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COMPLEX CYCLES AS OBSTRUCTIONS ON REAL ALGEBRAIC VARIETIES

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Abstract. Let Y be a compact nonsingular real algebraic variety of positive dimension. Then one can find a compact connected nonsingular real algebraic variety X, which admits a continuous map into Y that is not homotopic to any regular map. It is hard to determine the minimum dimension of such a variety X. In this paper, new upper bounds for dim X are obtained. The main role in the constructions is played by complex algebraic cycles on Y.

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1. Introduction. In the present paper, we investigate obstructions to representing homotopy classes of continuous maps, between real algebraic varieties, by regular maps. The term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^N$, for some N, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [**3**]). The class of real algebraic varieties is identical with the class of quasi-projective real algebraic varieties, cf. [**3**, Proposition 3.2.10, Theorem 3.4.4]. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

In [**6**], a numerical invariant $\beta(Y)$ was defined for any real algebraic variety Y. Recall that $\beta(Y)$ is the supremum of all nonnegative integers n with the following property: For every n-dimensional compact connected nonsingular real algebraic variety X, every continuous map from X into Y is homotopic to a regular map. Henceforth Y will be assumed to be compact and nonsingular. For any nonnegative integer k, let $H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2)$ denote the subgroup consisting of all algebraic cohomology classes in the cohomology group $H^k(Y;\mathbb{Z}/2)$, cf. [**7**] or [**1**, **3**, **5**]. According to [**6**, Theorem 2.9], $\beta(Y) \leq k$ if $H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2) \neq 0$ for some $k \geq 1$. As a consequence, one gets some upper bounds for $\beta(Y)$, which are independent of the algebraic-geometric structure of Y. For example, $\beta(Y) \leq \dim Y$ if $\dim Y \geq 1$. This assertion holds since $H^d_{\mathrm{alg}}(Y;\mathbb{Z}/2) = H^d(Y;\mathbb{Z}/2) \neq 0$ for $d = \dim Y$. Furthermore, $\beta(Y) \leq k$ if the kth Stiefel–Whitney class $w_k(Y)$ of Y is nonzero for some $k \geq 1$. Indeed, it suffices to note that $w_k(Y)$ is in $H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2)$, cf. [**7**] or [**1**]. In particular, $\beta(Y) \leq 1$ if Y

is nonorientable (as a smooth, that is, $\mathcal{C}^\infty$ manifold), the orientability of Y being equivalent to $w_1(Y) = 0$.

The reader may consult [**9**] for other results related to the problem under consideration.

In this paper, new upper bounds for $\beta(Y)$ are obtained. For Y orientable they are frequently sharper than those given in [**6**]. The key role is played by $\mathbb{C}$-algebraic cohomology classes. Let $H^{2k}_{\mathbb{C}\text{-alg}}(Y;\mathbb{Z})$ denote the subgroup consisting of all $\mathbb{C}$-algebraic cohomology classes in the cohomology group $H^{2k}(Y;\mathbb{Z})$, cf. [**2**]. For the convenience of the reader, the definition of $H^{2k}_{\mathbb{C}\text{-alg}}(Y;\mathbb{Z})$ is recalled in Section 2.

THEOREM 1.1. *Let Y be a compact nonsingular real algebraic variety. If the group $H^{2k}_{\mathbb{C}-\text{alg}}(Y;\mathbb{Z})$ is infinite for some $k \geq 1$, then $\beta(Y) \leq 2k - 1$.*

The proof is postponed until Section 2. It is worthwhile to record two consequences of Theorem 1.1. Let $p_k(Y) \in H^{4k}(Y;\mathbb{Z})$ denote the kth Pontryagin class of Y.

COROLLARY 1.2. *Let Y be a compact nonsingular real algebraic variety. If for some integer $k \geq 1$, the Pontryagin class $p_k(Y)$ is an element of infinite order in the group $H^{4k}(Y;\mathbb{Z})$, then $\beta(Y) \leq 4k - 1$.*

Proof. In view of [**2**, Theorem 5.3], $p_k(Y)$ is in $H^{4k}_{\mathbb{C}\text{-alg}}(Y;\mathbb{Z})$, and hence it suffices to apply Theorem 1.1. □

A compact oriented smooth manifold is said to be an *oriented boundary* if it is the boundary, with induced orientation, of a compact oriented smooth manifold with boundary.

