Bull. Aust. Math. Soc. **93** (2016), 467–472
doi:10.1017/S0004972715001306

CHARACTERISATION OF THE FOURIER TRANSFORM ON COMPACT GROUPS

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(Received 21 July 2015; accepted 13 August 2015; first published online 13 November 2015)

Abstract

Let G be a compact group. The aim of this note is to show that the only continuous *-homomorphism from $L^1(G)$ to ℓ^∞- $\bigoplus_{[\pi]\in\hat{G}} \mathcal{B}_2(\mathcal{H}_\pi)$ that transforms a convolution product into a pointwise product is, essentially, a Fourier transform. A similar result is also deduced for maps from $L^2(G)$ to ℓ^2- $\bigoplus_{[\pi]\in\hat{G}} \mathcal{B}_2(\mathcal{H}_\pi)$.

2010 *Mathematics subject classification*: primary 43A30; secondary 22C05, 43A32, 43A77.
Keywords and phrases: compact groups, group Fourier transform.

1. Introduction

The study of the Fourier transform on function spaces over $\mathbb{R}^n$ is a classical topic in harmonic analysis and the behaviour of the Fourier transform under various operations is well known. A most striking aspect is that these properties can also characterise the Fourier transform. One of the well-known properties of the Fourier transform is that it takes a convolution product into a pointwise product. So, it is natural to ask: *suppose that there exists a map which converts convolution products into pointwise products. Does it have any relation to the Fourier transform?*

In [1, 2], Alesker *et al.* tried to characterise the Fourier transform in this way. In [5], Jaming proved such a characterisation for the Fourier transform on the groups $\mathbb{Z}, \mathbb{Z}_n, \mathbb{R}^n$ and $\mathbb{T}^n$. A similar characterisation of the Fourier transform on the Heisenberg group was proved by Lakshmi Lavanya and Thangavelu [6]. In fact, their work serves as a motivation for the proof of the main results of this article.

Now let G be a compact group. In Section 3, after some preliminaries in Section 2, we prove a similar result for the Fourier transform on a compact group. We also characterise the Fourier transform on $L^2(G)$.

2. Preliminaries

Throughout this paper, G will always denote a compact group. It is well known that G possesses a unique Haar measure dx such that $\int_G dx = 1$. The convolution of two functions $f, g \in L^1(G)$, denoted $f * g$, is defined by

$$f * g(x) = \int_G f(xy^{-1})g(y)\,dy, \quad x \in G.$$

An irreducible unitary representation of G is always finite dimensional. Let $\widehat{G}$ denote the set of unitary equivalence classes of irreducible unitary representations of G. Then $\widehat{G}$ is called the unitary dual of G and $\widehat{G}$ is given the discrete topology.

Given a representation π and $u, v \in \mathcal{H}_\pi$, the mapping $x \mapsto \langle \pi(x)u, v\rangle_{\mathcal{H}_\pi}$ is called a coefficient function of π. Let $\mathcal{E}_\pi$ denote the space of all coefficient functions of the representation π. The space $\mathcal{E}_\pi$ depends only on the equivalence class containing π and not on the choice of a particular representative.

Let $\{(X_\alpha, \|\cdot\|_\alpha)\}_{\alpha\in\wedge}$ be a collection of Banach spaces. For $1 \le p < \infty$, we shall denote by ℓ^p- $\bigoplus_{\alpha\in\wedge} X_\alpha$ the Banach space

$$\Big\{(x_\alpha) \in \prod_{\alpha\in\wedge} X_\alpha : \sum_{\alpha\in\wedge} \|x_\alpha\|_\alpha^p < \infty\Big\}$$

equipped with the norm $\|(x_\alpha)\|_p := \big(\sum_{\alpha\in\wedge} \|x_\alpha\|_\alpha^p\big)^{1/p}$. Similarly, we shall denote by ℓ^∞- $\bigoplus_{\alpha\in\wedge} X_\alpha$ the Banach space $\{(x_\alpha) \in \prod_{\alpha\in\wedge} X_\alpha : \sup_{\alpha\in\wedge} \|x_\alpha\|_\alpha < \infty\}$ equipped with the norm $\|(x_\alpha)\|_\infty := \sup_{\alpha\in\wedge} \|x_\alpha\|_\alpha$.

