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Nagoya Math. J.
Vol. 43 (1971), 137–149

# ON THE TRACE OF HECKE OPERATORS FOR CERTAIN MODULAR GROUPS

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## Introduction.

The trace of Hecke operators with respect to a unit group of an order in a quaternion algebra has been given in Eichler [1], [2] in the case when the order is of square-free level. The purpose of this note is to study the order of type $(q_1, q_2, q_3)$ (see text 1.1), in the case, of cube-free level, and to give a formula for the trace of Hecke operators in the case $q_3 = 2$.

## Notation.

$\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ denote the ring of rational integers, the field of rational numbers, and the field of real numbers, respectively. $\boldsymbol{Q}_p$ denotes the $p$-adic closure of $\boldsymbol{Q}$ and $\boldsymbol{Z}_p$ the ring of integers in $\boldsymbol{Q}_p$. $R$ being a ring, $M_2(R)$ denotes the full matrix ring over $R$ of degree 2.

## 1. The order of type $(q_1, q_2, q_3)$

1.1. Let $A$ be a quaternion algebra over $\boldsymbol{Q}$ and $q_1^2 = d(A/\boldsymbol{Q})$ be its discriminant. For every prime number $p$, $A_p \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{Q}_p$ is a division algebra over $\boldsymbol{Q}_p$ or $A_p \simeq M_2(\boldsymbol{Q}_p)$ according as $p \mid q_1$ or $p \nmid q_1$. Let $q_2$, $q_3$ be square-free positive integers such that $(q_i, q_j) = 1$ for $i \neq j$, $1 \leq i, j \leq 3$. We then define the order $\mathfrak{O}$ of type $(q_1, q_2, q_3)$ which satisfies the following properties:

i) $\mathfrak{O}_p = \mathfrak{O} \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Z}_p$ is a maximal order in $A_p$, if $p \nmid q_1q_2q_3$,

ii) $\mathfrak{O}_p$ is the unique maximal order in the division algebra $A_p$, if $p/q_1$,

iii) $\mathfrak{O}_p \cong \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\boldsymbol{Z}_p) \mid c \equiv 0 \pmod{p}\right\}$, if $p \mid q_2$,

iv) $\mathfrak{O}_p \cong \left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_2(\boldsymbol{Z}_p) \mid c \equiv 0 \pmod{p^2}\right\}$, if $p \mid q_3$,

In this note we consider the order of type $(q_1, q_2, q_3)$ exclusively.

Received August 27, 1970.
Revised November 18, 1970.

**1.2.** The local properties of the order of type $(q_1, q_2, 1)$, in our notation, have been investigated by [1], [2]. So we study the property of $\mathfrak{O}_p$ for $p \mid q_3$. After fixing the isomorphism we assume $\mathfrak{O}_p = \left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_2(\boldsymbol{Z}_p) \mid c \equiv 0 \pmod{p^2}\right\}$, and write symbolically $\mathfrak{O}_p = \begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ (p^2) & \boldsymbol{Z}_p \end{pmatrix}$. Let $U_p$ be the unit group of $\mathfrak{O}_p$; then according to the elementary divisor theory, we find that every double coset $U_p \alpha U_p$ modulo scalar matrix $(\alpha \in \mathfrak{O}_p)$ is one of the following types:

(1) $U_p\begin{pmatrix} p^a & 0 \\ 0 & 1 \end{pmatrix}U_p$, (2) $U_p\begin{pmatrix} 1 & 0 \\ 0 & p^a \end{pmatrix}U_p$,

(3) $U_p\begin{pmatrix} p^a & 1 \\ 0 & p \end{pmatrix}U_p$, $(a \geq 1)$, (4) $U_p\begin{pmatrix} p & 1 \\ 0 & p^a \end{pmatrix}U_p$, $(a \geq 1)$,

(5) $U_p\begin{pmatrix} 0 & p^a \\ p^2 & 0 \end{pmatrix}U_p$, $(a \geq 0)$, (6) $U_p\begin{pmatrix} 0 & 1 \\ p^a & 0 \end{pmatrix}U_p$, $(a \geq 2)$,

(7) $U_p\begin{pmatrix} 0 & p^a \\ p^2 & p \end{pmatrix}U_p$, $(a \geq 0)$, (8) $U_p\begin{pmatrix} p & 0 \\ p^2 & p^a \end{pmatrix}U_p$, $(a \geq 1)$,

(9) $U_p\begin{pmatrix} p & 1+p^a \\ p^2 & p \end{pmatrix}U_p$,

and the degree (the number of left representatives) of $U_p \alpha U_p$ is calculated for the above nine cases as follows:

(1) $p^a$, (2) $p^a$, (3) $p^a - p^{a-1}$, (4) $p^a - p^{a-1}$,

(5) $p^a$, (6) $p^{a-2}$, (7) $p^{a+1} - p^a$, (8) $p^a - p^{a-1}$,

(9) $\dfrac{p(p-1)(p^{a+1}-1)}{p+1}$, if a is odd,

$\dfrac{(p-1)(p^{a+2}-p-2)}{p+1}$, if a is even.