COROLLARY 1.3. *Let Y be a compact nonsingular real algebraic variety. If Y is oriented, and the disjoint union of two copies of Y is not an oriented boundary, then $\beta(Y) \leq \dim Y - 1$.*

Proof. Let M be the disjoint union of two copies of Y. Then M is the unoriented boundary of $Y \times [0, 1]$, and hence its Stiefel - Whitney numbers are all equal to 0. Since M is not an oriented boundary, it follows that at least one Pontryagin number of M is different from 0 (cf. Wall's refinement [**14**] of earlier results due to Thom [**13**] and other authors). Consequently, $\dim Y = 4d$, for some positive integer d, and there exists an element of infinite order in $H^{4d}(Y;\mathbb{Z})$, which is a polynomial in the Pontryagin classes $p_k(Y)$ for $k \geq 0$. In particular, $p_k(Y)$ is an element of infinite order in the group $H^{4k}(Y;\mathbb{Z})$ for some $k \geq 1$. Now it suffices to apply Corollary 1.2. □

2. Complex algebraic cycles. First some basic definitions will be recalled. Let V be a nonsingular projective complex algebraic variety. For any nonnegative integer k, a cohomology class in $H^{2k}(V;\mathbb{Z})$ is said to be *algebraic* if it corresponds via the cycle map to an algebraic cycle of codimension k on V, cf. [**8**, Chapter 19]. The set $H^{2k}_{\text{alg}}(V;\mathbb{Z})$ of all algebraic cohomology classes in $H^{2k}(V;\mathbb{Z})$ forms a subgroup. This construction can be transferred in a suitable way to the real algebraic-geometric setting.

Let X be a compact nonsingular real algebraic variety. A *nonsingular projective complexification* of X is a pair (W, j), where W is a nonsingular projective complex algebraic variety defined over $\mathbb{R}$ and $j\colon X \to W$ is an injective map, such that $W(\mathbb{R})$ is Zariski dense in W, $j(X) = W(\mathbb{R})$ and j induces a biregular isomorphism between X and $W(\mathbb{R})$. Here $W(\mathbb{R})$ denotes the set of real points of W. The existence of (W, j)

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follows from Hironaka's theorem on resolution of singularities [**10**, **11**]. The subgroup

$$H^{2k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) := j^*(H^{2k}_{\text{alg}}(W;\mathbb{Z}))$$

of $H^{2k}(X;\mathbb{Z})$ does not depend on the choice of (W, j). Any cohomology class in $H^{2k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$ is said to be $\mathbb{C}$-*algebraic*. The groups $H^{2k}_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ have the expected functorial property: If $f\colon X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then

$$f^*(H^{2k}_{\mathbb{C}\text{-alg}}(Y;\mathbb{Z})) \subseteq H^{2k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}).$$

These properties of $H^{2k}_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ are proved in [**2**].

In order to interpret $\mathbb{C}$-algebraic cohomology classes as obstructions, one needs a somewhat refined version of Thom's representability theorem [**13**, Théorème III.4]. For any n-dimensional compact oriented smooth manifold N, let $[N]$ denote its fundamental class in $H_n(N;\mathbb{Z})$.

THEOREM 2.1. *Let Y be a CW-complex and let α be a homology class in $H_n(Y;\mathbb{Z})$, with $n \geq 1$. Then there exist an n-dimensional compact oriented smooth manifold N, a continuous map $f\colon N \to Y$ and a positive integer l such that $f_*([N]) = l\alpha$ and N is an oriented boundary. Furthermore, if α is represented by a singular cycle with support contained in a connected component of Y, then the manifold N can be chosen connected.*

