Canad. J. Math. Vol. **61** (6), 2009 pp. 1325–1340

Uniqueness of Shalika Models

Chufeng Nien

Abstract. Let $\mathbb{F}_q$ be a finite field of q elements, $\mathcal{F}$ a p-adic field, and D a quaternion division algebra over $\mathcal{F}$. This paper proves uniqueness of Shalika models for $\mathrm{GL}_{2n}(\mathbb{F}_q)$ and $\mathrm{GL}_{2n}(D)$, and re-obtains uniqueness of Shalika models for $\mathrm{GL}_{2n}(\mathcal{F})$ for any $n \in \mathbb{N}$.

1 Introduction

Let $\mathbb{F}_q$ denote a finite field of q elements, and $\mathcal{F}$ a p-adic field. Let F be one of the above fields, and $D = D_{\mathcal{F}}$ a quaternion division algebra over $\mathcal{F}$. Denote by Mat_n the space of n-by-n matrices over F. Throughout, ψ_0 denotes a nontrivial, complex, additive character of F.

Given D, a quaternion division algebra over $\mathcal{F}$, there exists a basis $\{1, i, j, k\}$ for D with multiplication table given by

	1	i	j	k
1	1	i	j	k
i	i	α	k	αj
j	j	$-k$	β	$-\beta i$
k	k	$-\alpha j$	βi	$-\alpha\beta$

for suitable $\alpha, \beta \in \mathcal{F}^*$.

For $z = a + bi + cj + dk \in D$ with $a, b, c, d \in \mathcal{F}$, define the *conjugation* of z by $\bar{z} = a - bi - cj - dk$. Note that $\overline{z_1 \cdot z_2} = \bar{z}_2 \cdot \bar{z}_1$, *i.e.*, it is an anti-involution on D. (An *anti-involution* τ of an algebra (or a group) G is an operator on G so that $(gh)^\tau = h^\tau g^\tau, g, h \in G$, and $\tau^2 = \mathrm{id}$) The *reduced norm* N and *reduced trace* Tr on D are defined as usual by $Nz = z\bar{z}$ and $\mathrm{Tr}\, z = z + \bar{z}$. There is an embedding $\iota\colon D \hookrightarrow \mathrm{GL}(2, K)$ defined by

$$z = a + bi + cj + dk = (a + bi) + (c + di)j = z_1 + z_2 j \mapsto \begin{pmatrix} z_1 & z_2\beta \\ \overline{z}_2 & \overline{z}_1 \end{pmatrix},$$

where $K = \mathcal{F}(\sqrt{\alpha})$. Then $\mathrm{Tr}\, z = \mathrm{tr}(\iota z), z \in D$, where tr is the trace map on matrices. Under this embedding, D is a closed subgroup of $\mathrm{GL}(2, K)$. This embedding can be naturally extended to $\iota\colon\ \mathrm{GL}(n, D) \hookrightarrow \mathrm{GL}(2n, K)$.

Received by the editors February 2, 2007.
The project was supported by NSC 95-2115-M-006-022, Taiwan.
AMS subject classification: 22E50.
Keywords: Shalika models, linear models, uniqueness, multiplicity free.

Let A be either the field F (one of the $\mathbb{F}_q$, $\mathcal{F}$) or the quaternion division algebra D. In $\mathrm{GL}_{2n}(A)$, denote

$$d(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \ g \in \mathrm{GL}_n \quad \text{and} \quad u(X) = \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix}, \ X \in \mathrm{Mat}_n .$$

Let $\mathrm{M}_n = \{d(g) \mid g \in \mathrm{GL}_n\}$, $\mathrm{U}_n = \{u(X) \mid X \in \mathrm{Mat}_n\}$; and $\mathrm{S}_n = \mathrm{M}_n\mathrm{U}_n$. A *Shalika character* on S_n is given by

$$\psi_n(d(g)u(X)) = \begin{cases} \psi_0(\operatorname{tr} X) & \text{for } s \in \mathrm{S}_n(\mathrm{F}), \\ \psi_0(\operatorname{tr}(\iota X)) & \text{for } s \in \mathrm{S}_n(D). \end{cases}$$

Abusing notation, we abbreviate $\psi_0(\operatorname{tr}(\iota X))$ by $\psi_0(\operatorname{tr} X)$ for $X \in \mathrm{Mat}_n(D)$, since no confusion should occur. Moreover, we will always refer to smooth representations when we talk about representations of groups other than finite groups.

Let ρ be an irreducible representation of $\mathrm{GL}_{2n}(A)$.

Definition 1.1 A linear functional $\Lambda_\rho \colon V_\rho \mapsto \mathbb{C}$ is called a *Shalika functional* of V_ρ if it satisfies $\Lambda_\rho(\rho(s)v) = \psi_n(s)\Lambda_\rho(v)$ for all $s \in \mathrm{S}_n$ and $v \in V_\rho$. We say that V_ρ has a *Shalika model* if there exists a nontrivial Shalika functional Λ_ρ satisfying the above equation. This definition is equivalent to

$$\dim \mathrm{Hom}_{\mathrm{GL}_{2n}}(\rho, \mathrm{Ind}_{\mathrm{S}_n}^{\mathrm{GL}_{2n}} \psi_n) \geq 1,$$

since $\mathrm{Hom}_{\mathrm{GL}_{2n}}(\rho, \mathrm{Ind}_{\mathrm{S}_n}^{\mathrm{GL}_{2n}} \psi_n) \cong \mathrm{Hom}_{\mathrm{S}_n}(\rho|_{\mathrm{S}_n}, \psi_n)$ by reciprocity.

Definition 1.2 Given a representation π of a group G, we say that π is *multiplicity free*, or possesses the *uniqueness* property, if $\dim \mathrm{Hom}_G(\rho, \pi) \leq 1$ for any irreducible representation ρ of G.

Definition 1.3 Let $\pi = \mathrm{Ind}_{L_n}^{\mathrm{GL}_{2n}} 1$, where

$$\left\{ L_n = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}, g_i \in \mathrm{GL}_n \right\}$$

and 1 denotes the trivial representation of $\mathrm{GL}_n \times \mathrm{GL}_n$. We say that V_ρ has a *linear model* if there exists a nontrivial intertwining operator from V_ρ to π.

For general linear groups over non-archimedean local fields, uniqueness of Shalika models was proved by H. Jacquet and S. Rallis [JR] via the verification of the multiplicity freeness of linear models and the fact that existence of Shalika models of GL_{2n} implies existence of linear models. Classification of Shalika models is not yet completely established. Y. Sakellaridis [Sa] showed necessary and sufficient conditions for an irreducible unramified principal series admitting Shalika models. D. Jiang and D. Soudry [JiS2] showed a certain group of representations possessing Shalika models. D. Jiang and Y. J. Qin [JiQ] defined a generalized Shalika model for SO($4n$) and

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

found the relationship between this model and the Shalika model of $GL(2n)$. The study of Shalika models interacts intensively with other related subjects. Let π be an irreducible cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of a number field F. Then the following statements are equivalent:

(i) π is the image of Langlands' functorial lifting from SO_{2n+1};
(ii) π has a nonzero Shalika period;
(iii) the exterior square L-function $L(s, \pi, \wedge^2)$ has a simple pole at $s = 1$.

The theory has been established over years through the work of many authors [JaS, CKPS, GRS, Ki, JiS1, Ji1].

