1$.)

In § 3 of this paper, we sharpen Kinney's theorem in several respects, Frostman's theorem appearing as a special case; and then we prove that the sharpened result is the best possible.

Our approach is somewhat different from that of Kinney in that we first study tangential limits locally (§ 2) and then use this information to handle the global situation.

Finally, in § 4, we prove a round of analogous theorems for the successive derivatives of a Blaschke product. Certain known theorems dealing with the Carathéodory angular derivative of a Blaschke product appear as special cases.

2. Local boundary behaviour. A well-known theorem of Frostman **(8)** is a special case (when $\gamma = 1$) of the following theorem:

THEOREM 1. *Let* $\{a_n\}$ *be a Blaschke sequence such that*

$$\sum_{n=1}^{\infty} (1 - |a_n|)/|e^{i\theta} - a_n|^{\gamma} < \infty \tag{5}$$

for some fixed number $\gamma (\gamma \geqslant 1)$. *Then* $B(z; \{a_n\})$ *and all its subproducts have* T_γ*-limits of modulus one at* $e^{i\theta}$.

https://doi.org/10.4153/CJM-1962-026-2 Published online by Cambridge University Press

Preliminary remarks. The core of the proof consists in proving that $\Pi b(z; a_n)$ converges uniformly on $R(m, \theta, \gamma)$ for every positive number m. Since

$$B(z; \{a_n e^{-i\theta}\}) = B(ze^{i\theta}; \{a_n\})$$

and

$$(1 - |a_n e^{-i\theta}|)/|1 - a_n e^{-i\theta}|^\gamma = (1 - |a_n|)/|e^{i\theta} - a_n|^\gamma,$$

it will suffice to prove the theorem for the special case $\theta = 0$.

Proof. By virtue of (5), it is clear that at most a finite number of the zeros a_n are on the radius terminating at $z = 1$; hence, one can assume without loss of generality that no zeros lie on the radius.

Then, using standard techniques, one can prove that

$$\sum_{n=1}^{\infty} (1 - |a_n|)/|\arg a_n|^\gamma < \infty. \tag{6}$$

We shall merely sketch the proof of (6). Suppose that

$$\sum_{n=1}^{\infty} (1 - |a_n|)/|\arg a_n|^\gamma = \infty,$$

and let us prove that

$$\sum_{n=1}^{\infty} (1 - |a_n|)/|1 - a_n|^\gamma = \infty.$$

Let $K = \{z: |z - 1| < 2^{-1/2}\}$. Then, because of (3), it is clear that

$$\sum_{a_n \in K} (1 - |a_n|)/|\arg a_n|^\gamma = \infty.$$

Consider the angle of measure $\pi/2$ with its vertex at $z = 1$ which is bisected by the radius terminating at $z = 1$. If infinitely many of the a_n in K are interior to this angle, then a simple calculation shows that, for such a_n, $1 - |a_n| > |1 - a_n|^\gamma \cdot 2^{-3/2}$. If only a finite number of a_n in K are in the angle, consider just those a_n which are in K and outside the angle; and let p_n be the perpendicular distance from $z = 1$ to the radius through a_n. Clearly, $p_n < |\arg a_n|$ and $p_n/|1 - a_n| > \cos(\pi/4)$. Therefore, $|1 - a_n| < p_n \cdot 2^{1/2} < |\arg a_n| 2^{1/2}$, and $|1 - a_n|^\gamma < |\arg a_n|^\gamma \cdot 2^{\gamma/2}$. Finally,

$$\sum_{a_n \in K} (1 - |a_n|)/|1 - a_n|^\gamma = \infty.$$

Next, using a theorem due to Dini (see **10**, p. 293), we select a sequence $\{w_n\}$ $(0 < w_n \leqslant 1; w_n \to 0)$ in such a way that

$$\sum_{n=1}^{\infty} (1 - |a_n|)/w_n |\arg a_n|^\gamma < \infty; \tag{7}$$

and we set

$$S_n = \{z: |z - a_n| < w_n |\arg a_n|^\gamma\} \qquad (n = 1, 2, \ldots).$$

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Let us prove that, if k is any fixed positive integer, then

$$\prod_{n=1}^{\infty} b(z; a_n)$$

converges uniformly on

$$D - \bigcup_{j=k}^{\infty} S_j.$$

Let $b(z; a) = 1 + c(z; a)$. Then, by (1),

$$c(z; a_n) = \frac{(|a_n| - 1)(a_n + |a_n|z)}{a_n(1 - \bar{a}_n z)};$$

and, consequently, for $z \in D - S_n$, we have

$$\begin{aligned} |c(z; a_n)| &\leqslant \frac{(1 + |z|)(1 - |a_n|)}{|1 - \bar{a}_n z|} \\ &< \frac{2(1 - |a_n|)}{|a_n - z|} \left| \frac{a_n - z}{1 - \bar{a}_n z} \right| \\ &< \frac{2(1 - |a_n|)}{|a_n - z|} \\ &\leqslant \frac{2(1 - |a_n|)}{w_n |\arg a_n|^{\gamma}}. \end{aligned}$$