By decomposing these double cosets into the sum of left representatives, we see that every integral left $\mathfrak{O}_p$-ideal with norm $p^n$ is one of the following types;

(i) $\mathfrak{O}_p\begin{pmatrix} p^a & t \\ 0 & p \end{pmatrix}$, $t \bmod p^b$, $a+b=n$, $a, b \geqq 0$,

(ii) $\mathfrak{O}_p\begin{pmatrix} 0 & p^a \\ p^b & t \end{pmatrix}$, $t \bmod p^{a+2}$, $a+b=n$, $a \geqq 0$, $b \geqq 2$,

(iii) $\mathfrak{O}_p\begin{pmatrix} p^a & 0 \\ p^{a+1}v & p^b \end{pmatrix}$, $1 \leqq v \leqq p-1$, $a+b=n$, $a \geqq 1$, $b \geqq 0$,

(iv) $\mathfrak{O}_p\begin{pmatrix} p^{b-2}v & p^a \\ p^b & 0 \end{pmatrix}$, $1 \leqq v \leqq p-1$, $a+b=n$, $a \geqq 0$, $b \geqq 2$,

(v) $\mathfrak{O}_p\begin{pmatrix} p^{a+1}x & p^b \\ p^{a+2} & p^{b+1}y \end{pmatrix}$, $xy-1\equiv 0 \pmod{p^{n-a-b-2}}$,
$xy-1 \not\equiv 0 \pmod{p^{n-a-b-1}}$,
$x, y \bmod p^{n-a-b-1}$, $x, y$ : units,
$a+b+2 \leqq n$, $a, b \geqq 0$.

## 2. The case $q_3 = 2$

**2.1.** Hereafter we assume $q_3 = 2$, hence $\mathfrak{O}$ is of type $(q_1, q_2, 2)$.

LEMMA 1. *The group of integral two-sided* $\mathfrak{O}_2 = \mathfrak{O} \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Z}_2$ *ideals modulo scalar ideals is isomorphic to the symmetric group of degree* 3, *hence its order is* 6.

*Proof.* Since for any integral two-dsied $\mathfrak{O}_2$ ideal $\mathfrak{M} = \mathfrak{O}_2\alpha = \alpha\mathfrak{O}_2$ $(\alpha \in \mathfrak{O}_2)$, the degree of $U_2\alpha U_2$ should be 1, hence the generator $\alpha$ of $\mathfrak{M}$ is, according to the elementary divisor theory given in **1, 2**, one of the following forms:

$$\iota = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \pi = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \ \xi = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \ \pi\xi = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix},$$

$$\xi\pi = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}, \ \xi\pi\xi = \pi\xi\pi = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}.$$

We see easily that $(\pi\xi)^3 = (\xi\pi)^3 = \iota$, and $\pi^2 = \xi^2 = \iota$ modulo scalar matrix. Hence we obtain Lemma 1.

**2.2.** Let $\mathfrak{g}$ be an order in a quadratic field $K = \boldsymbol{Q}(\sqrt{d})$ ($d$ : a squarefree integer); then we may put $\mathfrak{g} = \boldsymbol{Z}[1, \omega]$ and $\omega = f\omega_0$ $(f > 0)$ where $[1, \omega_0]$ is the canonical $\boldsymbol{Z}$-basis of the maximal order $\mathfrak{g}_0$ in $K$, namely

$$\omega_0 = \begin{cases} (1+\sqrt{d})/2, & \text{if } d \equiv 1 \pmod 4, \\ \sqrt{d} \quad . & \text{if } d \equiv 2, 3 \pmod 4). \end{cases}$$

The discriminat $D$ of $\mathfrak{g}$ is $D = f^2D_0$, where $D_0 = d$ or $4d$ according as $d \equiv 1$ or 2, 3 mod 4. Now for a prime $p$, we define the modified Legendre symbol as follows:

$$\left\{\frac{D}{p}\right\} = \begin{cases} 1, \text{ if } Dp^{-2} \in \boldsymbol{Z} \text{ and } Dp^{-2} \equiv 0, 1 \pmod 4, \\ \left(\frac{D}{p}\right), \text{ the Legendre symbol, otherwise.} \end{cases}$$

**2.3.** Let $K$ be a quadratic subfield of $A$ and $\mathfrak{g}$ be an order in $K$; then we say $\mathfrak{g}$ is optimally embedded in $\mathfrak{O}$ if $\mathfrak{g} = \mathfrak{O} \cap K$. It is easy to see that $\mathfrak{g}$ is optimally embedded in $\mathfrak{O}$ if and only if $\mathfrak{g}_p = \mathfrak{O}_p \cap K_p$ for every $p$. Now we shall prove the following theorem which is essential to give a formula for the trace of Hecke operators, and this was proved for the order of type $(q_1, q_2, 1)$ by [2].