Generalizations to the case of (quaternion) division algebras have been studied by various mathematicians. D. Prasad and A. Raghuram [PR] showed the uniqueness of Shalika models and established the self-duality of irreducible representations admitting Shalika models on $GL_2(D)$. Extensions to the above theory regarding the poles of the exterior square L-function and non-vanishing of Shalika periods in the case of $GL_2(D)$ were given by H. Jacquet and K. Martin [JM]. Moreover, they stated a conjecture (the Jacquet–Martin conjecture) relating the existences of Shalika models for representations of $GL_{2n}(D)$ and $GL_{4n}(\mathcal{F})$, which is now a theorem by W. T. Gan and S. Takeda [GT] in the case of $GL_2(D)$ and $GL_4(\mathcal{F})$. In this paper we will prove uniqueness of Shalika models for GL_{2n}, $n \in \mathbb{N}$ in the setting of p-adic fields, finite fields, and p-adic quaternion division algebras. We expect that the uniqueness of Shalika models for $GL_{2n}(D)$ (or even $GL_{2n}(\mathbb{F}_q)$) could prove useful in the future. Here we present the main theorems.

Theorem 4.1 *For any $n \in \mathbb{N}$, let $G = GL_{2n}(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field. Then*

$$\dim \operatorname{Hom}_G(\rho, \operatorname{Ind}_{S_n}^G \psi_n) \leq 1$$

for any irreducible representation of G.

Theorem 4.3 *Let $G = GL_{2n}$ over either a p-adic field $\mathcal{F}$ or a quaternion division algebra D over $\mathcal{F}$. Then*

$$\dim \operatorname{Hom}_G(\rho, \operatorname{Ind}_{S_n(A)}^G \psi_n) \leq 1,$$

for any irreducible representation of G.

2 Common Strategy

Given a finite group G, it is known that a representation V_π of G is multiplicity free if and only if the endomorphism algebra $\operatorname{Hom}_G(V_\pi, V_\pi)$ is abelian. Moreover, when $V_\pi = \operatorname{Ind}_H^G \rho$ is an induced representation, $\operatorname{Hom}_G(V_\pi, V_\pi)$ is explicitly characterized by Mackey's Theorem.

Theorem 2.1 **(Mackey)** *Let G be a finite group, H_i its subgroups and π_i representations of H_i, $i = 1, 2$. Denote by*

$$\mathfrak{S} = \{\triangle\colon G \mapsto \operatorname{Hom}_{\mathbb{C}}(\pi_1, \pi_2) \mid \triangle(h_2 g h_1) = \pi_2(h_2) \circ \triangle(g) \circ \pi_1(h_1), h_i \in H_i\}.$$

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

As a vector space, $\operatorname{Hom}_G(\operatorname{Ind}_{H_1}^G \pi_1, \operatorname{Ind}_{H_2}^G \pi_2)$ *is isomorphic to* $\mathfrak{S}$*. Given any* $\triangle \in \mathfrak{S}$*, the corresponding intertwining operator* $T_\triangle \in \operatorname{Hom}_G(\operatorname{Ind}_{H_1}^G \pi_1, \operatorname{Ind}_{H_2}^G \pi_2)$ *is given by* $T_\triangle(f_1) = \triangle * f_1$ *for* $f_1 \in \operatorname{Ind}_{H_1}^G \pi_1$*, where the convolution is given by*

$$\triangle * f_1(x) = \frac{1}{|G|} \sum_{g \in G} \triangle(xg^{-1}) f_1(g).$$

In particular, when $H_1 = H_2, \pi_1 = \pi_2$*, the algebra* $\operatorname{Aut}_G(\operatorname{Ind}_{H_1}^G \pi_1)$ *is isomorphic to* $(\mathfrak{S}, \cdot)$*, where the multiplication* $\cdot$ *is given by*

$$\triangle_1 \cdot \triangle_2(g) = \sum_{x \in G} \triangle_1(gx^{-1}) \circ \triangle_2(x), \triangle_i \in \mathfrak{S}.$$

In order to show that the endomorphism algebra is abelian, identifying an anti-involution to interchange the order of factors is a common strategy. The analogue of this method in the p-adic case is the Gelfand–Kazhdan criterion, which was first investigated in [GK] and further developed in [BZ, Gr].

Let $C_c^\infty(X)$ denote the space of smooth, compactly supported functions on an l-adic space X (in the sense of [BZ]). Let $\mathfrak{D}(X)$ denote the space of linear functionals on $C_c^\infty(X)$. Given a p-adic group G, define actions L_g and R_g on G, $C_c^\infty(G)$, and $\mathfrak{D}(G)$ as the following:

$$L_g \cdot x = gx, \quad R_g \cdot x = xg^{-1};$$
$$(L_g \cdot f)(x) = f(g^{-1}x), \quad (R_g \cdot f)(x) = f(xg);$$
$$(L_g \cdot T)(f) = T(L_{g^{-1}} \cdot f), \quad (R_g \cdot T)(f) = T(R_{g^{-1}} \cdot f),$$

where $g,\ x \in G$, $f \in C_c^\infty(G)$, and $T \in \mathfrak{D}(G)$.

Theorem 2.2 **(Gelfand–Kazhdan Criterion[Ga1, Ga2])** *Let* ψ *and* ψ^τ *be characters of a closed unimodular subgroup* H *of* G*. Suppose that there is an anti-involution* τ *of* G *such that* τ *stabilizes* H*,* $\psi(h^\tau) = \psi^\tau(h)$*, and* τ *acts trivially on all distributions* T*, so that*

$$T(L_h\eta) = \psi(h) \cdot T(\eta), \qquad T(R_h\eta) = \psi^\tau(h)^{-1} \cdot T(\eta) \text{ for } \eta \in C_c^\infty(G).$$

Then $\dim \operatorname{Hom}_G(\pi; \operatorname{Ind}_H^G \psi) \cdot \dim \operatorname{Hom}_H(Res_H^G \tilde{\pi}; \psi^\tau) \leq 1$*, where* π *is any irreducible representation of* G *and* $\tilde{\pi}$ *its contragradient.*

3 Key Proposition

For $k \in \mathbb{N}$, denote by

$$w_k = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

the matrix representative of the longest Weyl elements of GL_k, and set $w_0 = \mathrm{id}$. Define anti-involutions on $\mathrm{GL}_n(\mathrm{F})$ and $\mathrm{GL}_n(D)$ respectively by

$$\tau_n\colon \mathrm{GL}_{2n}(\mathrm{F}) \mapsto \mathrm{GL}_{2n}(\mathrm{F}), g \mapsto w_{2n} g^t w_{2n}^{-1},$$
$$\tau_n : \mathrm{GL}_{2n}(D) \mapsto \mathrm{GL}_{2n}(D), g \mapsto w_{2n} \bar{g}^t w_{2n}^{-1}.$$

When n is understood, we will abbreviate τ_n by τ. Note that τ_n stabilizes S_n and ψ_n. Throughout, B_n will denote the *Borel subgroup* of GL_n and W_n the *Weyl group* of GL_n.

Let $d_1(\alpha) = \begin{pmatrix} \alpha & \\ & I_n \end{pmatrix}$ and $d_2(\beta) = \begin{pmatrix} I_n & \\ & \beta \end{pmatrix}$ for $\alpha, \beta \in \mathrm{GL}_n$.

