DIAMETER OF A (0, 1)-MATRIX*

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1. Introduction. Let A be an $m \times n$ (0, 1)-matrix. Let $C_1, C_2, \ldots, C_n$ denote its columns. A sequence of distinct columns $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ is said to form a chain if the inner product of C_{i_t} and $C_{i_{t+1}}$ (for $1 \leq t \leq k-1$) is at least one. $k-1$ is called the length of the chain and this chain is said to connect C_{i_1} and C_{i_k}, and C_{i_1} and C_{i_k} are said to be connected. As can be easily seen, connectedness is an equivalence relation on the set of columns. A matrix is called connected if all its columns belong to the same equivalence class. If C_i and C_j belong to the same equivalence class, then $s(C_i, C_j)$ will denote the length of the shortest chain between C_i and C_j. We define the distance between any two columns C_i and C_j, to be denoted by $d(C_i, C_j)$, in the following manner.

$$d(C_i, C_i) = 0$$

and if $i \neq j$

$$d(C_i, C_j) = \begin{cases} s(C_i, C_j) & \text{if } C_i \text{ and } C_j \text{ belong to the same connected component,} \\ \infty, & \text{otherwise.} \end{cases}$$

The diameter of the matrix, to be denoted by $d(A)$, is defined as

$$d(A) = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} d(C_i, C_j) .$$

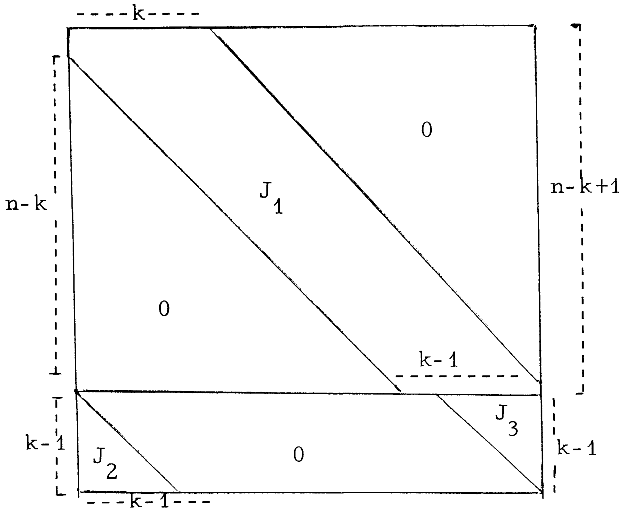
The diameter of a disconnected matrix is infinite. $\mathfrak{U}(R, S)$ denotes the class of matrices with R and S as row and column sum vectors respectively (as in [1]), and $\tilde{d}$ denotes $\min_{A \in \mathfrak{U}(R, S)} d(A)$.

*This work was done while the author was at the University of Alberta, Edmonton.

https://doi.org/10.4153/CMB-1968-035-6 Published online by Cambridge University Press

The aim of this note is to obtain some bounds of $\tilde{d}$ of $\mathfrak{N}(\tilde{K},\tilde{K})$ where $\tilde{K}$ is the n coordinate vector $(k, k, \ldots, k)$, k being a positive integer ≥ 2. We shall also consider a related extremal problem involving generalized inner products. We shall observe that incidence matrices of (v, k, λ)-designs are in some sense extremal matrices of diameter 1.

2. Bounds for $\tilde{d}$ and a related problem. In this section we shall consider the class $\mathfrak{N}(\tilde{K},\tilde{K})$ of (0, 1)-matrices. The matrix exhibited below belongs to $\mathfrak{N}(\tilde{K},\tilde{K})$.



where J_1, J_2, and J_3 are blocks of 1's, and 0's are blocks of zeros.

The diameter of this matrix is $\left[\frac{[\frac{n}{2}]}{k-1}\right]^*$. ([x] denotes the greatest integer $\leq x$ and $[x]^*$ denotes the least integer $\geq x$.) Hence we have

$$(1) \qquad \tilde{d} \leq \left[\frac{[\frac{n}{2}]}{k-1}\right]^* .$$

The upper bound in (1) is attained in the case of $k = 2$ but if $k \geq 3$ this upper bound may be higher than the actual value. For example if $n = 7$ and $k = 3$ the number on the right side of (1) is equal to 2 whereas $\tilde{d}$ is equal to 1, as the incidence matrix of projective plane of order 2 is of diameter 1. It may easily be noted that if $n \leq 2k-1$ then $\tilde{d} = 1$.