Theorem 2.1. *Let G be a compact group.*

(i) *The coefficient function arising out of an irreducible unitary representation belongs to* $L^2(G)$.

(ii) *(Schur's orthogonality relations.) If* $[\pi], [\sigma] \in \widehat{G}$ *and* $[\pi] \neq [\sigma]$, *then the spaces* $\mathcal{E}_\pi$ *and* $\mathcal{E}_\sigma$ *are mutually orthogonal subspaces of* $L^2(G)$.

(iii) *(Peter–Weyl theorem.) The space* $L^2(G)$ *is equal to the closure of the direct sum of the coefficient spaces of the irreducible unitary representations of G, that is,*

$$L^2(G) = \ell^2\text{-} \bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi.$$

Definition 2.2. Let $f \in L^1(G)$. Then the Fourier transform of f is defined by

$$\hat{f}(\pi) = \dim(\pi) \int_G f(x)\pi^*(x)\,dx, \quad [\pi] \in \widehat{G}.$$

Let $\mathcal{B}_2(\mathcal{H})$ denote the Hilbert space of all Hilbert–Schmidt operators on a Hilbert space $\mathcal{H}$, with the inner product defined by

$$\langle T, S\rangle_{\mathcal{B}_2(\mathcal{H})} := \operatorname{tr}(TS^*), \quad T, S \in \mathcal{B}_2(\mathcal{H}).$$

As a consequence of the Peter–Weyl theorem, we have the following theorems.

Theorem 2.3 (Plancherel theorem). *The Fourier transform is a unitary map from* $L^2(G)$ *onto* ℓ^2- $\bigoplus_{[\pi]\in\widehat{G}} \mathcal{B}_2(\mathcal{H}_\pi)$ *and*

$$\|f\|_2^2 = \sum_{[\pi]\in\widehat{G}} \frac{1}{\dim(\pi)} \|\hat{f}(\pi)\|^2_{\mathcal{B}_2(\mathcal{H}_\pi)}, \quad f \in L^2(G).$$

https://doi.org/10.1017/S0004972715001306 Published online by Cambridge University Press

Theorem 2.4 (Fourier inversion formula). *Let $f \in L^2(G)$. The inversion formula*

$$f(x) = \sum_{[\pi]\in\widehat{G}} \mathrm{tr}(\hat{f}(\pi)\pi(x))$$

holds in the $L^2(G)$ norm.

We refer to [3, 4] for more details of harmonic analysis on compact groups.

3. Characterisation of the Fourier transform

In this section, we characterise the Fourier transform on compact groups. The main theorems of this section generalise the results of [5, 6] to the context of compact groups. The idea behind the proof is analogous to the one given in [6].

Before stating the main theorem, we introduce some notation. For $[\pi] \in \widehat{G}$, let $\mathcal{H}_\pi$ be the representation space of π of dimension d_π and $\{e_1^\pi, e_2^\pi, \ldots, e_{d_\pi}^\pi\}$ an orthonormal basis for $\mathcal{H}_\pi$. For $1 \le i, j \le d_\pi$, let E_{ij}^π be the linear transformation on $\mathcal{H}_\pi$ given by $E_{ij}^\pi(e_k^\pi) = \delta_{jk} e_i^\pi$. Again, for $1 \le i, j \le d_\pi$, let $\pi_{ij} = \langle \pi(.)e_j^\pi, e_i^\pi \rangle$ be coefficient functions of π. Notice that the space $\mathcal{E}_\pi$ is equal to $\mathrm{span}\{\pi_{ij} : 1 \le i, j \le d_\pi\}$. Further, the π_{ij} have the following properties:

(i) $\hat{\pi}_{\alpha\beta}(\sigma) = \delta_{[\sigma][\pi]} E_{\alpha\beta}^\pi$ $(1 \le \alpha, \beta \le d_\pi)$;
(ii) $\pi_{\alpha\beta} * \pi_{\gamma\eta} = \delta_{\alpha\eta} \pi_{\gamma\beta}$ $(1 \le \alpha, \beta, \gamma, \delta \le d_\pi)$.