Lemma 3.1 *The representatives of* $\mathrm{S}_n \backslash \mathrm{GL}_{2n} / \mathrm{S}_n$ *can be expressed by*

$$\left\{ d_1(\alpha)\sigma_k d_2(\beta) \;\middle|\; k = 0, \dots, n, \alpha, \beta \in \mathrm{GL}_n \right\}, \text{ where } \sigma_k = \begin{pmatrix} & & w_k \\ & I_{2n-2k} & \\ w_k & & \end{pmatrix}.$$

Proof Let $\mathrm{P}_n, n \in \mathbb{N}$ denote the parabolic subgroup of GL_{2n} corresponding to the partition $\{n, n\}$, and $\mathrm{W}_{\mathrm{P}_n}$ its Weyl group. Then

$$\mathrm{P}_n \backslash \mathrm{GL}_{2n} / \mathrm{P}_n \cong \mathrm{W}_{\mathrm{P}_n} \backslash \mathrm{W}_{2n} / \mathrm{W}_{\mathrm{P}_n} \leftrightarrow \mathfrak{L}_n,$$

where

$$\mathfrak{L}_n = \left\{ (a_{i,j}) \in \mathrm{Mat}_2(\mathbb{Z}) \mid 0 \le a_{i,j} \le n, \sum_{k=1}^{2} a_{k,j} = \sum_{k=1}^{2} a_{i,k} = n, 1 \le i, j \le 2 \right\}.$$

The last bijection refers to [GaRe]. Notice that the cardinality of $\mathfrak{L}_n$ is $n+1$. Moreover, we can choose representatives of $\mathrm{W}_{\mathrm{P}_n} \backslash \mathrm{W}_{2n} / \mathrm{W}_{\mathrm{P}_n}$ to be τ-invariant, given by $\sigma_0 = id, \sigma_k = (1, 2n)(2, 2n-1) \cdots (k, 2n+1-k), k = 1, \dots, n$. That is,

$$\sigma_k = \begin{pmatrix} & & w_k \\ & I_{2n-2k} & \\ w_k & & \end{pmatrix},$$

where

$$w_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

is the permutation matrix representative for the longest Weyl element of GL_k. ■

Let $\mathbb{H}$ denote $\mathrm{S}_n(A)$. Put $\hat{\mathbb{H}} = \mathbb{H} \times \mathbb{H}$. Let $(h_1, h_2) \in \hat{\mathbb{H}}$ act on $g \in G$ and $\eta \in C_c^\infty(G)$ by

$$(h_1, h_2) \cdot g = h_1 g h_2^{-1} \quad \text{and} \quad (h_1, h_2) \cdot \eta(g) = \psi_n(h_1^{-1} h_2) \eta(h_1^{-1} g h_2).$$

We also denote $\hat{\mathrm{M}}_n = \mathrm{M}_n \times \mathrm{M}_n \subset \hat{\mathbb{H}}$ and $\hat{\mathrm{U}}_n = \mathrm{U}_n \times \mathrm{U}_n \subset \hat{\mathbb{H}}$.

Define a character $\hat{\psi}_n$ of $\hat{\mathbb{H}}$ by $\hat{\psi}_n(\hat{h}) = \psi_n(h_1 h_2^{-1})$ for $\hat{h} = (h_1, h_2) \in \hat{\mathbb{H}}$. Denote by $\hat{\mathbb{H}}_g$ the *stabilizer* of g in $\hat{\mathbb{H}}$.

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

Definition 3.2 Define an equivalence relation $\sim$ on $g, g' \in \mathrm{GL}_{2n}$, where $g \sim g'$ means that $\hat{h} \cdot g = g'$ for some $\hat{h} \in \hat{\mathbb{H}}$. We also write $g \sim_{\hat{h}} g'$ to indicate the connecting map $\hat{h}$ satisfying $\hat{h} \cdot g = g'$.

Definition 3.3 We call a double coset $\mathrm{S}_n \gamma \mathrm{S}_n$, $g \in \mathrm{GL}_{2n}$ *admissible* if $\hat{\psi}_n$ is trivial on $\hat{\mathbb{H}}_\gamma$. In this case, the element γ is also called *admissible*.

Definition 3.4 We call a double coset $\mathrm{S}_n \gamma \mathrm{S}_n$ *$\hat{\psi}$-τ-invariant* if there exist $\hat{h} \in \hat{\mathbb{H}}$ such that $\hat{h} \cdot \gamma = \gamma^\tau$ and $\hat{\psi}_n(\hat{h}) = 1$. In this case, the element γ is also called *$\hat{\psi}$-τ-invariant*.

Note that when $\gamma \sim \beta$, γ is admissible (respectively $\hat{\psi}$-τ-invariant) if and only if β is admissible (respectively $\hat{\psi}$-τ-invariant).

The rest of this section is devoted to the proof of the following proposition.

Proposition 3.5 *Every admissible double coset $\mathrm{S}_n g \mathrm{S}_n$ is $\hat{\psi}$-τ-invariant.*

First we need some auxiliary Lemmas.

Lemma 3.6 *For $k \in \{0, \dots, n\}$, let*

$$\gamma_k = \gamma_k(\alpha, \beta) = d_1(\alpha)\sigma_k d_2(\beta) \in \mathrm{S}_n \backslash \mathrm{GL}_{2n} / \mathrm{S}_n, \alpha, \beta \in \mathrm{GL}_n,$$

where

$$\alpha = \begin{pmatrix} * & \alpha_k \\ * & * \end{pmatrix}, \ \beta = \begin{pmatrix} * & \beta_k \\ * & * \end{pmatrix}, \text{ and } \alpha_k, \beta_k \in \mathrm{Mat}_{n-k}.$$

If $\alpha_k \neq \beta_k$, then γ_k is non-admissible.

Proof For $\alpha_k \neq \beta_k$, let $u_k = u_k(X) = \left(\begin{smallmatrix} I_n & X \\ & I_n \end{smallmatrix}\right)$, where $X = \left(\begin{smallmatrix} 0 & 0 \\ \tilde{X} & 0 \end{smallmatrix}\right)$, $\tilde{X} \in \mathrm{Mat}_{n-k}$. Then $\sigma_k^{-1} u_k \sigma_k = u_k$. Let

$$n_1 = d_1(\alpha) u_k d_1(\alpha)^{-1} \in \mathrm{U}_n \quad \text{and} \quad n_2 = \gamma_k^{-1} n_1 \gamma_k = d_2(\beta)^{-1} u_k d_2(\beta) \in \mathrm{U}_n.$$

Note that $\psi_n(n_1) = \psi_0(\mathrm{tr}(\alpha X))$ and $\psi_n(n_2) = \psi_0(\mathrm{tr}(X\beta)) = \psi_0(\mathrm{tr}(\beta X))$. Then

$$\alpha X = \begin{pmatrix} \alpha_k \tilde{X} & 0 \\ * & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{tr}(\alpha X) = \mathrm{tr}(\alpha_k \tilde{X}).$$

Similarly, $\mathrm{tr}(\beta X) = \mathrm{tr}(\beta_k \tilde{X})$. If $\alpha_k \neq \beta_k$, there exists $\tilde{X} \in \mathrm{Mat}_{n-k}$ such that

$$\psi_0(\mathrm{tr}(\alpha_k \tilde{X})) \neq \psi_0(\mathrm{tr}(\beta_k \tilde{X})),$$

and hence $\gamma_k = \gamma_k(\alpha, \beta)$ is not admissible. ■

Lemma 3.7 *If $\alpha_k = \beta_k = 0_k$,*

$$\gamma_k(\alpha, \beta) = \begin{pmatrix} & 0_{n-k} & * & * \\ & * & * & * \\ & & * & 0_{n-k} \\ w_k & & & \end{pmatrix}.$$

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

In this case, $k \geq n-k$ and

$$\gamma_k \sim \begin{pmatrix} & 0 & 0 & \eta_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \lambda_k & & & \end{pmatrix},$$

for some $\lambda_k, \eta_k \in \mathrm{Mat}_k$. Moreover, for γ_k to be admissible, it is necessary that the upper-right-hand square $n-k$-blocks of η and λ^{-1} are negatives of each other.