Proof. One can assume without loss of generality that Y is compact and connected. The argument used in [**13**, pp. 57, 58] implies the existence of a compact connected orientable smooth manifold P containing Y as a retract. One can find such a manifold P with $p := \dim P \geq 2n + 1$. Let $i\colon Y \hookrightarrow P$ be the inclusion map and let $r\colon P \to Y$ be a retraction. According to Thom's theorem [**13**, Théorème II.29], there exists a positive integer l such that the homology class $li_*(\alpha)$ in $H_n(P;\mathbb{Z})$ can be represented by an n-dimensional compact oriented smooth submanifold M of P. Since the manifold P is connected and $1 \leq n \leq p - 2$, the connected components of M can be joined with n-dimensional tubes in P. Hence M can be assumed to be connected. Let U be an open subset of $P \setminus M$ that is diffeomorphic to $\mathbb{R}^p$. Since $p \geq 2n + 1$, there exists a smooth submanifold M' of U, diffeomorphic to M. Choosing an orientation of M' and joining the submanifolds M and M' with an n-dimensional tube in P, one obtains a compact connected oriented smooth submanifold N of P representing the homology class $li_*(\alpha)$. Furthermore, if the orientation of M' is suitably chosen, then N is diffeomorphic to the connected sum $M\#(-M)$, where $-M$ stands for M with the opposite orientation. Since $M\#(-M)$ represents the same oriented bordism class as the disjoint union of M and $-M$, it follows that N is an oriented boundary. If $j\colon N \hookrightarrow P$ is the inclusion map and $f := r \circ j$, then $j_*([N]) = li_*(\alpha)$ and

$$f_*([N]) = r_*(j_*([N])) = lr_*(i_*(\alpha)) = l(r \circ i)_*(\alpha) = l\alpha.$$

The proof is complete. □

Proof of Theorem 1.1. Let u be an element of infinite order in the group $H^{2k}_{\mathbb{C}\text{-alg}}(Y;\mathbb{Z})$. Then there exists a homology class α in $H_{2k}(Y;\mathbb{Z})$ such that the Kronecker index $\langle u, \alpha\rangle$ is different from 0. One can choose α in such a way that it is represented by a singular cycle with support contained in a connected component of Y. By Theorem 2.1, one can find a $2k$-dimensional compact connected oriented smooth manifold N, a continuous

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map $f\colon N \to Y$ and a positive integer l such that $f_*([N]) = l\alpha$ and N is an oriented boundary. The last named property of N is crucial. According to [**4**, Theorem 3.3], it implies the existence of a nonsingular real algebraic variety X, which is diffeomorphic to N and satisfies $H^{2k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = 0$. If $\varphi\colon X \to N$ is a smooth diffeomorphism and $g := f \circ \varphi$, then

$$g_*([X]) = f_*(\varphi([X])) = f_*([N]) = l\alpha.$$

Consequently,

$$\langle g^*(u), [X]\rangle = \langle u, g_*([X])\rangle = \langle u, l\alpha\rangle = l\langle u, \alpha\rangle \neq 0,$$

and hence $g^*(u) \neq 0$ in $H^{2k}(X;\mathbb{Z})$. Since $H^{2k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = 0$, the functoriality of $H^{2k}_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ implies that the map g is not homotopic to any regular map. Thus, $\beta(Y) \leq 2k - 1$, as required. □

Any projective complex algebraic variety V can be regarded as a real algebraic variety, denoted $V_{\mathbb{R}}$ (identify $\mathbb{C}$ with $\mathbb{R}^2$). Obviously, $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2\dim_{\mathbb{C}} V$. Furthermore, V and $V_{\mathbb{R}}$ coincide as topological spaces endowed with the Euclidean topology.

EXAMPLE 2.2. If V is a nonsingular projective complex algebraic variety of positive dimension, then $\beta(V_{\mathbb{R}}) \leq 1$. Indeed, there are two subgroups, $H^{2k}_{\text{alg}}(V;\mathbb{Z})$ and $H^{2k}_{\mathbb{C}\text{-alg}}(V_{\mathbb{R}};\mathbb{Z})$, of the cohomology group $H^{2k}(V_{\mathbb{R}};\mathbb{Z})$. It is well known that $H^{2k}_{\text{alg}}(V;\mathbb{Z}) \subseteq H^{2k}_{\mathbb{C}\text{-alg}}(V_{\mathbb{R}})$, cf. [**12**]. Since the group $H^{2}_{\text{alg}}(V;\mathbb{Z})$ is infinite, the inequality $\beta(V_{\mathbb{R}}) \leq 1$ follows from Theorem 1.1. Consequently, if V is connected and simply connected, then $\beta(V_{\mathbb{R}}) = 1$. It does not seem that these facts can be established using methods developed in [**6**].

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