Moreover, the space $\mathcal{L} = \mathrm{span}\{\pi_{ij} : [\pi] \in \widehat{G}, 1 \le i, j \le d_\pi\}$ is dense in $L^1(G)$.

Theorem 3.1. *Suppose that the map $T : L^1(G) \to \ell^\infty\text{-}\bigoplus_{[\pi]\in\hat{G}} \mathcal{B}_2(\mathcal{H}_\pi)$ is nonzero continuous and *-preserving and such that:*

(i) *$T(f * g)(\pi) = T(f)(\pi)T(g)(\pi)$ for all $f, g \in L^1(G)$; and*
(ii) *$T(R_x f)(\pi) = T(f)(\pi)\pi^*(x)$ for all $f \in L^1(G), x \in G, [\pi] \in \hat{G}$.*

Let $E := \{[\pi] \in \hat{G} : T(f)(\pi) \neq 0$ for some $f \in L^1(G)\}$. Then, for each $[\pi] \in E$, $T(f)(\pi)$ is equal to $\hat{f}(\pi)$ for all $f \in L^1(G)$.

Proof. In order to prove the theorem, it is enough to prove it for a dense subset of $L^1(G)$. In the light of the comments above, we prove the theorem for the dense subspace $\mathcal{L}$. Again, as $\mathcal{L}$ is just the span of all $\mathcal{E}_\pi$ for $[\pi] \in \hat{G}$, it is enough to study the action of T on each $\mathcal{E}_\pi$.

Since T is nonzero, there exist $[\pi], [\sigma] \in \hat{G}$ and $f \in \mathcal{E}_\pi$ such that $T(f)(\sigma) \neq 0$. For each $1 \le \alpha, \beta \le d_\pi$, let $Q_{\alpha\beta}^\sigma := T(\bar{\pi}_{\alpha\beta})(\sigma)$. Note that $Q_{\alpha\beta}^\sigma$ has the following properties:

(i) $Q_{\alpha\beta}^\sigma Q_{\gamma\eta}^\sigma = \delta_{\alpha\eta} Q_{\gamma\beta}^\sigma$;
(ii) $(Q_{\alpha\beta}^\sigma)^* = Q_{\beta\alpha}^\sigma$.

Further, for each $1 \le \alpha \le d_\pi$, we claim that $Q_{\alpha\alpha}^\sigma \neq 0$. On the contrary, suppose that $Q_{\alpha\alpha}^\sigma = 0$ for some α with $1 \le \alpha \le d_\pi$. Then, for any $v \in \mathcal{H}_\sigma$,

$$Q_{\alpha\beta}^\sigma v = Q_{\alpha\beta}^\sigma Q_{\alpha\alpha}^\sigma v = 0 \quad (1 \le \beta \le d_\pi).$$

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Similarly,

$$Q^{\sigma}_{\beta\alpha} v = Q^{\sigma}_{\alpha\alpha} Q^{\sigma}_{\beta\alpha} v = 0 \quad (1 \le \beta \le d_\pi).$$

Thus,

$$Q^{\sigma}_{\gamma\beta} = Q^{\sigma}_{\alpha\beta} Q^{\sigma}_{\gamma\alpha} = 0 \quad (1 \le \gamma, \beta \le d_\pi).$$

This implies that $T(f)(\sigma) = 0$ for $f \in \mathcal{E}_\pi$, which is a contradiction. Therefore, by (i) and (ii), it follows that $Q^{\sigma}_{\alpha\alpha}$ is a nonzero projection on $\mathcal{H}_\sigma$ $(1 \le \alpha \le d_\pi)$.