Proof Let $e = 2k - n$. Then for suitable

$$s_1 = d\left(\begin{pmatrix} v_{n-k} & \\ & v'_k \end{pmatrix}\right),\ s_2 = d\left(\begin{pmatrix} r_k & \\ & r'_{n-k} \end{pmatrix}\right) \in \mathrm{M}_n,$$

$u, u' \in \mathrm{U}_n$, $v'_k, r_k \in \mathrm{Mat}_k$, and $v_{n-k}, r'_{n-k} \in \mathrm{Mat}_{n-k}$, we can reduce γ_k to

$$\gamma_k \sim_{(s_1,s_2)} \begin{pmatrix} & 0 & * & * \\ & I_{n-k} & * & * \\ & & I_{n-k} & 0 \\ \lambda_k & & & \end{pmatrix} \sim_{(u,u')} \begin{pmatrix} & 0 & 0 & \eta_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \lambda_k & & & \end{pmatrix} = \gamma'_k,$$

for some $\lambda_k, \eta_k \in \mathrm{Mat}_k$. Denote by η^2 (respectively D^2) the upper-right-hand square $n-k$-block of η_k (respectively λ_k^{-1}).

Let

$$s = d\left(\begin{pmatrix} I_{n-k} & & \\ & I_e & \\ A & & I_{n-k} \end{pmatrix}\right) \in \mathrm{S}_n, A \in \mathrm{Mat}_{n-k}.$$

Then

$$\gamma'_k s \gamma_k'^{-1} = \left(\begin{array}{ccc|ccc} I_{n-k} & & & \eta_k\begin{pmatrix} 0 \\ \hline A \end{pmatrix} & & \\ & I_e & & & & \\ & & I_{n-k} & & (A|0)\lambda^{-1} & \\ \hline & & & I_{n-k} & & \\ & & & & I_e & \\ & & & & & I_{n-k} \end{array}\right).$$

For γ'_k(hence γ_k) to be admissible, it is necessary that

$$\psi_n(\gamma'_k s \gamma_k'^{-1}) = \psi_0(\operatorname{tr} A(\eta^2 + D^2)) = 1$$

for all $A \in \mathrm{Mat}_{n-k}$, *i.e.*, $D^2 = -\eta^2$. ∎

Lemma 3.8 *Let*

$$\gamma'_k = \begin{pmatrix} & 0 & 0 & \eta_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \lambda_k & & & \end{pmatrix},$$

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

where $\lambda_k, \eta_k \in \mathrm{Mat}_k$ and the upper-right-hand square $n-k$-blocks of η and λ^{-1} are negatives of each other with rank n'. Then

$$\gamma_k' \sim \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \tilde{\lambda}_k & & & \end{pmatrix},$$

where

$$\tilde{\eta}_k = \begin{pmatrix} 0 & 0 & 0 & -I_{n'} \\ 0 & I_{n''} & 0 & 0 \\ 0 & 0 & I_{n''} & 0 \\ I_{2k-n-n''} & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\lambda}_k^{-1} = \begin{pmatrix} 0 & 0 & I_{n'} \\ B^1_{n-k,2k-n} & 0 & 0 \\ B^3_{2k-n} & B^4_{2k-n,n''} & 0 \end{pmatrix},$$

and $n'' = n-k-n'$.

Proof Suppose that $\eta^2 = -D^2$ in γ_k' (with notations as in the proof of the previous lemma) and the rank of η^2 equals n'. Choose $g, h \in \mathrm{Mat}_{n-k}$ such that $g\eta^2 h^{-1} = \left(\begin{smallmatrix} 0 & -I_{n'} \\ 0 & 0 \end{smallmatrix}\right)$, and let $m = d(g, I_e, h)$. Then γ_k'

$$\sim_{(m,m)} \left(\begin{array}{c|cc} 0 & 0 & \begin{pmatrix} * & * & 0 & -I_{n'} \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \\ I_{n-k} & 0 & 0 \\ \hline & I_{n-k} & 0 \\ \lambda_k' & & \end{array}\right) \sim \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ * & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \lambda_k'' & & * & \end{pmatrix},$$

where

$$\tilde{\eta}_k = \begin{pmatrix} 0 & 0 & 0 & -I_{n'} \\ 0 & I_{n''} & 0 & 0 \\ 0 & 0 & I_{n''} & 0 \\ I_{2k-n-n''} & 0 & 0 & 0 \end{pmatrix},$$

and $n'' = n-k-n'$. For suitable $\hat{u} \in \hat{\mathrm{U}}_n$, further reduction shows that

$$\gamma_k' \sim_{\hat{u}} \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \tilde{\lambda}_k & & & \end{pmatrix} = \tilde{\gamma}_k.$$

Next we consider

$$\tilde{\gamma}_k^{-1} = \begin{pmatrix} & 0 & 0 & \tilde{\lambda}_k^{-1} \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ \tilde{\eta}_k^{-1} & & & \end{pmatrix}.$$

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

By similar reduction and the fact that $\eta^2 = -D^2$, we may assume that

$$\tilde{\lambda}_k^{-1} = \begin{pmatrix} 0 & 0 & I_{n'} \\ B^1_{n-k,2k-n} & 0 & 0 \\ B^3_{2k-n,2k-n} & B^4_{2k-n,n''} & 0 \end{pmatrix}.$$

■

Lemma 3.9 *Let* $e = 2k - n \geq 0$, $\theta_k = \begin{pmatrix} & 0 & 0 & R_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ Q_k & & & \end{pmatrix}$ *with*

$$R_k = \left(\begin{array}{c|cc} 0 & 0 & I_{n'} \\ B^1_{n-k,2k-n} & 0 & 0 \\ \hline B^3_{2k-n,2k-n} & B^4_{2k-n,n''} & 0 \end{array}\right) = \left(\begin{array}{c|c} \eta^1_{n-k,e} & \eta^2_{n-k,n-k} \\ \hline \eta^3_{e,e} & \eta^4_{e,n-k} \end{array}\right)$$

and

$$Q_k = \left(\begin{array}{cc|cc} 0 & 0 & 0 & I_{2k-n-n''} \\ 0 & I_{n''} & 0 & 0 \\ \hline 0 & 0 & I_{n''} & 0 \\ -I_{n'} & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \lambda^1_{e,n-k} & \lambda^2_{e,e} \\ \hline \lambda^3_{n-k,n-k} & \lambda^4_{n-k,e} \end{array}\right).$$

If θ_k *is admissible, then*

$$\theta_k \sim \begin{pmatrix} & & & \begin{pmatrix} 0 & 0 & 0 & I_{n'} \\ T^2 & 0 & 0 & 0 \\ & E & 0 & 0 \\ & & T^2 & 0 \end{pmatrix} \\ & I_{n-k} & & \\ & & I_{n-k} & \\ Q_k & & & \end{pmatrix},$$

for some $T^2 \in \mathrm{Mat}_{n''}$, $E \in \mathrm{GL}_{2k-n-2n''}$. *In this case,* θ_k *is* $\hat{\psi}$-τ*-invariant.*

