A CLASS OF NON-DESARGUESIAN PROJECTIVE PLANES

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1. Introduction. In **(7)**, Veblen and Wedderburn gave an example of a non-Desarguesian projective plane of order 9; we shall show that this plane is self-dual and can be characterized by a collineation group of order 78, somewhat like the planes associated with difference sets. Furthermore, the technique used in **(7)** will be generalized and we will construct a new non-Desarguesian plane of order p^{2n} for every positive integer n and every odd prime p. To do this, we need a result due to Zassenhaus **(8)** that there exists a near-field which is not a field of every order p^{2n}, p an odd prime, whose center is a field of order p^n. However there are some significant differences between the case $p^{2n} = 9$ and all other cases, and these lead to some unsolved problems.

Furthermore, we shall show that none of the planes constructed in this fashion can be coordinatized by Veblen-Wedderburn systems (with either distributive law) and that each such plane possesses a (non-linear) planar ternary ring whose additive loop is an elementary abelian group.

2. Construction of planes. A finite *left Veblen-Wedderburn system* (left V-W system) is a finite set R containing at least the two distinct elements 0 (zero) and 1 (one), together with two binary operations, addition $(+)$ and multiplication $(\cdot)$ (where we often write ab for $a \cdot b$), all satisfying:

(1) R is a group under addition, with "identity" 0;

(2) The non-zero elements of R form a loop under multiplication, with identity 1;

(3) $0x = x0 = 0$, all $x \in R$;

(4) The left distributive law is valid in R:

$$a(b + c) = ab + ac, \qquad a, b, c \in R.$$

Similarly, a right V-W system satisfies the right distributive law, in place of (4). A left (right) V-W system with associative multiplication is a *left (right) near-field*. Throughout this paper we shall omit the term "finite," as all V-W systems considered will be finite; it has been shown that the addition of any V-W system (indeed, even the infinite ones) is commutative, and (for the finite case only) is elementary abelian. Hence R always has order equal to a power of a prime. (See **(4)** for proofs.)

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Any (left or right) V-W system is a linear planar ternary ring **(2; 3)**. If R is a (left or right) V-W system which is not a field then the plane π coordinatized by R is non-Desarguesian and contains a distinguished line or point (depending on which distributive law is present) which is moved by no collineation; a pair of anti-isomorphic V-W systems coordinatize planes which are duals. In **(8)** Zassenhaus has determined all (finite) near-fields, as well as their automorphism groups, and in particular, has shown that for any odd prime p and any positive integer n there is a near-field R (which is not a field) of order p^{2n} whose center is a field of order p^n. (By "center" we mean here the set of all elements $z \in R$ such that $zx = xz$ for all $x \in R$.)

Let R be a left near-field (which is not a field) of order p^{2n} containing a field F of order p^n as its center; let $q = p^{2n} + p^n + 1$. For the sake of simplicity in the argument, we introduce the set V of all ordered triples (x, y, z) where x, y, z are in R, and let V_0 be the subset of V consisting of all triples all of whose entries are in F. Then V is a left vector space over R (or over F), and V_0 is a left vector space over F. Suppose A is a (non-singular) linear transformation of V, as a vector space over R, with the following additional properties:

(i) $V_0 A = V_0$,

(ii) if v is in V, then $vA^q = kv$ for some k in R, $k \neq 0$,

(iii) if v_0 is in V_0, $v_0 \neq (0, 0, 0)$, and if $v_0 A^m = kv$, where k is in F, $k \neq 0$, then $m \equiv 0 \pmod q$.

Then, using (i), if x, y, z are in R, we can write:

$$(1)\quad (x, y, z)A^m = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z),$$

where the a_{ij} are in F (and the a_{ij} depend only on m, of course). Furthermore, if A has the form given by (1), as a linear transformation only of V_0 over F, and if A satisfies (ii) and (iii) (as they apply to V_0 and F), then certainly A is a linear transformation of V over R and satisfies (i), (ii), and (iii). So the existence of A depends only on the existence of a linear transformation A_0 of V_0 over F satisfying:

(iv) if v_0 is in V_0, $v_0 \neq (0, 0, 0)$, then $v_0 A_0{}^m = kv_0$, where k is in F, $k \neq 0$, if and only if $m \equiv 0 \pmod q$.

