The n^{th} root of a prime number cannot be the root of an equation of degree less than n with rational coefficients.

By DAVID MAIR.

Suppose possible the relation

$$a_0 + a_1 p^{\frac{1}{n}} + a_2 p^{\frac{2}{n}} + \ldots\ldots a_{n-1} p^{\frac{n-1}{n}} = 0$$

where n is an integer and a_0, a_1, ... a_{n-1}, p are rational.

If p is fractional $= \frac{a}{\beta}$ put $p = \frac{q}{\beta^n}$; then by multiplying throughout by a suitable integer we get a relation of the above form in which all symbols represent integers.

If a_0 contained a factor p the equation could be reduced to the form

$$a_1 + a_2 p^{\frac{1}{n}} + \ldots\ldots a_{n-1} p^{\frac{n-2}{n}} + a_0 p^{\frac{n-1}{n}} = 0 \,;$$

suppose such reduction to have been carried out so that the first term does not contain p.

The relation may be put in the various forms

$$a_0 \quad + a_1 p^{\frac{1}{n}} + \ldots\ldots\ldots + a_{n-1} p^{\frac{n-1}{n}} = 0$$

$$a_{n-1} p + a_0 p^{\frac{1}{n}} + \ldots\ldots\ldots + a_{n-2} p^{\frac{n-1}{n}} = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_1 p \quad + a_2 p \,.\, p^{\frac{1}{n}} + \ldots\ldots + a_0 p^{\frac{n-1}{n}} \quad = 0 \,;$$

whence by eliminating $p^{\frac{1}{n}}$, $p^{\frac{2}{n}}$, ...

$$\begin{vmatrix} a_0 & a_1 & a_2 \ldots\ldots\ldots a_{n-1} \\ a_{n-1} p & a_0 & a_1 \ldots\ldots\ldots a_{n-2} \\ a_{n-2} p & a_{n-1} p & a_0 \ldots\ldots\ldots a_{n-3} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_1 p & a_2 p & a_3 p \ldots\ldots\ldots a_0 \end{vmatrix} = 0.$$

The only term of the determinant not containing p is $a_0{}^n$;

$$\therefore \quad a_0{}^n + p\mathrm{M} = 0.$$

From which it follows that if p is a product of prime factors, *all different*, a_0 is a multiple of p.

Hence the assumed relation can not be true if p is a prime number or a product of different primes.

https://doi.org/10.1017/S0013091500032569 Published online by Cambridge University Press