The above example shows that even if $n > 2k-1$, $\tilde{d}$ can be equal to 1. We may now ask ourselves the question: What is the maximum value of n such that there exists a matrix of diameter 1 in the class $\mathfrak{N}(\tilde{K},\tilde{K})$. We shall see that this problem is a particular case of the following more general problem.

Let $A = \{a_{ij}\}$ be an $m \times n$ matrix. The generalized inner product of r columns, say $C_{j_1}, C_{j_2}, \ldots, C_{j_r}$ is defined as $\sum_{i=1}^{m} \prod_{k=1}^{r} a_{ij_k}$. An integer n is said to have the property $P(k, r, t)$ if there exists a matrix A in the class $\mathfrak{A}(\tilde{K}, \tilde{K})$ such that the generalized inner product of any r columns of A is at least t. Obviously if n is too large compared to k, r and t it will not have the property $P(k, r, t)$. What then is the maximum value of the integer n which has the property $P(k, r, t)$? We shall denote this maximum by $M(k, r, t)$.

THEOREM 1. $M(k, r, t) \leq \frac{k(k-r+1)}{t} + r - 1$.

Proof. The above problem is equivalent to asking for the maximum value of n such that there exists a matrix in the class $\mathfrak{A}(\tilde{K}, \tilde{K})$ in which the partial row sum vector of every set of r columns has at least t r's. Let A be a matrix in $\mathfrak{A}(\tilde{K}, \tilde{K})$ which has this property. Consider the first $r-1$ columns of A. Suppose that the row sum vector of the submatrix constituted by these $r-1$ columns has p $r-1$'s. Each of these p rows will have $k-r+1$ 1's in the rth, (r+1)th, ..., nth columns. And each of the rth, (r+1)th, ..., nth columns will have at least t 1's in these p rows. We therefore have

$$p(k-r+1) \geq (n-r+1)t \tag{2}$$

as $k \geq p$, it follows from (2) that

$$k(k-r+1) \geq (n-r+1)t$$

or

$$n \leq \frac{k(k-r+1)}{t} + r - 1 \quad .$$

This completes the proof of the theorem.

We have in particular proved that $M(k, 2, \lambda) \leq \frac{k(k-1)}{\lambda} + 1$. But the incidence matrix of a (v, k, λ)-design, if it exists, will have the property $P(k, 2, \lambda)$ and will have $\frac{k(k-1)}{\lambda} + 1$ columns and will imply $M(k, 2, \lambda) = \frac{k(k-1)}{\lambda} + 1$. On the other hand if there is a matrix with property $P(k, 2, \lambda)$ and having as many as $\frac{k(k-1)}{\lambda} + 1$ columns then it is easy to observe that it will have to be incidence matrix of a BIB design. This establishes that

$$M(k, 2, \lambda) = \frac{k(k-1)}{\lambda} + 1$$

https://doi.org/10.4153/CMB-1968-035-6 Published online by Cambridge University Press

if and only if there exists a (v, k, λ)-design (k and λ being the fixed quantities).

THEOREM 2.

$$(3) \qquad 2^{\frac{\log\left(\frac{n(k^2-2)}{2(k^2-1)}+1\right)}{\log(k^2-1)}} - 2 \leq \tilde{d} \quad .$$

Proof. With each matrix in the class $\mathfrak{U}(\tilde{K}, \tilde{K})$ we can associate a graph in the following way. We have a vertex corresponding to each column and two vertices are joined if and only if the corresponding columns have a 1 in the same row. The diameter of a matrix is the same as the diameter of the graph thus associated with it. In the class of graphs corresponding to different matrices in $\mathfrak{U}(\tilde{K}, \tilde{K})$, the degree of any vertex is at most $k^2 - 1$. It can be verified that no such graph can have diameter less than the left hand side of (3). This proves the result.

REFERENCE

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https://doi.org/10.4153/CMB-1968-035-6 Published online by Cambridge University Press