We shall return to the question of the existence of A_0 after some discussion of the use of A.

Given R, F, and A as above, let us construct a set π of points and lines, with an incidence relation (i.e., point on line or line contains point, etc.), as follows. The points of π will be the elements of V, excepting the element $(0, 0, 0)$, with the identification $(x, y, z) = (kx, ky, kz)$, for any non-zero k in R. The lines of π are the formal symbols $L_t A^m$, where either $t = 1$ or t is in R, t not in F, and where the only identifications will be $L_t A^k = L_t A^m$ if $k \equiv m \pmod q$. Incidence is as follows: $v = (x, y, z)$ is on L_t if $x + yt + z = 0$, while $L_t A^m$ contains just those points vA^m such that v is on L_t. By simple counting, it is seen that π contains $p^{4n} + p^{2n} + 1$ points and the same number of lines, and that each line is incident with $p^{2n} + 1$ distinct points (we have not yet shown, of course, that an arbitrary pair of distinct lines differ at all as sets of points). Thus, from **(5)**, if we can show that each two distinct lines have exactly one point in common, then π will be a projective plane of order p^{2n}.

Now the existence of A_0 is assured by the work of Singer **(6)**, since the cyclic collineation given by him is easily seen to be nothing but a linear transformation of the vector space V_0, and to satisfy (iv). Indeed, Singer states as much, since he represents his projective plane by "homogeneous" coordinates from the field F (i.e., ordered triples with the identification we have used above), and then, without explicitly making use of the vector space, shows the existence of a linear transformation A_0 whose qth power (and no smaller positive power) maps an element (x, y, z) onto an element (kx, ky, kz), $k \neq 0$.

Thus if we succeed in showing that the set π constructed above is a projective plane, it will even contain a subplane π_0 of order p^n, consisting exactly of those points (x, y, z) for which x, y, z are in F, and of the lines L_1A^m. (Thus π_0 will be Desarguesian.) Furthermore, whether π is a projective plane or not, each mapping A^m is a collineation of π: if the point P is on the line L, then PA^m is on LA^m. From this last remark, to show that any pair of distinct lines intersect in exactly one point, it will be sufficient to show that any pair of distinct lines L_tA^m and L_s intersect in exactly one point. In what follows, we assume that equation (1) takes the following form for A^{-m}:

$$(2)\quad (x, y, z)A^{-m} = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z).$$

Let us consider the intersection of the distinct lines L_tA^m and L_s; if (x, y, z) is on both lines, then $(x, y, z)A^{-m}$ is on L_t, so we have:

$$(3)\quad (a_{11}x + a_{12}y + a_{13}z) + (a_{21}x + a_{22}y + a_{23}z)t + (a_{31}x + a_{32}y + a_{33}z) = 0,$$

$$(4)\quad x + ys + z = 0,$$

where each a_{ij} is in F. Solving equation (4) for x and substituting in (3), we have:

$$(5)\qquad yu + za + (yv + zb)t = 0,$$

where:

$$(6)\qquad \begin{aligned} u &= a_{12} + a_{32} - (a_{11} + a_{31})s, & v &= a_{22} - a_{21}s,\\ a &= a_{13} + a_{33} - (a_{11} + a_{31}), & b &= a_{23} - a_{21}. \end{aligned}$$

Note that a, b are in F. We now have several cases.