This inequality, in conjunction with (7) and the fact that $|c(z; a_n)| < 2$ $(n = 1, 2, \ldots, k - 1)$, implies that

$$\prod_{n=1}^{\infty} [1 + c(z; a_n)]$$

converges uniformly on

$$D - \bigcup_{j=k}^{\infty} S_j$$

(see **16**, p. 291).

Next, we want to prove that, for each $m (m > 0)$, $R(m, 0, \gamma)$ meets at most a finite number of the disks S_n. Given any $m (m > 0)$, it will suffice to prove that, for j sufficiently large, $z_0 \in S_j \cap D$ entails $1 - |z_0| < m\, |\arg z_0|^{\gamma}$.

Clearly,

$$1 - |z_0| < 1 - |a_j| + w_j\, |\arg a_j|^{\gamma}. \tag{8}$$

Also, for j sufficiently large,

$$|\arg z_0 - \arg a_j| < \arc\sin \{w_j |\arg a_j|^{\gamma}/|a_j|\} < \pi w_j |\arg a_j|^{\gamma}.$$

Consequently, for j sufficiently large,

$$m|\arg z_0|^{\gamma} \geqslant m(|\arg a_j| - |\arg z_0 - \arg a_j|)^{\gamma} > m(|\arg a_j| - \pi w_j |\arg a_j|^{\gamma})^{\gamma}.$$

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Hence, if $m|\arg a_j|^\gamma\{1 - \pi w_j|\arg a_j|^{\gamma-1}\}^\gamma \geqslant 1 - |a_j| + w_j|\arg a_j|^\gamma$ for j sufficiently large, then, by virtue of (8), we shall be finished. We might just as well prove that

$$m\{1 - \pi w_j|\arg a_j|^{\gamma-1}\}^\gamma \geqslant (1 - |a_j|)/|\arg a_j|^\gamma + w_j$$

for j sufficiently large. Condition (6) implies that $(1 - |a_j|)/|\arg a_j|^\gamma \to 0$ as $j \to \infty$. Also $\gamma - 1 \geqslant 0$, $|\arg a_j| \leqslant \pi$, and $w_j \to 0$ as $j \to \infty$; the desired conclusion follows at once.

Now, we are ready to prove that $B(z; \{a_n\})$ has a T_γ-limit of modulus one at $z = 1$. Let $m(m > 0)$ be fixed. Since $R(m, 0, \gamma)$ meets at most a finite number of disks S_n,

$$D - \bigcup_{n=k}^{\infty} S_n \supset R(m, 0, \gamma)$$

for some integer k; and, hence,

$$\prod_{n=1}^{\infty} b(z; a_n)$$

converges uniformly on $R(m, 0, \gamma)$.

Since

$$\prod_{n=1}^{N} b(z; a_n)$$

(N any fixed positive integer) is a rational function with only a finite number of poles, all of which are outside of $D \cup C$,

$$\prod_{n=1}^{N} b(z; a_n) \to \prod_{n=1}^{N} b(1; a_n)$$

as $z \to 1$ on $R(m, 0, \gamma)$. Consequently, by virtue of the uniform convergence (see **12**, p. 42 or **13**, p. 551),

$$B(z; \{a_n\}) = \lim_{N\to\infty} \prod_{n=1}^{N} b(z; a_n) \to \lim_{N\to\infty} \prod_{n=1}^{N} b(1; a_n) = B(1; \{a_n\})$$

as $z \to 1$ on $R(m, 0, \gamma)$. Finally, we observe that the limit is of modulus one since $|b(1; a_n)| = 1$ $(n = 1, 2, \ldots)$.

That the same conclusion holds for any subproduct is obvious.

3. Global boundary behaviour. We next prove that, if a Blaschke sequence is sparsely distributed in the sense that (4) holds, then the associated Blaschke product has T_γ-limits $(1 \leqslant \gamma < 1/\alpha)$ almost everywhere.