Proof Let

$$C = \begin{pmatrix} C^1_{e,n-k} & C^2_{e,e} \\ C^3{}_{n-k,n-k} & C^4_{n-k,e} \end{pmatrix} = R_k^{-1} \text{ and } D = \begin{pmatrix} D^1_{n-k,e} & D^2_{n-k,n-k} \\ D^3_{e,e} & D^4_{e,n-k} \end{pmatrix} = Q_k^{-1}$$

and let $s_1 = \begin{pmatrix} g & Y \\ & g \end{pmatrix} \in S_n$, with $g = \begin{pmatrix} I_{n-k} & & \\ g^1 & I_e & \\ & g^3 & I_{n-k} \end{pmatrix}$,

$$g^1 = \begin{pmatrix} 0 & r_{n'',n''} \\ 0 & 0 \end{pmatrix}, \qquad g^3 = \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix}, \tag{3.1}$$

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

$$Y = \begin{pmatrix} Y^1 & & & \\ Y^2 & & & \\ & & Y^3 & Y^4 \end{pmatrix} \text{ with } \lambda^3 Y^1 + \lambda^4 Y^2 = Y^3 C^1 + Y^4 C^3 =: p,$$

where the subscripts denote the sizes of matrices. Then $s_2 = \theta s_1 \theta^{-1} =$

$$\left(\begin{array}{c|c} R_k \begin{pmatrix} I_e & \\ g^3 & I_{n-k} \end{pmatrix} R_k^{-1} \quad & R_k \begin{pmatrix} g^1 \\ \hline 0 \end{pmatrix} \\ \begin{pmatrix} Y^3 \mid Y^4 \end{pmatrix} R_k^{-1} \quad I_{n-k} & \begin{pmatrix} 0 & g^3 \end{pmatrix} Q_k^{-1} \\ \hline & I_{n-k} \\ & Q_k \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \quad Q_k \begin{pmatrix} I_{n-k} & \\ g^1 & I_e \end{pmatrix} Q_k^{-1} \end{array}\right).$$

Therefore $s_2 \in \mathrm{S}_n$ if and only if

$$Q_k \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} \eta^4 g^3 C^1 \\ p \end{pmatrix}, \quad \begin{pmatrix} Y^3, & Y^4 \end{pmatrix} R_k^{-1} = \begin{pmatrix} p, & \lambda^4 g^1 D^1 \end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = Q_k^{-1} \begin{pmatrix} \eta^4 g^3 C^1 \\ p \end{pmatrix}, \quad \begin{pmatrix} Y^3, & Y^4 \end{pmatrix} = \begin{pmatrix} p, & \lambda^4 g^1 D^1 \end{pmatrix} \eta_k.$$

Also,

$$\begin{aligned} \psi_n(s_1) &= \psi_0(\mathrm{tr}(Y^1 + Y^4)) = \psi_0(\mathrm{tr}(D^1 \eta^4 g^3 C^1 + D^2 p + p\eta^2 + \lambda^4 g^1 D^1 \eta^4)), \\ &= \psi_0(\mathrm{tr}(p(D^2 + \eta^2) + D^1 \eta^4 g^3 C^1 + \lambda^4 g^1 D^1 \eta^4)) \\ &= \psi_0(\mathrm{tr}(D^1 \eta^4 g^3 C^1 + \lambda^4 g^1 D^1 \eta^4)), \\ \psi_n(s_2) &= \psi_0(\mathrm{tr}(\eta^1 g^1 + g^3 D^4)) = \psi_0(tr(\eta^1 g^1 + g^3 D^4)) = \psi_0(tr(\eta^1 g^1 + g^3 D^4)). \end{aligned}$$

For θ_k to be admissible, it is necessary that $D^1 \eta^4 \lambda^4 g^1 = \eta^1 g^1$ and $g^3 C^1 D^1 \eta^4 = g^3 D^4$ for all g^1, g^3 as in equation (3.1).

Now we assume that θ_k satisfies

$$D^1 \eta^4 \lambda^4 g^1 = \eta^1 g^1 \quad \text{and} \quad g^3 C^1 D^1 \eta^4 = g^3 D^4 \tag{3.2}$$

for all g^1, g^3 as in equation (3.1). Write

$$\eta^4 = \begin{pmatrix} T^1 & 0 \\ T^2 & 0 \end{pmatrix}, \quad \eta^1 = \begin{pmatrix} 0 & 0 \\ T^3 & T^4 \end{pmatrix}, \quad \text{and} \quad C^1 = \begin{pmatrix} 0 & V^1 \\ 0 & V^2, \end{pmatrix}$$

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

where $T^2, T^3, V^1 \in \mathrm{Mat}_{n''}$. Then

$$D^1\eta^4\lambda^4 g^1 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n''} \end{pmatrix}\begin{pmatrix} T^1 & 0 \\ T^2 & 0 \end{pmatrix}\begin{pmatrix} I_{n''} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & r_{n'',n''} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T^2 r_{n'',n''} \end{pmatrix};$$

$$\eta^1 g^1 = \begin{pmatrix} 0 & 0 \\ T^3 & T^4 \end{pmatrix}\begin{pmatrix} 0 & r_{n'',n''} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T^3 r_{n'',n''} \end{pmatrix};$$

$$g^3 C^1 D^1 \eta^4 = \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix}\begin{pmatrix} 0 & V^1 \\ 0 & V^2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I_{n''} \end{pmatrix}\begin{pmatrix} T^1 & 0 \\ T^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix}\begin{pmatrix} V^1 T^2 & 0 \\ V^2 T^2 & 0 \end{pmatrix};$$

$$g^3 D^4 = \begin{pmatrix} *_{n'',2k-n} \\ 0 \end{pmatrix}\begin{pmatrix} I_{n''} & 0 \\ 0 & 0 \end{pmatrix}.$$

Condition (3.2) implies that $T^2 = T^3, V^1T^2 = I_{n''}, V^2 = 0$. That is ,

$$\theta_k = \begin{pmatrix} & 0 & 0 & \tilde{\eta}_k \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & 0 \\ Q_k & & & \end{pmatrix}, \text{ with } \tilde{R}_k = \begin{pmatrix} 0 & 0 & 0 & I_{n'} \\ T^2 & T^4 & 0 & 0 \\ * & * & T^1 & 0 \\ * & * & T^2 & 0 \end{pmatrix}.$$

Note that $V^1T^2 = I_{n''}$, implying $\det T^2 \neq 0$, and hence

$$\theta_k \sim \begin{pmatrix} & & & \begin{pmatrix} 0 & 0 & 0 & I_{n'} \\ T^2 & 0 & 0 & 0 \\ & E & 0 & 0 \\ & & T^2 & 0 \end{pmatrix} \\ & I_{n-k} & & \\ & & I_{n-k} & \\ Q_k & & & \end{pmatrix},$$

for some $E \in \mathrm{GL}_{2k-n-2n''}$.