Case 1; $b \neq 0$. Then equation (5) can be written :

$$(yv + zb)b^{-1}a + y(u - vb^{-1}a) + (yv + zb)t = 0,$$

utilizing the fact that a, b^{-1} are in the center. This becomes:

$$(7)\qquad (yv + zb)\ (b^{-1}a + t) + y(u - vb^{-1}a) = 0.$$

It is easy to see that if $t = 1$ then (5) and **(4)** have a unique common solution for the point (x, y, z). So we assume $t \neq 1$, and thus in (7), t is not in F, and hence $w = b^{-1}a + t \neq 0$. So (7) becomes $(yv + zb)w = -y(u - vb^{-1}a)$, or, multiplying through by w^{-1} and collecting terms,

$$(8) \qquad y[v + (u - vb^{-1}a)w^{-1}] + zb = 0.$$

Since not both the coefficients in (8) can be zero (i.e., $b \neq 0$), (8) and (4) have a unique common solution for the point (x, y, z).

Case II; $b = 0$, $a \neq 0$. Then (5) becomes:

$$(9) \qquad y(u + vt) + za = 0.$$

But since $a \neq 0$, (9) and (4) have also a unique common point.

Case III; $a = b = 0$. That is to say:

$$(10) \qquad a_{13} + a_{33} = a_{11} + a_{31}, \qquad a_{23} = a_{21}.$$

But now consider the element $v_0 = (1, 0, -1)$; from (10) it is immediate that $v_0A^{-m} = cv_0$, where $c = a_{11} - a_{13}$. Necessarily $c \neq 0$, since A^{-m} is not singular. Thus $m \equiv 0 \pmod q$, so $L_tA^m = L_t$, and then it is easy to see that the distinct lines L_t and L_s have only the point $(1, 0, -1)$ in common.

We have thus completed the proof that π is a projective plane of order p^{2n}, and we note that π possesses a cyclic collineation group (generated by A) of order $q = p^{2n} + p^n + 1$. Furthermore, the collineation A fixes no point or line of π, so if we succeed in showing that π is non-Desarguesian, then it even has the stronger property that it cannot be coordinatized by a V-W system, in any manner whatsoever. For use in § 4, we note that the line L_tA^m of π can be represented by an equation:

$$(11) \qquad xa + yb + zc + (xa' + yb' + zc')t = 0,$$

where a, b, c, a', b', c', are all in F.

3. Collineation groups. As pointed out above, π possesses a cyclic group of collineations of order $q = p^{2n} + p^n + 1$. Furthermore, if θ is any automorphism of the near-field R, and if θ fixes every element in F, then the mapping $(x, y, z) \to (x\theta, y\theta, z\theta)$, $L_tA^m \to L_{t\theta}A^m$, is a collineation of π. The near-field of order 9 possesses the non-abelian group of order 6 as a group of automorphisms **(8)** and this group necessarily fixes every element in the subfield of order three. Thus the plane has a group of collineations of order 78, which is the direct product of the cyclic group of order 13 and the non-abelian group of order 6. In **(7)** this collineation group is given explicitly, and it is pointed out that the group is transitive and regular on the points and lines of π which are not in the subplane π_0 of order three (π_0 is mapped into itself by all of the 78 collineations). We shall abstract from this situation before analysing the plane of order 9 in more detail.

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Let π be a projective plane containing a proper subplane π_0 of finite order m. Suppose G is a group of collineations of π such that π_0 is mapped into itself by every element of G, and such that G is transitive and regular on the points and lines of π that are not in π_0. Since any line of π_0 contains a point P not in π_0, every point of π that is not in π_0 is on exactly one line of π_0; for any point of π, not in π_0, is an image of P under some element of G. Let π have order n; as every point of π is on at least one of the $m^2 + m + 1$ lines of π_0, n is finite, for no finite set of lines carries all of the points of an infinite plane. Each line of π_0 carries $n - m$ points not in π_0. So there are