THEOREM 2. *Let $\{a_n\}$ be a Blaschke sequence such that*

$$\sum_{n=1}^{\infty} (1 - |a_n|)^\alpha < \infty$$

for some fixed $\alpha(0 < \alpha < 1)$. Then, for each $\gamma(1 \leqslant \gamma < 1/\alpha)$,

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$$E_\gamma = \{e^{i\theta}: \sum_{n=1}^{\infty} (1 - |a_n|)/|e^{i\theta} - a_n|^\gamma = \infty\}$$

has zero capacity of order $\alpha\gamma$.

Preliminary remarks. Before commencing the proof, let us recall the definitions of capacity and Hausdorff outer measure.

Let U be a set in the complex plane, and let α be a positive number. Then the *α-dimensional Hausdorff outer measure of U* is defined to be the number

$$h_\alpha(U) = \sup_{\epsilon>0} \inf\left\{\sum_{i=1}^{\infty} (d(U_i))^\alpha: U \subset \bigcup_{i=1}^{\infty} U_i; d(U_i) < \epsilon, i = 1, 2, \ldots\right\},$$

where $d(U_i)$ denotes the diameter of U_i.

Let U be a bounded Borel set in the complex plane, and let α be a positive number. If there exists a positive mass distribution μ over U of total mass one such that

$$\int_U |z - z_0|^{-\alpha} d\mu(z)$$

is bounded away from infinity, the set U is said to be of *positive capacity of order α*; otherwise, it is said to be of *capacity zero of order α*, and we write $c_\alpha(U) = 0$.

There is a strong interrelation between these two concepts. For example, if U (a Borel set) is bounded and $h_\alpha(U) = 0$, then $c_\alpha(U) = 0$. The converse is not true; but, if U is compact (or merely bounded) and $h_\alpha(U) > 0$, then $c_\beta(U) > 0$ for all $\beta < \alpha$.

We shall need one more fact: If $\{U_n\}$ is a countable collection of Borel sets for which $c_\alpha(U_n) = 0$ $(n = 1, 2, \ldots)$, then

$$c_\alpha\left(\bigcup_{n=1}^{\infty} U_n\right) = 0.$$

For detailed discussions of these concepts, see (**8**; **12**, pp. 133–135; 7; and **3**).

Proof. Hold $\gamma (\gamma \geqslant 1)$ fixed; and, for each positive integer n, let O_n be an open arc on C with centre at $a_n/|a_n|$ and of length $(1 - |a_n|)^{1/\gamma}$. Let

$$G_n = \bigcup_{k=n}^{\infty} O_k$$

and $F_n = C - G_n$. Then clearly

$$\bigcup_{n=1}^{\infty} F_n$$

and

$$\bigcap_{n=1}^{\infty} G_n$$

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are disjoint sets whose union is C. Let $E_\gamma \cap F_n = f_n$ and

$$E_\gamma \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = G.$$

We shall prove that $h_{\alpha\gamma}(G) = 0$ and $c_{\alpha\gamma}(f_n) = 0$ $(n = 1, 2, \ldots)$.

First, we observe that

$$h_{\alpha\gamma}\left(\bigcap_{n=1}^{\infty} G_n \right) = 0$$

since

$$\bigcup_{k=N}^{\infty} O_k$$

is a cover for

$$\bigcap_{n=1}^{\infty} G_n$$

for each N and

$$\lim_{N\to\infty} \sum_{k=N}^{\infty} \{(1 - |a_k|)^{1/\gamma}\}^{\alpha\gamma} = 0.$$

Hence, $h_{\alpha\gamma}(G) = 0$.

At this point, the reader should convince himself that E_γ is a Borel set. Since the argument is straightforward, we omit the details.

Next, let us examine F_n (n fixed). If $e^{i\theta}$ is an arbitrary point of F_n, one has, for $k \geqslant n$, $|e^{i\theta} - a_k| > (1/\pi)(1 - |a_k|)^{1/\gamma}$; or $(1 - |a_k|)/|e^{i\theta} - a_k|^\gamma < \pi^\gamma$. Suppose that $c_{\alpha\gamma}(f_n) > 0$. Then, there exists a positive mass distribution $\mu(\theta)$ over f_n such that

$$\int_{f_n} |e^{i\theta} - z|^{-\alpha\gamma} d\mu(\theta) < M < \infty$$

for all z in the plane.