THEOREM 1. *Let $\mathfrak{O}$ and $\mathfrak{O}'$ be order of type $(q_1, q_2, 2)$ and $\mathfrak{g}$ be an order of a quadratic subfield of which is optimally embedded in both $\mathfrak{O}$ and $\mathfrak{O}'$. Then there exists an ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{O}\mathfrak{a} = \mathfrak{a}\mathfrak{O}'$. Conversely, if $\mathfrak{g}$ is optimally embedded in $\mathfrak{O}$ and if there exists an $\mathfrak{g}$-ideal such that $\mathfrak{O}\mathfrak{a} = \mathfrak{a}\mathfrak{O}'$ then $\mathfrak{g}$ is also optimally embedded in $\mathfrak{O}'$.*

*Proof.* The second assertion holds trivially as it is contained in [2]. So we examine the local behaviour of orders to prove the first assertion. For $p = 2$, we may assume $\mathfrak{O}_2 = \mathfrak{O} \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Z}_2 = \begin{pmatrix} \boldsymbol{Z}_2 & \boldsymbol{Z}_2 \\ (4) & \boldsymbol{Z}_2 \end{pmatrix}$. Since $\mathfrak{O}'_2$ is isomorphic to $\mathfrak{O}_2$, there exists $\alpha \in A_2$ such that $\alpha^{-1}\mathfrak{O}_2\alpha = \mathfrak{O}'_2$. Under this situation we shall show that there exists $\beta \in \mathfrak{g} \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Z}_2$ such that $\mathfrak{O}_2\beta = \beta\mathfrak{O}'_2$. First, we assume $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 2^r \end{pmatrix} (r > 0)$. Put $\mathfrak{g}_2 = \boldsymbol{Z}_2[1, \omega]$, and fix $\omega$ to be $\omega = \begin{pmatrix} 0 & b \\ 4c & d \end{pmatrix} \in \mathfrak{O}_2$ after a suitable translation; since $\mathfrak{g}_2$ is embedded in $\mathfrak{O}_2$ optimally, we see $(b, c, d)_2 = 1$, where $(\ ,\ ,\ )_2$ denotes the $g{\cdot}c{\cdot}d{\cdot}$in $\boldsymbol{Z}_2$. $\mathfrak{g}_2$ is also optimally embedded in $\alpha^{-1}\mathfrak{O}_2\alpha = \mathfrak{O}'_2$, and $\alpha\omega\alpha^{-1} = \begin{pmatrix} 0 & 2^{-r}b \\ 2^{r+2}c & d \end{pmatrix}$, hence $(2^{-r}b, 2^r c, d) = 1$. For the proof of existence of $\beta \in \mathfrak{g}_2$ such that $\mathfrak{O}_2\beta = \beta\mathfrak{O}'_2$, we consider three cases.

*case* 1. $(d, 2) = 1$. Take $\beta \in \mathfrak{g}_2$ such that $\beta = 2^r - d + \omega = \begin{pmatrix} 2^r - d & b \\ 4c & 2^r \end{pmatrix}$. Then $\beta\alpha^{-1} = \begin{pmatrix} 2^r - d & b \\ 4c & 2^r \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{-r} \end{pmatrix} = \begin{pmatrix} 2^r - d & 2^{-r}b \\ 4c & 1 \end{pmatrix}$. Since $2^r - d$ is a unit in $\boldsymbol{Z}_2$, $\beta\alpha^{-1} = \varepsilon \in U_2$ hence $\mathfrak{O}_2\beta = \mathfrak{O}_2\varepsilon\alpha = \mathfrak{O}\alpha = \alpha\mathfrak{O}' = \beta\mathfrak{O}'$.

*case* 2. $(d, 4) = 2$. Take $\beta = 2^{r+2} - d + \omega \in \mathfrak{g}_2$, then $\beta\alpha^{-1} = \begin{pmatrix} 2^{r+2} - d & 2^{-r}b \\ 4c & 4 \end{pmatrix}$. Now put $\eta = \xi\pi = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}$, ($\eta$ : an element which generates a two-sided $\mathfrak{O}_2$-ideal by Lemma 1), then $\beta\alpha^{-1}\eta^{-1} = \begin{pmatrix} 2^{-r}b & 2^{-1}(2^{-1}u - 2^{-r}b) \\ 4 & c-2 \end{pmatrix}$ where $u = 2^{r+2} - d$. As $(d, 4) = 2$, $2^{-r}b$ and $c - 2$ are both units in $\boldsymbol{Z}_2$ and $2^{-1}(2^{-1}u - 2^{-r}b) \in \boldsymbol{Z}_2$. Therefore $\beta\alpha^{-1}\eta^{-1} = \varepsilon \in U_2$, $\mathfrak{O}_2\beta = \mathfrak{O}_2\eta\pi = \eta\mathfrak{O}_2\alpha = \eta\alpha\mathfrak{O}'_2 = \beta\mathfrak{O}'_2$.