$$(n - m)(m^2 + m + 1) + m^2 + m + 1 = n^2 + n + 1$$

points in π. Solving this for n, and noting that $n \neq m$, it is elementary that $n = m^2$. If L_1 and L_2 are lines of π_0, containing the points P_1 and P_2, respectively, where $P_1, P_2 \notin \pi_0$, then since $P_1x = P_2$ for some $x \in G$, we have $L_1x = L_2$ since L_2 is the only line of π_0 containing P_2. Thus G is transitive on the lines, and similarly on the points, of π_0. The order of G is equal to the number of points of π which are not in π_0; i.e., G has order $m^4 + m^2 + 1 - (m^2 + m + 1) = m^4 - m$. Let the points and lines of π that are not in π_0 be called tangent points and tangent lines. Let P_0 be some fixed tangent point and let J_0 be some fixed tangent line; we can assume that P_0 is on J_0. Let K_0 be the unique line of π_0 which contains P_0, and let Q_0 be the unique point of π_0 which is on J_0. Let D be the subset of G consisting of all x such that P_0x is on J_0, let E consist of all x such that P_0x is on K_0, let F consist of all x such that J_0x contains Q_0, and let D_0 consist of all x such that Q_0x is on K_0.

We observe that E and F are subgroups of G. For if e is in E, then P_0e is a point of K_0, and must lie on exactly one line of π_0; hence this line must be K_0, so $K_0e = K_0$, and E is the subgroup of G which fixes K_0. Similarly, F is the subgroup which fixes Q_0. If K is any line of π_0, then K is fixed by some conjugate (in G) of E, and if Q is any point of π_0, then Q is fixed by some conjugate of F. Furthermore, $FD_0E = D_0$; E and F have order $m^2 - m$, D contains m^2 elements and D_0 contains $(m + 1)(m^2 - m)$ elements.

THEOREM 1. (i) *If a is not in E, then $a = d_1d_2^{-1}$ for a unique d_1, d_2 in D; if a is in E, $a \neq 1$, then $a \neq d_1d_2^{-1}$ for any d_1, d_2 in D.*

(ii) *The left cosets of F can all be represented as dF for a unique d in D, or as $d_0^{-1}F$ for d_0 in D_0, but not both.*

(iii) *If a is not in F, then $a = d_1d_2^{-1}$ for d_1, d_2 in D_0, where d_1, d_2 are uniquely determined up to a common right multiple by an element of E (i.e., d_1e, d_2e, where e is in E, are also in D_0 and $a = (d_1e)(d_2e)^{-1}$, but a only has representations of this form).*

(iv) *$FD_0E = D_0$.*

Proof. (i) If a is not in E, then $P_0 \cdot P_0 a$ is a line $J_0 b$ for a unique b. Hence $b^{-1} = d_2$ in D, $ab^{-1} = d_1$ in D, so $a = d_1 d_2^{-1}$; since b is unique, it is easy to see that d_1, d_2 are unique. Conversely, if a is in E, $a \neq 1$, then the impossibility of $a = d_1 d_2^{-1}$ is easy to demonstrate.

(ii) As in (i), consideration of the point $R = J_0 \cdot J_0 a$, where a is not in F, shows that $a = d_1^{-1} d_2$ for a unique pair d_1, d_2 in D, and if a is in F, $a \neq 1$, then $a \neq d_1^{-1} d_2$ for any d_1, d_2 in D. Hence if d_1, d_2 are in D and $d_1 F = d_2 F$, then $d_1^{-1} d_2$ is in F, so $d_1 = d_2$. Thus the m^2 cosets dF, for d in D, are all distinct.

Consider the points P_0 and $Q_0 a$; the line $L = P_0 \cdot Q_0 a$ is a line $J_0 b$ if and only if b^{-1} is in D and $Q_0 a = Q_0 b$; i.e., if and only if a^{-1} is in $b^{-1}F$, where b^{-1} is in D. On the other hand, if $L = K_0 b$, then b is in E and ab^{-1} is in D_0, so a is in $D_0 b \subseteq D_0 E = D_0 = FD_0$. So a^{-1} is in $d_0^{-1}F$, where d_0 is in D_0. This proves (ii).