In particular, for $k \geqslant n$,

$$\begin{aligned}\int_{f_n} \frac{1 - |a_k|}{|e^{i\theta} - a_k|^\gamma} d\mu(\theta) &= \int_{f_n} (1 - |a_k|)^\alpha \left(\frac{1 - |a_k|}{|e^{i\theta} - a_k|^\gamma} \right)^{1-\alpha} \cdot \frac{d\mu(\theta)}{|e^{i\theta} - a_k|^{\alpha\gamma}} \\ &< \int_{f_n} (1 - |a_k|)^\alpha \, \pi^{\gamma(1-\alpha)} \cdot \frac{d\mu(\theta)}{|e^{i\theta} - a_k|^{\alpha\gamma}} \\ &< (1 - |a_k|)^\alpha \, \pi^{\gamma(1-\alpha)} \cdot M.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{f_n} \left\{ \sum_{k=n}^{\infty} (1 - |a_k|)/|e^{i\theta} - a_k|^\gamma \right\} d\mu(\theta) \\ &\qquad = \sum_{k=n}^{\infty} \int_{f_n} \{(1 - |a_k|)/|e^{i\theta} - a_k|^\gamma\} d\mu(\theta) \\ &\qquad < \sum_{k=n}^{\infty} (1 - |a_k|)^\alpha \, \pi^{\gamma(1-\alpha)} \cdot M < \infty,\end{aligned}$$

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which contradicts the assumption that

$$\sum_{k=1}^{\infty} (1 - |a_k|)/|e^{i\theta} - a_k|^{\gamma} = \infty$$

on f_n. Hence, $c_{\alpha\gamma}(f_n) = 0$ $(n = 1, 2, \ldots)$.

Finally, $h_{\alpha\gamma}(G) = 0$ implies that $c_{\alpha\gamma}(G) = 0$; moreover, since G and f_n $(n = 1, 2, \ldots)$ are Borel sets, we conclude that $c_{\alpha\gamma}(E_\gamma) = 0$.

Combining Theorems 1 and 2, we get the following theorem:

THEOREM 3. *Let $\{a_n\}$ be a Blaschke sequence such that*

$$\sum_{n=1}^{\infty} (1 - |a_n|)^{\alpha} < \infty$$

for some fixed $\alpha(0 < \alpha < 1)$. Then, corresponding to each $\gamma(1 \leqslant \gamma < 1/\alpha)$, there is a set E_γ whose capacity of order $\alpha\gamma$ is zero such that $B(z; \{a_n\})$ and all its subproducts have T_γ-limits of modulus one at each point of $C - E_\gamma$.

Next, let us prove that Theorem 3 is the best possible result in the following sense:

THEOREM 4. *Let α be a fixed number $(0 < \alpha < 1)$, and let $\{d_n\}$ be a monotone sequence such that $0 < d_n < 1$ $(n = 1, 2, \ldots)$,*

$$\sum_{n=1}^{\infty} d_n < \infty,$$

and

$$\sum_{n=1}^{\infty} d_n^{\alpha} = \infty.$$

Then, given $\gamma(1 \leqslant \gamma < 1/\alpha)$, one can construct a Blaschke sequence $\{a_n\}$ where $1 - |a_n| = d_n$ and a set N_γ where $c_\beta(N_\gamma) > 0$ for all $\beta < \alpha\gamma$ in such a way that, at each point of N_γ, $B(z; \{a_n\})$ fails to have a non-zero T_γ-limit. Moreover, one can construct a subproduct of $B(z; \{a_n\})$ which, at each point of N_γ, fails to have a T_γ-limit.

Proof. We shall construct a perfect set N_γ and choose the arguments of the zeros a_k in such a way that each of the sets $R(m, \theta, \gamma)$ (for $e^{i\theta} \in N_\gamma$ and a certain positive number m) contains infinitely many a_k.

Hold γ fixed, and let $t_n = 2^{-n/\alpha\gamma} \cdot n^{-2/\alpha\gamma}$ $(n = 1, 2, \ldots)$. Let A_1 be an arbitrary closed arc of length t_1 on C; and, by removing an open arc from its centre, construct two subarcs A_2 and A_3, each of length t_2. In a similar fashion, select from A_2 and A_3 4 arcs A_4, A_5, A_6, A_7, each of length t_3, and so on. Note that this construction is possible since $2t_{n+1} < t_n$ $(n = 1, 2, \ldots)$. Let $N_\gamma = A_1 \cap (A_2 \cup A_3) \cap (A_4 \cup A_5 \cup A_6 \cup A_7) \cap \ldots$ be the resulting generalized Cantor set.