Let $\{u^j_{\alpha,\sigma}\}$ be an orthonormal basis for the range of $Q^{\sigma}_{\alpha\alpha}$ and let

$$v^j_{\alpha\beta,\sigma} := Q^{\sigma}_{\alpha\beta} u^j_{\alpha,\sigma}, \quad 1 \le \beta \le d_\pi.$$

The system $\{v^j_{\alpha\beta,\sigma}\}$ is an orthonormal system for each fixed α. Indeed,

$$\begin{aligned} \langle v^j_{\alpha\beta,\sigma}, v^k_{\alpha\gamma,\sigma}\rangle_{\mathcal{H}_\sigma} &= \langle Q^{\sigma}_{\alpha\beta} u^j_{\alpha,\sigma}, Q^{\sigma}_{\alpha\gamma} u^k_{\alpha,\sigma}\rangle_{\mathcal{H}_\sigma} = \langle Q^{\sigma}_{\gamma\alpha} Q^{\sigma}_{\alpha\beta} u^j_{\alpha,\sigma}, u^k_{\alpha,\sigma}\rangle_{\mathcal{H}_\sigma} \\ &= \delta_{\gamma\beta} \langle Q^{\sigma}_{\alpha\alpha} u^j_{\alpha,\sigma}, u^k_{\alpha,\sigma}\rangle_{\mathcal{H}_\sigma} = \delta_{\gamma\beta}\delta_{jk}. \end{aligned}$$

Define the Hilbert space $\mathcal{H}^j_\alpha = \operatorname{span}\{v^j_{\alpha\beta,\sigma} : 1 \le \beta \le d_\pi\}$ and define the operator $U_{\pi,\sigma} : \mathcal{H}_\pi \to \mathcal{H}^j_\alpha$ by $U_{\pi,\sigma}(e^\pi_\beta) = v^j_{\alpha\beta,\sigma}$. Then $U_{\pi,\sigma}$ is a unitary operator. Further, let $S^{\,j}_{\alpha,\sigma}(f) := U_{\pi,\sigma}\hat{f}(\pi)(U_{\pi,\sigma})^*$ for $f \in \mathcal{E}_\pi$. Then

$$\begin{aligned} S^{\,j}_{\alpha,\sigma}(\bar{\pi}_{\gamma\eta})(v^j_{\alpha\beta,\sigma}) &= U_{\pi,\sigma}\hat{\bar{\pi}}_{\gamma\eta}(\pi)(U_{\pi,\sigma})^*(v^j_{\alpha\beta,\sigma}) \\ &= U_{\pi,\sigma}\hat{\bar{\pi}}_{\gamma\eta}(\pi)e^\pi_\beta = U_{\pi,\sigma}E_{\eta\gamma}e^\pi_\beta \\ &= U_{\pi,\sigma}\delta_{\gamma\beta}e^\pi_\eta = \delta_{\gamma\beta}v^j_{\alpha\eta,\sigma}. \end{aligned}$$

On the other hand,

$$T(\bar{\pi}_{\gamma\eta})(\sigma)(v^j_{\alpha\beta,\sigma}) = Q^{\sigma}_{\gamma\eta} Q^{\sigma}_{\alpha\beta} u^j_\alpha = \delta_{\gamma\beta} Q^{\sigma}_{\alpha\eta} u^j_\alpha = \delta_{\gamma\beta} v^j_{\alpha\eta,\sigma}.$$

Hence, for any $f \in \mathcal{E}_\pi$, we have $T(f)(\sigma) = U_{\pi,\sigma}\hat{f}(\pi)(U_{\pi,\sigma})^*$. Further, note that the action of $T(f)(\sigma)$ on the orthogonal complement of $\mathcal{H}^j_\alpha$ in $\mathcal{H}_\sigma$ is 0.