(iii) Consider the line $K_0 b = Q_0 \cdot Q_0 a$, where a is not in F. We have, as in (i), $a = d_1 d_2^{-1}$, where d_1, d_2 are in D_0, but d_1, d_2 are not unique: since $K_0 b = K_0 eb$ for any e in E, it is easy to see that $d_1 e$, $d_2 e$ (both in D_0) also represent a as $a = (d_1 e)(d_2 e)^{-1}$, and that all such representations are of this form.

(iv) It has already been pointed out that (iv) is satisfied.

Now, without giving the proofs (which are straightforward but time-consuming), we remark that the existence of a group G of order $m^4 - m$, containing subgroups E and F of order $m^2 - m$ and two subsets D and D_0, with m^2 and $(m + 1)(m^2 - m)$ elements respectively, all satisfying (i)-(iv) of Theorem 1, implies the existence of the projective plane π. Points are designated by (a) for all a in G, and (Fa) for all cosets Fa of F; lines are $[Db]$ for all b in G, and $[Eb]$ for all cosets Eb of E. Incidence is given by the rules: (a) is on $[Db]$ if a is in Db; (a) is on $[Eb]$ if a is in Eb; (Fa) is on $[Db]$ if b is in Fa; (Fa) is on $[Eb]$ if a is in $D_0 b$.

Suppose such a group possesses an automorphism λ with the properties $E\lambda = E$, $F\lambda = F$, $d^{-1}\lambda$ is in D if d is in D, $d_0^{-1}\lambda$ is in D_0 if d_0 is in D_0. Then the mapping T defined below is a duality of the plane π, the simple proof of which statement we omit.

$$T\colon\quad \begin{array}{ll} (a) \to [D \cdot a\lambda] & \qquad [Db] \to (b\lambda) \\ (Fa) \to [E \cdot a\lambda] & \qquad [Eb] \to (F \cdot b\lambda). \end{array}$$

In the case of the plane of order 9 given by Veblen and Wedderburn, G can be taken as the group of order 78 mentioned above, and $E = F$ is the subgroup of order 6; note that E is normal in G. The subsets D and D_0 depend upon our choice of P_0 and J_0, so we reproduce the plane below. The points are the symbols $A_i, B_i, \ldots, G_i$, $i = 0, 1, \ldots, 12$, and we give seven of the lines; the remaining lines are found by successively adding one to the subscripts, reducing modulo 13 (indeed, this operation corresponds to the collineation A):

$$\begin{array}{llllllllllll}
L_1 & : A_0 & A_1 & A_3 & A_9 & B_0 & C_0 & D_0 & E_0 & F_0 & G_0 \\
L_j & : A_0 & B_1 & B_8 & D_3 & D_{11} & E_2 & E_5 & E_6 & G_7 & G_9 \\
L_{2j} & : A_0 & C_1 & C_8 & E_7 & E_9 & F_3 & F_{11} & G_2 & G_5 & G_6 \\
L_{1+j} & : A_0 & B_7 & B_9 & D_1 & D_8 & F_2 & F_5 & F_6 & G_3 & G_{11} \\
L_{2+2j} & : A_0 & B_2 & B_5 & B_6 & C_3 & C_{11} & E_1 & E_8 & F_7 & F_9 \\
L_{1+2j} & : A_0 & C_7 & C_9 & D_2 & D_5 & D_6 & E_3 & E_{11} & F_1 & F_8 \\
L_{2+j} & : A_0 & B_3 & B_{11} & C_2 & C_5 & C_6 & D_7 & D_9 & G_1 & G_8 .
\end{array}$$

Then the group $E = F$ corresponds to the permutations:

$$E = \{1, (BDG)(CEF), (BGD)(CFE), (BC)(DF)(EG), (BE)(CD)(FG), (BF)(CG)(DE)\}.$$

Let us choose B_2 as P_0 and L_{2+2j} as J_0. Then K_0 is L_1A^2, Q_0 is A_0, and:

$$D = \{1, A^3, A^4, (BC)(DF)(EG)A, (BC)(DF)(EG)A^9, (BE)(CD)(FG)A^6, (BE)(CD)(FG)A^{12}, (BF)(CG)(DE)A^5, (BF)(CG)(DE)A^7\},$$
$$D_0 = \{A^2, A^6, A^{10}, A^{11}\} \cdot E.$$