Next, select arg a_k in such a way that a_k is on the radius through the centre

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of A_k. Then, consider any point $e^{i\theta} \in A_k$, where $2^{n-1} \leqslant k < 2^n$. We see that $|e^{i\theta} - a_k| < d_k + t_n/2$. Hence,

$$|e^{i\theta} - a_k|^\gamma < 2^\gamma[d_k^\gamma + t_n^\gamma \cdot 2^{-\gamma}],$$

or

$$\frac{d_k}{|e^{i\theta} - a_k|^\gamma} > \frac{1}{2^\gamma[d_k^{\gamma-1} + t_n^\gamma \cdot 2^{-\gamma} \cdot d_k^{-1}]}$$
$$\geqslant \frac{1}{2^\gamma[d_k^{\gamma-1} + t_n^\gamma \cdot 2^{-\gamma} \cdot d_{2^n}^{-1}]}.$$

Next, we notice that

$$\sum_{k=1}^{\infty} d_k^\alpha \leqslant \sum_{n=0}^{\infty} 2^n \cdot d_{2^n}^\alpha .$$

The first series diverges by hypothesis; consequently, the second must also diverge. It follows that

$$2^n \cdot d_{2^n}^\alpha > n^{-2}$$

for an infinite number of indices n, say, $\{n_j\}$, where $n_1 < n_2 < n_3 < \ldots$. Then,

$$(d_{2^{n_j}})^\alpha > 2^{-n_j} \cdot n_j^{-2} \; (j = 1, 2, \ldots);$$

and, hence,

$$(d_{2^{n_j}})^{1/\gamma} > 2^{-n_j/\alpha\gamma} \cdot n_j^{-2/\alpha\gamma} = t_{n_j} \; (j = 1, 2, \ldots).$$

This yields

$$t_{n_j}^\gamma \cdot 2^{-\gamma} \cdot d_{2^{n_j}}^{-1} < 2^{-\gamma} \; (j = 1, 2, \ldots).$$

Now, suppose that $e^{i\theta} \in N_\gamma$. Then, for each n_j $(j = 1, 2, \ldots)$, $e^{i\theta}$ is in one of the arcs

$$A_k \; (2^{n_j-1} \leqslant k < 2^{n_j}),$$

say,

$$A_{k_j}.$$

Consequently,

$$\frac{d_{k_j}}{|e^{i\theta} - a_{k_j}|^\gamma} > \frac{1}{2^\gamma[d_{k_j}^{\gamma-1} + t_{n_j}^\gamma \cdot 2^{-\gamma} \cdot d_{2^{n_j}}^{-1}]}$$
$$> \frac{1}{2^\gamma[d_{k_j}^{\gamma-1} + 2^{-\gamma}]}$$
$$\geqslant 1/(1 + 2^\gamma) = s_\gamma > 0$$

for $j = 1, 2, \ldots$.

Such being the case,

$$1 - |a_{k_j}| > s_\gamma|e^{i\theta} - a_{k_j}|^\gamma$$
$$> (2/\pi)^\gamma \cdot s_\gamma|\arg a_{k_j} - \theta|^\gamma \quad (j = 1, 2, \ldots).$$

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Consequently,

$$a_{k_j} \in R(m, \theta, \gamma)$$

for $j = 1, 2, \ldots$ if $m = (2/\pi)^\gamma \cdot s_\gamma$. Set

$$z_j = a_{k_j} \qquad (j = 1, 2, \ldots).$$

Then,

$$\lim_{j\to\infty} B(z_j; \{a_n\}) = 0,$$

and $z_j \to e^{i\theta}$ as $j \to \infty$; thus, if $B(z; \{a_n\})$ has a T_γ-limit at $e^{i\theta}$, the limit must be zero.

Finally, we observe that

$$h_\beta(N_\gamma) = \lim_{n\to\infty} 2^{n-1} t_n^\beta = \lim_{n\to\infty} 2^{-1} 2^{n(1-\beta/\alpha\gamma)} \cdot n^{-2\beta/\alpha\gamma} = \infty$$

if $\beta < \alpha\gamma$. This, combined with the fact that N_γ is closed, implies that $c_\beta(N_\gamma) > 0$ for all $\beta < \alpha\gamma$.

This completes the proof of the first part of the theorem.

Next, we shall construct a subproduct of $B(z; \{a_n\})$ which does not have a T_γ-limit on N_γ. Select a sequence $\{c_n\}$ in such a way that $0 < c_n < 1$ $(n = 1, 2, \ldots)$ and

$$\prod_{n=1}^{\infty} c_n > 0.$$

We shall define a sequence of real numbers $\{r_k\}$ $(0 < r_k < r_{k+1} < 1)$ and an increasing sequence of positive integers $\{j_k\}$.