In the case of fields, there exists some $g \in \mathrm{GL}_{2k-n-2n''}$ such that $gEg^{-1} = E^t$, where E^t denotes the transpose of E. In the case of quaternion division algebras, there exists some $g \in \mathrm{GL}_{2k-n-2n''}$ such that $gEg^{-1} = \bar{E}^t$ [Ra, Lemma 3.1]. In either case, there exists $g \in \mathrm{GL}_{2k-n-2n''}$ such that $gEg^{-1} = E^\tau$. Let

$$\zeta = \mathrm{diag}(I_{n'+n''}, g, I_{n-k+2n''}, I_{n'+n''}, g, I_{n-k+2n''}).$$

Then $\zeta\theta_k\zeta^{-1} = \theta_k^\tau$ and θ_k is $\hat{\psi}$-τ-invariant. ■

Proposition 3.10 *Every admissible double coset* $\mathrm{S}_n g \mathrm{S}_n$ *is* $\hat{\psi}$-τ*-invariant.*

Proof The proof is by induction on the index n of GL_{2n}. By Bruhat decomposition, $\mathrm{GL}_2 = \mathrm{B}_2\mathrm{W}_2\mathrm{B}_2 = \mathrm{S}_2 D \mathrm{W}_2 D \mathrm{S}_2$, where $D = \{\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \mid a \in A^*\}$. Representatives of $\mathrm{S}_2 \backslash \mathrm{GL}_2 / \mathrm{S}_2$ can be expressed by $\xi_1(a) = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ or $\xi_2(a) = \left(\begin{smallmatrix} & a \\ 1 & \end{smallmatrix}\right)$, $a \in A^*$. Since

https://doi.org/10.4153/CJM-2009-062-1 Published online by Cambridge University Press

$\xi_1(a)\begin{pmatrix}1&x\\&1\end{pmatrix}\xi_1(a)^{-1}=\begin{pmatrix}1&ax\\&1\end{pmatrix}$, $x\in A$, and ψ_0 is nontrivial, there exists $x\in A$ such that $\psi_0(ax)\neq\psi_0(x)$ for $a\neq 1$. Hence $\xi_1(a)$ is not admissible for $a\neq 1$. For $a=1$, $\xi_1(a)=\mathrm{id}$ is $\hat{\psi}$-τ-invariant. In the case of $A=\mathrm{F}$, $\xi_2(a), a\in\mathrm{F}^*$ is τ-invariant. In the case of $A=D$, there exists $b\in D^*$ such that $bab^{-1}=\bar{a}$ for $a\in D^*$. (Either refer to [Ra, Lemma 3.1] or check by direct computation.) Therefore, there exists $\begin{pmatrix}b&\\&1\end{pmatrix}\in\mathrm{Mat}_2$ such that $\begin{pmatrix}b&\\&1\end{pmatrix}\xi_2(a)\begin{pmatrix}b&\\&1\end{pmatrix}^{-1}=\xi_2(a)^\tau$ and $\xi_2(a)$ is $\hat{\psi}$-τ-invariant. The conclusion is then true for $n=1$. Now we assume that it is also true for $1,\dots,n-1$.

Lemma 3.6 deals with the case of $\alpha_k\neq\beta_k$, and others do the same for $\alpha_k=\beta_k=0_k$, therefore it suffices to show that $\gamma_k=d_1(\alpha)\sigma_k d_2(\beta)$, with $\alpha_k=\beta_k\neq 0_k$, is either non-admissible or $\hat{\psi}$-τ-invariant.

Since $\mathrm{rank}(\alpha_K)\neq 0$, there exist $g,h\in\mathrm{GL}_{n-k}$ such that

$$\gamma_k=\begin{pmatrix}*&\alpha_k&\\ *&*&\\ &&I_n\end{pmatrix}\sigma_k\begin{pmatrix}I_n&&\\&*&\alpha_k\\&*&*\end{pmatrix}=\begin{pmatrix}&\alpha_k&&&\\&*&&*&\\&&&*&\alpha_k\\w_k&&&&\end{pmatrix}$$

$$\sim d\left(\begin{pmatrix}g&\\&I_{n+k}\end{pmatrix}\right)\gamma_k d\left(\begin{pmatrix}I_{n+k}&\\&h\end{pmatrix}\right)=\begin{pmatrix}&0&1&*&*&*\\&\delta_{k-1}&0&*&*&*\\&*&*&*&*&*\\&&&*&0&1\\&&&*&\delta_{k-1}&0\\w_k&&&&&\end{pmatrix},$$

for some $\delta_{k-1}\in\mathrm{Mat}_{n-k-1}$. With suitable choices of $\hat{m}\in\hat{\mathrm{M}}_n$, $\hat{u}\in\hat{\mathrm{U}}_n$,

$$\gamma_k\sim_{\hat{m}}\begin{pmatrix}*&0&1&*&*&*\\&\delta'_{k-1}&0&*&*&*\\&*&0&*&*&*\\&&&0&0&1\\&&&*&\delta'_{k-1}&0\\w_k&&&&&*\end{pmatrix}\sim_{\hat{u}}\begin{pmatrix}&0&1&&&\\&\delta''_{k-1}&0&*&*&\\&*&0&*&*&\\&&&0&0&1\\&&&*&\delta''_{k-1}&0\\w_k&&&&&\end{pmatrix}=\tilde{\gamma}_k.$$

Let

$$N=\begin{pmatrix}N_1&N_2\\N_3&N_4\end{pmatrix}\in\mathrm{GL}_{2n-2}$$

be embedded in GL_{2n} as

$$\begin{pmatrix}&1&&\\N_1&&N_2&\\&&&1\\N_3&&N_4&\end{pmatrix},\ N_i\in\mathrm{Mat}_{n-1}\,.$$

By induction assumption, there exist $\hat{s}=(s_1,s_2)$ with

$$s_1=\begin{pmatrix}1&&&\\&q&&Y\\&&1&\\&&&q\end{pmatrix}\quad\text{and}\quad s_2=\begin{pmatrix}p&&Z&\\&1&&\\&&p&\\&&&1\end{pmatrix},$$

where $p, q \in \mathrm{GL}_{n-1}, Y, Z \in \mathrm{Mat}_{n-1}$ such that either $\hat{s} \in \hat{\mathbb{H}}_{\tilde{\gamma}_k}$ and $\hat{\psi}(s) \neq 1$ or $\hat{s} \cdot \tilde{\gamma}_k = (\tilde{\gamma}_k)^\tau$ and $\hat{\psi}(\hat{s}) = 1$. Note that the above embedding is consistent between (τ_{n-1}, ψ_{n-1}) and (τ_n, ψ_n). Hence the conclusion holds by induction. ∎

4 Main Theorems

Theorem 4.1 *For any $n \in \mathbb{N}$, let $G = \mathrm{GL}_{2n}(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field. Then*

$$\dim \mathrm{Hom}_G(\rho, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n) \leq 1$$

for any irreducible representation of G.

Proof Let $\pi = \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n$. Proposition 3.5 implies that an element

$$\triangle\colon G \mapsto \mathrm{Hom}_{\mathbb{C}}(\mathrm{Ind}_{\mathrm{S}_n}^G \psi_n, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n)$$

satisfying $\triangle(s'gs) = \pi(s') \circ \triangle(g) \circ \pi(s)$ for $s, s' \in \mathrm{S}_n$ is τ-invariant. By Theorem 2.1 $\mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{S}_n}^G \psi_n, \mathrm{Ind}_{\mathrm{S}_n}^G \psi_n)$ is abelian and the result follows. ∎

Lemma 4.2 *Let G denote $\mathrm{GL}_{2n}(A)$, where A is a p-adic field $\mathcal{F}$ or a quaternion division algebra D over $\mathcal{F}$. If T is a distribution on G satisfying*

$$T(L_{h_1} \circ R_{h_2}(\eta)) = \psi_n(h_1 h_2^{-1}) T(\eta)$$

for $h_1, h_2 \in \mathrm{S}_n(A), \eta \in C_c^\infty(G)$, then T is τ-invariant.