Every element of G can be represented in the form eA^k, where e is in E. Let λ be the automorphism $eA^k \to eA^{-k}$; then $E\lambda = E$, $d^{-1}\lambda$ is in D if d is in D and $d_0^{-1}\lambda$ is in D_0 if d_0 is in D_0. So (see above) the plane is self-dual.

This plane has some further interesting properties, some of which we will mention. Each one of the following sets of three points is on a line, and it is easy to see that the whole set of seven points, together with the seven lines joining them, forms a subplane of order two:

$$A_3 A_9 B_0;\ A_2 A_3 D_2;\ A_2 A_9 G_6;\ B_0 D_2 G_6;\ A_3 D_{11} G_6;\ A_2 B_0 D_{11};\ A_9 D_2 D_{11}.$$

Since no element of G fixes all of the seven points, this means that there are at least 78 distinct subplanes of order two and every point of the plane is in at least four subplanes of order two.

The duality T given above has 22 absolute points (i.e., points on their image line): $A_1, A_8, A_9, A_{12}, B_2, B_4, B_{10}, C_2, C_4, C_{10}, \ldots, G_2, G_4, G_{10}$. The line L_1A^5 contains two absolute points A_1, A_8, while L_jA contains four absolute points A_1, B_2, D_4, G_{10}. Thus the duality is not "regular," giving a negative answer to a question raised by Baer **(1)**.

Finally, π contains "ovals"; i.e., sets of ten points, no three of which are on a line. An example of one is the set of points:

$$A_0, A_1, A_6, A_7, B_2, C_8, C_9, D_8, D_9, E_2.$$

The oval given above even has the strong property that there is another duality of the plane (distinct from T) whose absolute points are exactly the points of the oval (the duality can, in fact, be constructed from the oval). Hence the collineation group of the plane consists of more than the 78 coll-neations given above; for if T_1 and T_2 are dualities of any projective plane, then T_1T_2 is a collineation of the plane. The author has not been able to determine the full collineation group of the plane given above.

4. Coordinate rings. We shall use the planar ternary ring coordinatizing a projective plane to derive some more properties of the class of planes constructed in § 2; in particular, we shall show that these planes are all non-Desarguesian, and cannot be coordinatized by Veblen-Wedderburn systems. The particular formulation of the ternary ring will follow the lines of **(3)**, but we briefly review the basic idea, which differs a bit from Hall's technique in **(2)**. In **(4)** will also be found a good deal of material on these ternary rings. R is a nonempty set containing at least the two distinct elements 0 and 1, and points are symbols (x, y), (m), (∞), where $m, x, y \in R$, and ∞ is a symbol not in R; lines are symbols $[m, k]$, $[\infty, (k, 0)]$, L_∞, where $m, k \in R$. A ternary operation F is defined so that (x, y) is on $[m, k]$ if and only if $F(m, x, y) = k$; the other rules of incidence are: (x, y) is on $[\infty, (x, 0)]$, (m) is on $[m, k]$ and L_∞, (∞) is on $[\infty, (k, 0)]$ and L_∞. Then the ternary function F satisfies certain axioms, which will be found in **(2, 3; 4)**. Addition is defined by $a + b = F(1, a, b)$ and multiplication by $a \cdot b = ab = F(a, b, 0)$; the ring is called *linear* if $F(a, b, c) = ab + c$ for all $a, b, c \in R$. It is well known that every planar ternary ring for a Desarguesian plane is an associative division ring, and, in particular, is linear.