Let

$$W(j) = \{a_m : 2^{n_j - 1} \leqslant m < 2^{n_j}\}$$

where $\{n_1, n_2, \ldots\}$ is the set of indices defined above. For the sake of simplicity, set $W(j_k) = W_k$ once j_k has been defined for a fixed integer k.

Let $j_1 = 1$. Then

$$B(z; W_1) = \prod_{W_1} b(z; a_n)$$

is a *finite* Blaschke product; and, hence, we can choose r_1 $(0 < r_1 < 1)$ in such a way that

$$|B(z; W_1)| \geqslant c_1 \tag{9}$$

for $r_1 \leqslant |z| \leqslant 1$.

Next, we want to select j_2 in such a way that $|B(z; W(j_2))| \geqslant c_2$ for $|z| \leqslant r_1$. It suffices to take j_2 so large that

$$\sum_{W(j_2)} (1 - |a_m|) < \frac{1 - r_1}{1 + r_1} \log\left\{1 + \frac{1 - c_2}{2}\right\};$$

for, if $|z| \leqslant r_1$, we have

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$$\Big|\prod_{W_2} b(z;a_m) - 1\Big| \leqslant \prod_{W_2} (1 + |c(z;a_m)|) - 1$$
$$= \prod_{W_2}\left[1 + \left|\frac{(|a_m| - 1)(a_m + |a_m|z)}{a_m(1 - \bar{a}_m z)}\right|\right] - 1$$
$$\leqslant \prod_{W_2}\left[1 + (1 - |a_m|)\frac{1 + r_1}{1 - r_1}\right] - 1$$
$$< \exp\left[\sum_{W_2} (1 - |a_m|)\frac{1 + r_1}{1 - r_1}\right] - 1$$
$$< (1 - c_2)/2.$$

By virtue of (3), such a j_2 exists.

Next, choose r_2 $(r_1 < r_2 < 1)$ in such a way that

$$|B(z; W_1 \cup W_2)| \geqslant c_2 \tag{10}$$

for $r_2 \leqslant |z| \leqslant 1$.

Then, take j_3 to be so large that

$$|B(z; W(j_3))| \geqslant c_3 \tag{11}$$

for $0 \leqslant |z| \leqslant r_2$.

By induction, define $\{r_m\}$ and $\{j_m\}$ in such a way that

$$|B(z; W_m)| \geqslant c_m \tag{12}$$

for $0 \leqslant |z| \leqslant r_{m-1}$ $(m > 1)$, and

$$|B(z; \bigcup_{k=1}^{m} W_k)| \geqslant c_m \tag{13}$$

for $r_m \leqslant |z| \leqslant 1$ $(m \geqslant 1)$.

From (9), (11), and (12), we see that $|B(z; W_1)| \geqslant c_1$, $|B(z; W_2)| \geqslant c_2, \ldots,$ $|B(z; W_m)| \geqslant c_m, \ldots$ for $|z| = r_1$. Consequently,

$$|B(z; \bigcup_{m=1}^{\infty} W_m)| \geqslant \prod_{m=1}^{\infty} c_m$$

for $|z| = r_1$.

Likewise, (10), (11), and (12) yield $|B(z; W_1 \cup W_2)| \geqslant c_2$, $|B(z; W_3)| \geqslant c_3$, $\ldots$ for $|z| = r_2$; and, therefore,

$$|B(z; \bigcup_{m=1}^{\infty} W_m)| \geqslant \prod_{m=2}^{\infty} c_m$$

for $|z| = r_2$.

In general, (13) and (12) yield

$$|B(z; \bigcup_{m=1}^{\infty} W_m)| \geqslant \prod_{m=n}^{\infty} c_m$$

for $|z| = r_n$; and, therefore,

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$$\limsup_{r\to 1} |B(re^{i\theta}; \bigcup_{m=1}^{\infty} W_m)| = 1$$

for all θ $(0 \leqslant \theta < 2\pi)$. Let

$$W = \bigcup_{m=1}^{\infty} W_m.$$

Then, at each point of N_γ, the subproduct $B(z; W)$ of $B(z; \{a_n\})$ fails to have a T_γ-limit.

4. Boundary behaviour of the successive derivatives. Since the techniques used in this section are similar to those used in § 2, we shall merely sketch the proof of the following theorem:

THEOREM 5. *Let $\{a_n\}$ be a Blaschke sequence such that*

$$\sum_{n=1}^{\infty} (1 - |a_n|)/|e^{i\theta} - a_n|^{\gamma} < \infty$$

for some fixed number γ. Then, if $\gamma \geqslant 2k$ for some positive integer k, the kth derivative of $B(z; \{a_n\})$, as well as the kth derivative of any subproduct of $B(z; \{a_n\})$, has a $T_{\gamma/2k}$-limit at $e^{i\theta}$.