Proof We verify the assumptions of [BZ, Theorem 6.10].[1]

The assumptions of Theorem 6.10 in [BZ] in this case are the following:

(i) The action of $\hat{\mathbb{H}}$ is constructible (same as constructive in the sense of [BZ]), which means that the set of $\{(g, \hat{h} \cdot g) \mid g \in G, \hat{h} \in \hat{\mathbb{H}}\}$ is a union of finitely many locally closed subsets of $G \times G$.
(ii) For each $\hat{h} \in \hat{\mathbb{H}}$, there is $\hat{h}_\tau \in \hat{\mathbb{H}}$ such that $\hat{h} \cdot g^\tau = (\hat{h}_\tau \cdot g)^\tau$ for all $g \in G$.
(iii) $\tau^2 = \mathrm{id}$.
(iv) If T is a nonzero $\hat{\mathbb{H}}$-invariant distribution on an $\hat{\mathbb{H}}$-orbit Y, then $Y^\tau = Y$ and $T^\tau = T$.

The conclusion is that any $\hat{\mathbb{H}}$-invariant distribution on G is also τ-invariant.

By [BZ, Theorem A §6.15], the action of $\hat{\mathbb{H}}$ is constructible on $\mathrm{GL}_{2n}(\mathcal{F})$. Also $\iota\hat{\mathbb{H}}$ is constructible on $\mathrm{GL}_{4n}(K), K = \mathcal{F}(\sqrt{\alpha})$, and its closed subgroup $\mathrm{GL}_{2n}(D)$, where ι is the embedding defined earlier. Condition (i) is then verified. For condition (ii), take $\hat{h}_\tau = (h_2^{-\tau}, h_1^{-\tau})$ for $\hat{h} = (h_1, h_2) \in \hat{\mathbb{H}}$. Since $\mathbb{H}$ is τ-invariant and $(\hat{h}_\tau)_\tau = \hat{h}$, τ induces an anti-involution on $\hat{\mathbb{H}}$ (still denoted by τ) $\tau\colon \hat{\mathbb{H}} \mapsto \hat{\mathbb{H}}$ by $\hat{h} \mapsto \hat{h}_\tau$. The action of $\hat{h} \in \hat{\mathbb{H}}$ satisfies that $\hat{h} \cdot g^\tau = (\hat{h}_\tau \cdot g)^\tau$ for all $g \in G$. Condition (iii) is obvious. To verify condition (iv), let T be a nonzero $\hat{\mathbb{H}}$-invariant distribution on an $\hat{\mathbb{H}}$-orbit $Y = \mathbb{H}g\mathbb{H}$, *i.e.*, $T(\hat{h} \cdot (\eta)) = T(\eta)$ for all $\hat{h} = (h_1, h_2) \in \hat{\mathbb{H}}$ and $\eta \in C_c^\infty(Y)$. Then $Y \cong \hat{\mathbb{H}}/\hat{\mathbb{H}}_g$. ($\hat{\mathbb{H}}_g$ the stabilizer of g in $\hat{\mathbb{H}}$.) Define a character $\hat{\psi}_n$ of $\hat{\mathbb{H}}$ by

$$\hat{\psi}_n(\hat{h}) = \psi_n(h_1 h_2^{-1}) \text{ for } \hat{h} = (h_1, h_2) \in \hat{\mathbb{H}},$$

[1]This proof mimics [So, Theorem 2.3]. We keep it here for the sake of compleness.

then $\hat{\psi}_n$ is τ-invariant and $C_c^\infty(Y) \cong \operatorname{ind}_{\hat{\mathbb{H}}_g}^{\hat{\mathbb{H}}} 1$ (un-normalized compact induction). We have that

$$T \in \operatorname{Hom}_{\hat{\mathbb{H}}}(\operatorname{ind}_{\hat{\mathbb{H}}_g}^{\hat{\mathbb{H}}} 1, \hat{\psi}_n) \cong \operatorname{Hom}_{\hat{\mathbb{H}}_g}(\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}, \operatorname{Res}_{\hat{\mathbb{H}}_g} \hat{\psi}_n)$$

by Frobenius reciprocity, where $\delta_{\hat{\mathbb{H}}}, \delta_{\hat{\mathbb{H}}_g}$ are the modular functions of $\hat{\mathbb{H}}$ and $\hat{\mathbb{H}}_g$, respectively. Since $|\hat{\psi}_n| \equiv 1$ and $\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}$ is positive, by Schur's lemma we have either

$$\dim \operatorname{Hom}_{\hat{\mathbb{H}}_g}(\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}, \operatorname{Res}_{\hat{\mathbb{H}}_g} \hat{\psi}_n) = 0 \quad \text{or} \quad \delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1} = \operatorname{Res}_{\hat{\mathbb{H}}_g} \hat{\psi}_n \equiv 1.$$

Therefore we conclude that $\operatorname{Hom}_{\hat{\mathbb{H}}_g}(\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1}, \operatorname{Res}_{\hat{\mathbb{H}}_g} \hat{\psi}_n) = 0$ for those non-admissible g, *i.e.*, there is no nontrivial $\hat{\mathbb{H}}$-invariant distribution T on such Y.

Now we consider those $\hat{\psi}$-τ-invariant g. We may assume that $\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1} \equiv 1$, since otherwise $\hat{\mathbb{H}}$-invariant distribution on such Y is trivial. Note that $\hat{k} \cdot g = g^\tau$ for some $\hat{k} \in \mathbb{H}$ implies that the double coset $Y = \mathbb{H}g\mathbb{H}$ is τ-invariant. It remains to show that $T^\tau = T$. In our case T is proportional (see [BZ, 6.12]) to

$$T_1(\eta) = \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h} \cdot g)\hat{\psi}_n^{-1}(\hat{h})\, d\hat{h},$$

where $d\hat{h}$ is a left $\hat{\mathbb{H}}$-invariant measure on $\hat{\mathbb{H}}/\hat{\mathbb{H}}_g$. We have

$$\begin{aligned} T_1^\tau(\eta) = T_1(\eta^\tau) &= \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta((\hat{h} \cdot g)^\tau)\hat{\psi}_n^{-1}(\hat{h})\, d\hat{h} \\ &= \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}_\tau \cdot g^\tau)\hat{\psi}_n^{-1}(\hat{h})\, d\hat{h} \\ &= \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}_\tau \cdot \hat{k} \cdot g)\hat{\psi}_n^{-1}(\hat{h})\, d\hat{h} = \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}' \cdot g)\hat{\psi}_n^{-1}(\hat{h}')\hat{\psi}_n(\hat{k})\, d\hat{h}'. \end{aligned}$$

The last equality is obtained by the change of variables $\hat{h}' = \hat{h}_\tau \cdot \hat{k}$ along with our assumption that $\delta_{\hat{\mathbb{H}}}\delta_{\hat{\mathbb{H}}_g}^{-1} \equiv 1$ and the fact that $\hat{\psi}_n$ is τ-invariant. Since $\hat{\psi}_n(\hat{k}) = 1$, we have

$$T_1^\tau(\eta) = \int_{\hat{\mathbb{H}}/\hat{\mathbb{H}}_g} \eta(\hat{h}' \cdot g)\hat{\psi}_n^{-1}(\hat{h}')\, d\hat{h}' = T_1(\eta). \qquad \blacksquare$$