Throughout the rest of this section let π be a projective plane of order p^{2n}, constructed from the left near-field R, as in § 2. We shall coordinatize π so as to construct one of its planar ternary rings, and in what follows, the ordered triples have the same meaning as in § 2. Let (∞) be $(0, 0, 1)$; (0) be $(1, 0, 0)$; $(0, 0)$ be $(0, 1, 0)$; and let (1) be $(1, 0, -1)$. The x-axis, y-axis, and L_∞ are then all lines of the form $L_1 A^k$, and, in particular, they are respectively $z = 0$, $x = 0$, and $y = 0$. The points on the y-axis are (in the old representation) all of the form $(0, 1, v)$, so let $(0, 1, v)$ be $(0, v)$ in our new coordinate system. Every line through (1), or $(1, 0, -1)$, is a line of the form $x + yt + z = 0$. The point $(v, 0)$ on the x-axis will be the point $(u, 1, 0)$ which is collinear with $(1, 0, -1)$ and $(0, 1, v)$; but $(1, 0, -1)$ and $(0, 1, v)$ are on the line L_{-v} if $v \notin F$, whence $u + 1(-v) + 0 = 0$, or $u = v$. If $v \in F$, then it is immediate that $u = v$. So $(v, 0)$ is $(v, 1, 0)$.

The point (m) will be the point on L_∞ which is collinear with $(1, 1, 0)$ and $(0,1, m)$, and will be a point $(1, 0, v)$. Let $xa + yb + zc + (xa'+yb'+c'z)t=0$ be the line joining $(1, 1, 0)$ and $(0, 1, m)$. Then:

$$a + b + (a' + b')t = 0 \tag{1}$$
$$b + mc + (b' + mc')t = 0. \tag{2}$$

Since $a, a', b, b' \in F$, (1) implies that either $t = 1$ and $a + a' = -(b + b')$, or $t \neq 1$ and $a + b = a' + b' = 0$. If $t = 1$, then $a + a' + v(c + c') = 0$, and (2) becomes

$$a + a' + (-m)(c + c') = 0;$$

thus $v = -m$. If $t \neq 1$, then $a + vc + (a' + vc')t = 0$, and (2) becomes $-a + mc + (-a' + mc')t = 0$, so again $v = -m$. Thus (m) is the point $(1, 0, -m)$.

Consider the point (u, v) which lies on the line J_1 joining $(0, 0, 1)$ and $(u, 1, 0)$ and on the line J_2 joining $(1, 0, 0)$ and $(0, 1, v)$. Then:

$$(3) \qquad \begin{aligned} J_1 &: xa + yb + zc + (xa' + yb' + zc')t = 0, \\ J_2 &: xd + ye + zf + (xd' + ye' + zf')s = 0, \end{aligned}$$

where

$$(4) \qquad c + c't = d + d's = ua + b + (ua' + b')t = e + vf + (e' + vf')s = 0.$$

Thus as before, $t = 1$ and $c = -c'$, or $t \neq 1$ and $c = c' = 0$; similarly, $s = 1$ and $d = -d'$, or $s \neq 1$ and $d = d' = 0$.

There are four cases to check, but all of them are easy. If $t \neq 1$, $s \neq 1$, then it is a simple verification, using (4), that $(u, 1, v)$ is on both of the lines J_1 and J_2. If $t = 1$, $s \neq 1$, then J_1 is

$$x(a + a') + y(b + b') = 0,$$

where $u(a + a') + b + b' = 0$, so $(u, 1, \mathrm{v})$ is on J_1. The line J_2 becomes

$$ye + zf + (ye' + zf')s = 0,$$

where $e + vf + (e' + vf')s = 0$, so $(u, 1, v)$ is on J_2. In all cases, we find that $(u, 1, v)$ is on both J_1 and J_2, so (u, v) is $(u, 1, v)$.