Proof. For typographical reasons, let $B(z; \{a_n\}) = B(z) = B$. First, we establish the theorem for $k = 1$. By Theorem 1, B has a T_γ-limit at $e^{i\theta}$; hence, *a fortiori*, B has a $T_{\gamma/2}$-limit at $e^{i\theta}$.

A simple calculation yields

$$B' = B \cdot S \tag{14}$$

where

$$S = S(z) = \sum_{n=1}^{\infty} (1 - |a_n|^2)/(1 - \bar{a}_n z)(z - a_n).$$

As in § 2, we can assume without loss of generality that no zeros fall on the radius terminating at $e^{i\theta}$; and, as before, we select $\{w_n\}$ in such a way that (7) holds.

Let

$$Q_n = \{z\colon |z - a_n| < w_n^{1/2}|\arg a_n - \theta|^{\gamma/2}\}.$$

Then, as in the proof of Theorem 1, one proves that $R(m, \theta, \gamma/2)$ meets only a finite number of disks Q_n. Since the proof is not essentially different, we omit the details.

Next, let us prove that $S(z)$ approaches a (finite) limit as z approaches $e^{i\theta}$ on $R(m, \theta, \gamma/2)$. Select an integer n_0 in such a way that

$$R(m, \theta, \gamma/2) \cap \bigcup_{n=n_0}^{\infty} Q_n$$

is empty, and decompose S as follows: $S = S_F + S_R$ where

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$$S_R = S_R(z) = \sum_{k=n_0}^{\infty} (1 - |a_k|^2)/(1 - \bar{a}_k z)(z - a_k).$$

Since S_F is a rational function with only a finite number of poles, none of which is at $e^{i\theta}$, $S_F(z) \to S_F(e^{i\theta})$ as $z \to e^{i\theta}$ on $R(m, \theta, \gamma/2)$.

Next, consider $S_R(z)$. For $z \in D - Q_k$, we have

$$\left|\frac{1 - |a_k|^2}{(1 - \bar{a}_k z)(z - a_k)}\right| < \frac{1 - |a_k|^2}{|z - a_k|^2} \leqslant \frac{2(1 - |a_k|)}{w_k |\arg a_k - \theta|^{\gamma}}.$$

Consequently, by (7), $S_R(z)$ converges uniformly on

$$D - \bigcup_{k=n_0}^{\infty} Q_k,$$

and, *a fortiori*, uniformly on $R(m, \theta, \gamma/2)$; and $S_R(z) \to S_R(e^{i\theta})$ (finite) as $z \to e^{i\theta}$ on $R(m, \theta, \gamma/2)$. Thus

$$S(z) \to S_F(e^{i\theta}) + S_R(e^{i\theta}) = e^{-i\theta} \sum_{k=1}^{\infty} (1 - |a_k|^2)/|e^{i\theta} - a_k|^2$$

as $z \to e^{i\theta}$ on $R(m, \theta, \gamma/2)$.

Finally, combining this result with (14), we see that

$$B'(z; \{a_n\}) \to B(e^{i\theta}; \{a_n\}) e^{-i\theta} \sum_{k=1}^{\infty} \frac{1 - |a_k|^2}{|e^{i\theta} - a_k|^2}$$

as $z \to e^{i\theta}$ on $R(m, \theta, \gamma/2)$.

We are now ready to prove the general case. First, we make some preliminary observations.

Appealing to Weierstrass's theorem, one can easily prove that

$$S^{(k)}(z) = \sum_{n=1}^{\infty} \frac{d^k}{dz^k} \left\{\frac{1 - |a_n|^2}{(1 - \bar{a}_n z)(z - a_n)}\right\} \quad (k = 1, 2, \ldots)$$

provided $z \neq a_n$ $(n = 1, 2, \ldots)$ and $|z| < 1$.

Leibniz's rule, applied to (14), yields

$$B^{(t)} = \sum_{k=0}^{t-1} B^{(k)} S^{(t-k-1)} \binom{t-1}{k} \quad (z \neq a_n). \tag{15}$$

Our induction hypothesis is that, if (5) holds $(\gamma \geqslant 2k)$, then $S^{(k-1)}$ and $B^{(k)}$ have $T_{\gamma/2k}$-limits $(k = 1, 2, \ldots, t - 1)$ at $e^{i\theta}$.