Theorem 4.3 *Let $G = \mathrm{GL}_{2n}(A)$, where A is either a p-adic field $\mathcal{F}$ or a quaternion division algebra D over $\mathcal{F}$. Then* $\dim \operatorname{Hom}_G(\rho, \operatorname{Ind}_{S_n}^G \psi_n) \leq 1$ *for any irreducible representation ρ of G.*

Proof We have obtained $\dim \operatorname{Hom}_G(\pi; \operatorname{Ind}_{S_n}^G \psi_n) \cdot \dim \operatorname{Hom}_{S_n}(\operatorname{Res}_{S_n}^G \tilde{\pi}; \psi_n) \leq 1$ for any irreducible representation π of G from the previous theorem and the Gelfand–Kazhdan criterion. It suffices to show that if π has a nontrivial Shalika functional, then $\tilde{\pi}$ will also have one. Assume that Λ_π is a nontrivial Shalika functional for π, *i.e.*, $\Lambda_\pi(\pi(h)v) = \psi_n(h)\Lambda_\pi(v)$ for all $\in S_n$ and $v \in V_\pi$. Define a representation π' on the same vector space V_π by $\pi'(g)v = \pi(\xi g^{-\tau}\xi^{-1})v$, where $\xi = \operatorname{diag}(I_n, -I_n)$. Then $\psi(\xi s^{-\tau}\xi^{-1}) = \psi(s)$ for $s \in S_n$, and Λ_π is also a Shalika functional for π'.

In the case of $\mathcal{F}$, define another representation π'' on the same vector space V_π by $\pi''(g)v = \pi(g^{-t})v$. Then $\pi'' \cong \tilde{\pi}$ by [BZ, Theorem 7.3]. Since $\xi g^{-\tau}\xi^{-1}$ is conjugate to g^{-t} , we have $\pi' \cong \tilde{\pi}$.

In the case of D, define another representation π'' on the same vector space V_π by $\pi''(g)v = \pi(\eta\bar{g}^{-t}\eta^{-1})v$, where $\eta(i,j) = (-1)^i\delta_{i,2n-j+1}$. Then $\pi'' \sim \tilde{\pi}$ by [Ra, Theorem 3.1]. Since $\xi g^{-\tau}\xi^{-1}$ is conjugate to $\eta\bar{g}^{-t}\eta^{-1}$, we have $\pi' \sim \pi'' \sim \tilde{\pi}$.

In either case,

$$\begin{aligned}\dim \operatorname{Hom}_G(\tilde{\pi}; \operatorname{Ind}_{S_n}^G \psi_n) &= \dim \operatorname{Hom}_G(\pi'; \operatorname{Ind}_{S_n}^G \psi_n)\\ &= \dim \operatorname{Hom}_{S_n}(\pi'|_{S_n}; \psi_n)\\ &\geq 1,\end{aligned}$$

which completes the proof. ∎

References

[BZ] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group* GL$(n, \mathcal{F})$, *where* $\mathcal{F}$ *is a non-Archimedean local field.* Russ. Math. Surveys **31**(1976), no. 3, 1–68.

[CKPS] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the classical groups.* Publ. Math. Inst. Hautes Études Sci. No. 99 (2004), 163–233.

[Ga1] P. Garrett, *The Gelfand-Kazhdan criterion.* http://www.math.umn.edu/' garrett/m/v/.

[Ga2] ———, *Uniqueness of invariant distributions.* http://www.math.umn.edu/' garrett/m/v/.

[GaRe] A. M. Garsia and C. Reutenauer, *A decomposition of Solomons descent algebra.* Adv. Math. **77**(1989), no. 2, 189–262.

[Ge] S. I. Gelfand, *Representations of a general linear group over a finite field.* In: Lie Groups and Their Representations. Halsted, New York, 1975, pp. 119–132,

[GK] S.I. Gelfand and D. Kazhdan, *Representations of the group* GL(n, K), *where* K *is a local field.* In: Lie Groups and Their Representations. Halsted, New York, 1975, pp. 95–118.

[GRS] D. Ginzburg, S. Rallis, and D. Soudry, *Generic automorphic forms on* SO$(2n+1)$*: functorial lift to* GL$(2n)$*, endoscopy, and base change.* Internat. Math. Res. Notices **2001**, no. 14, 729–764.

[Gr] B. H. Gross, *Some applications of Gelfand pairs to number theory.* Bull. Amer. Math. Soc. **24**(1991), no. 2, 277–301.

[GT] W. T. Gan and S. Takeda, *On Shalika periods and a conjecture of Jacquet-Martin.* To appear in Amer. J. Math.

[JaS] H. Jacquet and J. Shalika, *Exterior square L-functions.* In: Automorphic Forms, Shimura Varieties, and L-functions. Perspect. Math. 11, Academic Press, Boston, MA, 1990 pp. 143–226, 143-225.

[Ji1] D. Jiang, *On the fundamental automorphic L-functions of* SO$(2n+1)$. Int. Math. Res. Notices **2006**. art. id. 64069.

[JiQ] D. Jiang and Y. Qin, *Residues of Eisenstein series and generalized Shalika models for* SO$(4n)$. J. Ramanujan Math. Soc. **22**,(2007), no. 2, 101–133.

[JiS1] D. Jiang and D. Soudry, *The local converse theorem for* SO$(2n+1)$ *and applications.* Ann. of Math. **157**(2003), no. 3, 743–806.

[JiS2] ———, *Generic representations and local Langlands reciprocity law for p-adic* SO$(2n+1)$. In: Contributions to Automorphic Forms, Geometry, and Number Theory. Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 457–519.

[JR] H. Jacquet and S. Rallis, *Uniqueness of linear periods*, Compositio Math **102**(1996), no. 1, 65–123.

[JM] H. Jacquet and K. Martin, *Shalika periods on* GL$_2(D)$ *and* GL$_4$. Pacific J. Math. **233**(2007), no. 2, 341–370.

[Ki] H. H. Kim, *Applications of Langlands' functorial lift of odd orthogonal groups.* Trans. Amer. Math. Soc. **354**(2002), no. 7, 2775–2796 (electronic).

[PR] D. Prasad and A. Raghuram, *Kirillov theory of* $\mathrm{GL}_2(\mathcal{D})$ *where* $\mathcal{D}$ *is a division algebra over a non-Archimedean local field.* Duke J. of Math, **104**(2000), no. 1, 19–44.

[Ra] A. Raghuram, *On representations of* p*-adic* $\mathrm{GL}(2, \mathcal{D})$. Pacific J. Math. **206**(2002), no. 2, 451-464.

[Sa] Y. Sakellaridis, *A Casselman-Shalika formula for the Shalika model of* $\mathrm{GL}(n)$. Canad. J. Math. **58**(2006), no. 5, 1095–1120.

[So] D. Soudry, *A uniqueness theorem for representations of GSO(6) and the strong multiplicity one theorem for generic representations of GSp(4)*,Israel Journal of Mathematics, Vol. 58, no. 3 (1987).

Department of Mathematics,, National Cheng Kung University, Tainan 701, Taiwan
e-mail: nienpig@mail.ncku.edu.tw,