We now claim that $\mathcal{H}^j_\alpha$ is invariant under σ. Since σ is a unitary representation, it is enough to prove that the complement $(\mathcal{H}^j_\alpha)^\perp$ is invariant under σ. To see this, take $v \in (\mathcal{H}^j_\alpha)^\perp$. Then, for any $f \in \mathcal{E}_\pi$, $T(f)(\sigma)(v) = 0$. As $\mathcal{E}_\pi$ is invariant under translations, it follows that, for all $x \in G$, $T(R_x f)(\sigma)v = 0$, which by our assumption is equivalent to $T(f)(\sigma)\sigma^*(x)v = 0$ for all $f \in \mathcal{E}_\pi$ and $x \in G$. Thus, $\sigma^*(x)v \in \ker(T(f)(\sigma))$. Hence, $(\mathcal{H}^j_\alpha)^\perp$ is invariant under σ. It now follows that $U_{\pi,\sigma}$ is a unitary isomorphism between $\mathcal{H}_\pi$ and $\mathcal{H}_\sigma$.

We next claim that $U_{\pi,\sigma}$ is an intertwining operator between the representations π and σ. Note that, by our assumption, for any $f \in \mathcal{E}_\pi$ and $x \in G$,

$$T(R_x f)(\sigma) = T(f)(\sigma)\sigma^*(x) = U_{\pi,\sigma}\hat{f}(\pi)(U_{\pi,\sigma})^*\sigma^*(x).$$

On the other hand,

$$T(R_x f)(\sigma) = U_{\pi,\sigma}\widehat{R_x f}(\pi)(U_{\pi,\sigma})^* = U_{\pi,\sigma}\hat{f}(\pi)\pi^*(x)(U_{\pi,\sigma})^*.$$

Therefore, $\pi^*(x)(U_{\pi,\sigma})^* = (U_{\pi,\sigma})^*\sigma^*(x)$ for all $x \in G$, or equivalently,

$$U_{\pi,\sigma}\pi(x) = \sigma(x)U_{\pi,\sigma} \quad \text{for all } x \in G.$$

Therefore, $[\sigma] = [\pi]$ and $T(f)(\pi) = \hat{f}(\pi)$ for all $f \in \mathcal{E}_\pi$. Thus, if $T|_{\mathcal{E}_\pi} \neq 0$, then, for each $f \in \mathcal{E}_\pi$, $T(f)(\sigma) = \delta_{[\sigma][\pi]}\hat{f}(\pi)$ for all $[\pi] \in \hat{G}$. □

Our next result is about maps from L^2. It is worth mentioning that we do not assume continuity of the map but, rather, this is one of the consequences.

Corollary 3.2. *Let $T : L^2(G) \to \ell^2\text{-}\bigoplus_{[\pi]\in\hat{G}} \mathcal{B}_2(\mathcal{H}_\pi)$ be a surjective *-preserving linear operator such that:*

(i) *$T(f * g)(\pi) = T(f)(\pi)T(g)(\pi)$ for all $f, g \in L^2(G)$; and*
(ii) *$T(R_x f)(\pi) = T(f)(\pi)\pi^*(x)$ for all $f \in L^2(G), x \in G, [\pi] \in \hat{G}$.*

Then $T(f)(\pi) = \hat{f}(\pi)$ for all $[\pi] \in \hat{G}, f \in L^2(G)$.

Proof. Although the proof given for the case of the Heisenberg group [6] works very well in our case also if we assume boundedness of T, we would like to give a proof based on Theorem 3.1.

We know that, by the Peter–Weyl theorem, $L^2(G) = \ell^2\text{-}\bigoplus_{[\pi]\in\hat{G}} \mathcal{E}_\pi$. Since T is surjective, it is nonzero. Thus, there exists $[\pi] \in \hat{G}$ such that $T|_{\mathcal{E}_\pi} \neq 0$. Hence, by Theorem 3.1, for all $f \in \mathcal{E}_\pi$, $T(f)(\sigma) = \delta_{[\sigma][\pi]}\hat{f}(\sigma)$ for $[\sigma] \in \hat{G}$. Again, since T is surjective, it follows that T is nonzero on each $\mathcal{E}_\pi$. Hence, the proof is completed. □

Acknowledgement

The authors would like to thank the anonymous referee for his useful suggestions.

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