We now prove this assertion for $k = t$. Clearly, under this hypothesis, $B^{(k)}$ $(k = 0, 1, \ldots, t - 1)$ and $S^{(k)}$ $(k = 0, 1, \ldots, t - 2)$ have $T_{\gamma/2t}$-limits at $e^{i\theta}$. In view of (15), once we prove that $S^{(t-1)}$ has a $T_{\gamma/2t}$-limit at $e^{i\theta}$, it will follow that $B^{(t)}$ has a $T_{\gamma/2t}$-limit at $e^{i\theta}$; and, accordingly, our induction will be completed.

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Let $u_n = (1 - \bar{a}_n z)(z - a_n)$. Then one can easily verify (see **4**, p. 131) that

$$S^{(k)} = \sum_{n=1}^{\infty} (1 - |a_n|^2)\left[k! \sum \frac{(-1)(-2)\ldots(-\sigma)}{\alpha_1!\,\alpha_2!\ldots\alpha_k!} u_n^{k-\sigma} \prod_{j=1}^{k} \left(\frac{u_n^{(j)}}{j!}\right)^{\alpha_j}\right] \Big/ u_n^{k+1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers, $\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k = k$, and $\sigma = \alpha_1 + \ldots + \alpha_k$.

When $k = t - 1$, a simple calculation shows that the nth term in the above sum is bounded by $\alpha(1 - |a_n|)|z - a_n|^{-2t}$ where α is a constant which is independent of n and z provided $|z| < 1$. (Note that

$$u_n^{(j)}(z) \equiv 0 \; (j \geqslant 3).)$$

Next, we let

$$Q_n^{(t)} = \{z\colon |z - a_n| < w_n^{1/2t}|\arg a_n - \theta|^{\gamma/2t}\}$$

where w_n is defined as in § 2. Then, as in § 2, we prove that $R(m, \theta, \gamma/2t)$ meets only a finite number of the disks $Q_n^{(t)}$ $(n = 1, 2, \ldots)$. Since the rest of the proof is not essentially different from the case when $k = 1$, we omit the details.

Combining Theorems 2 and 5, we get the following result:

THEOREM 6. *Let $\{a_n\}$ be a Blaschke sequence such that*

$$\sum_{n=1}^{\infty} (1 - |a_n|)^{\alpha} < \infty$$

for some fixed α $(0 < \alpha < 1)$. Then, corresponding to each γ $(2k \leqslant \gamma < 1/\alpha$; k some positive integer), there is a set E_γ whose capacity of order $\alpha\gamma$ is zero such that the kth *derivative of $B(z; \{a_n\})$, as well as the* kth *derivative of any sub-product of $B(z; \{a_n\})$, has a $T_{\gamma/2k}$-limit at each point of $C - E_\gamma$.*

5. Conclusion. Several concluding remarks seem to be in order.

First, it should be pointed out that, although, by Theorem 3, the exceptional set E_γ is metrically small, it need not be topologically small. (Clearly it may be topologically small, as was the exceptional set constructed in Theorem 4.) Indeed, one can easily construct a Blaschke sequence in such a way that (4) holds and the union of its elements has C as its derived set. Then, by known results from cluster set theory (see **1** and **5**), one can infer that the radial cluster set of the associated Blaschke product is equal to $D \cup C$ at each point of a residual set in C.

Second, there are good reasons for believing that the converses of Theorems 1 and 5 are valid. In fact, Frostman (**8**) has proved that converse of Theorem 1 for the case of radial limits. Since the converses were not essential for the purpose of this paper, and since it appeared that proofs would necessarily be tedious, we have intentionally relegated the question; we hope to settle the matter in a future paper.

Third, Theorem 5 assures us that, if (5) holds, then any rectifiable curve

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in $R(m, \theta, \gamma/2)$ is mapped onto a rectifiable curve by $B(z; \{a_n\})$ or by any subproduct thereof. The present author conjectures that this result can be sharpened in the sense that $R(m, \theta, \gamma/2)$ can be replaced by the larger set $R(m, \theta, \gamma)$. The author can prove this when $\gamma = 1$ and the curves are line segments in D terminating at $e^{i\theta}$.

Fourth, in his thesis **(3)**, Carleson singles out certain subclasses of the class of functions of bounded characteristic, among which are all Blaschke products satisfying (4), and proves that these functions have radial limits off certain sets of capacity zero of order α. In other words, he extends the special case of Theorem 3 when $\gamma = 1$ to a much larger class of functions. It seems natural to conjecture that Theorem 3 can likewise be extended.

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