So we have:

THEOREM 2. *If π is coordinatized as above, then (u, v) is $(u, 1, v)$, (m) is $(1, 0, -m)$ and (∞) is $(0, 0, 1)$.*

Now we shall investigate the ternary ring for π, where we use $T(a, b, c)$ for the ternary operation, and let $a \oplus b = T(1, a, b)$, $a \bigcirc b = T(a, b, 0)$. In order to find the value of $T(m, u, v)$, we consider the line L which contains $(1, 0, -m)$ and $(u, 1, v)$, and let $(0, 1, k)$ be the intersection of L with the y-axis; then $k = T(m, u, v)$. Let L be the line $xa + yb + zc + (xa' + yb' + zc')t = 0$. Then:

$$(5) \qquad a - mc + (a' - mc')t = 0$$

$$(6) \qquad ua + b + vc + (ua' + b' + vc')\mathrm{t} = 0.$$

THEOREM 3. *For all a and b, $a \oplus b = a + b$.*

Proof. Let $m = 1$ in (5). Then $a - c + (a' - c')t = 0$. So if $t = 1$, we have $a + a' = c + c'$, and (6) becomes $u(a + a') + (b + b') + v(c + c') = 0$, or $(u + v)(a + a') + (b + b') = 0$. But then the point $(0, 1, u + v)$ is on L, so $k = u \oplus v = u + v$. If $t \neq 1$, then $a = c$ and $a' = c'$, and (6) becomes

$$(u + v)\, a + b + [(u + v)a' + b']t = 0,$$

whence again $(0, 1, u + v)$ is on L, so $u \oplus v = u + v$.

THEOREM 4. *The ternary ring (R,T) is not linear.*

Proof. Referring to (5) and (6), let m and u be arbitrary, $u \neq 0$, and let v be chosen so that $k = T(m, u, v) = 0$. Then $(0, 1, 0)$ is on L, so $b + b't = 0$.

Suppose $t = 1$, whence $b = -b'$. Then (5) is $a + a' - m(c + c') = 0$, and (6) is

$$u(a + a') + v(c + c') = 0,$$

and this implies $u^{-1}v = -m$, or $um + v = 0$. Suppose $t \neq 1$, whence $b = b' = 0$. Then (5) is $a - mc + (a' - mc')t = 0$, and (6) can be written

$$a + u^{-1}vc + (a' + u^{-1}vc')t = 0,$$

and so again $u^{-1}v = -m$, or $um + v = 0$.

Now assume that (R,T) is linear. For arbitrary m and u, $u \neq 0$, let $p = m \bigcirc u$. Then $m \bigcirc u \oplus (-p) = T(m, u, -p) = 0$, so by the above, $um + (-p) = 0$, or $um = p = m \bigcirc u$. Thus (R,T) is anti-isomorphic to the near-field R, and so (R,T) is itself a near-field. As pointed out earlier, a (finite) near-field plane possessing a collineation moving every point and line is necessarily Desarguesian. But π does possess such a collineation; it is obvious that A fixes no point or line of π. Hence (R,T) is a field, so R is also a field, and this is contradictory; thus (R,T) cannot be linear.

CORLLOARY. *The plane π is non-Desarguesian.*

Proof. See the proof of Theorem 4. Or note that every ternary ring for a Desarguesian plane is linear, whence by Theorem 4, π is not Desarguesian.

Besides the class of projective planes given in this paper, all finite planes known at the present time are coordinatizable by ternary rings which are (at least) V-W systems. So, in a sense, the class of planes given here are the "weakest" finite planes known. Aside from having prime-power order, these planes share a much stronger property with the V-W system planes however: they can be coordinatized by a planar ternary ring whose addition forms an elementary abelian group, or equivalently, there is a complete set of mutually orthogonal latin squares associated with the plane which contains among its squares the Cayley table of an elementary abelian group.

Other questions about the class of planes, which are answered in § 3 for the plane of order 9, include: are the planes self-dual and does there exist a collineation group G with the properties discussed in § 3? The automorphism groups of the near-fields used in this paper, with the exception of the one of order 9, are always too small to yield enough collineations to construct the group G. The author has checked a plane of the class, of order 25, and it does not possess such a group G of collineations. Finally, what other finite planes possess such a collineation group G?

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