*case* 3. $(d, 4) = 4$. Take $\beta = -d + \omega \in \mathfrak{g}_2$, $\beta = \begin{pmatrix} -d & b \\ 4c & 0 \end{pmatrix}$, and $\pi = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$; then $\beta\alpha^{-1}\pi^{-1} = \begin{pmatrix} 2^{-r}b & 4^{-1}d \\ 0 & c \end{pmatrix}$, in this case $2^{-r}b$, and $c$ are units in $\boldsymbol{Z}_2$, $\beta\alpha^{-1}\pi^{-1} = \varepsilon \in U_2$ hence $\mathfrak{O}_2\beta = \mathfrak{O}_2\pi\alpha = \pi\mathfrak{O}_2\alpha = \pi\alpha\mathfrak{O}'_2 = \beta\mathfrak{O}'_2$. Thus we have proved the existence of $\beta \in \mathfrak{g}_2$ such that $\mathfrak{O}_2\beta = \beta\mathfrak{O}'_2$ for the case $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 2^r \end{pmatrix}$. As for the second step, we shall show that if the above assertion is true for an $\alpha \in A_2$, then the assertion is true also for the following elements: (1) $\alpha_i$ : the left representative of $U_2\alpha U_2$, (2) $\alpha\gamma$ : here $\gamma$ a generator of a two-sided integral $\mathfrak{O}_2$-ideal (3) $\alpha^{-1}$. Because, for the type (1) by a suitable element $\varepsilon\in U_2$, $\alpha = \alpha_i\varepsilon$, and $\mathfrak{g}$ is optimally embedded in $\mathfrak{O}$ and in $\alpha_i{}^{-1}\mathfrak{O}\alpha_i$. Then $\varepsilon^{-1}\mathfrak{g}\varepsilon$ is optimally embedded in $\mathfrak{O}$ and in $\alpha^{-1}\mathfrak{O}\alpha = \mathfrak{O}'$. For the type (2), $\gamma\mathfrak{g}_2\gamma^{-1}$ is optimally embedded in $\gamma\mathfrak{O}\gamma^{-1} = \mathfrak{O}$ and in $\alpha^{-1}\mathfrak{O}\alpha = \mathfrak{O}'$. For the type (3), $\alpha^{-1}\mathfrak{g}_2\alpha$ is optimally embedded in $\alpha^{-1}\mathfrak{O}\alpha = \mathfrak{O}'$ and in $\mathfrak{O}$. Hence in any case there exists $\beta\in\mathfrak{g}_2$ such that $\mathfrak{O}_2\beta = \beta\mathfrak{O}_2'$. Let $\pi = \begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix}$, $\xi = \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$ be as in Lemma 1. Then for $\alpha = \begin{pmatrix}p^a & 0\\ 0 & p^b\end{pmatrix}$, our assertion is true by (1), this is also valid for the following elements and the left representatives of their double cosets with $U_2$ on account of (1) and (2). $\alpha\pi = \begin{pmatrix}0 & 2^a\\ 2^{b+2} & 0\end{pmatrix}$, $\alpha\xi = \begin{pmatrix}2^{a+1} & 2^a\\ 0 & 2^{b+1}\end{pmatrix}$, $\alpha\pi\xi = \begin{pmatrix}0 & 2^a\\ 2^{b+2} & 2^{b+1}\end{pmatrix}$, $\alpha\xi\pi\xi = \begin{pmatrix}2^{a+1} & 0\\ 2^{b+2} & 2^{b+1}\end{pmatrix}$. After all we only have to check for $\alpha = \begin{pmatrix}2 & 1+2^r\\ 4 & 2\end{pmatrix}$. According to the condition that $\mathfrak{g}_2 = \boldsymbol{Z}_2[1, \omega]$, with $\omega = \begin{pmatrix} & b\\ 4c & d\end{pmatrix}\in\mathfrak{O}_2$, $\mathfrak{g}_2$ is optimally embedded in $\mathfrak{O}_2$ and in $\alpha$, and we see easily that $d$ is a unit in $\boldsymbol{Z}_2$. Take $\beta = -d + \omega\in\mathfrak{g}_2$; then $-2^{r+2}\beta\alpha^{-1} = \begin{pmatrix}-2(d+2b) & 2b+(1+2^r)d\\ 8c & -4c(1+2^r)\end{pmatrix}$, hence there exists $\varepsilon\in U_2$ such that $-2^{r+2}\varepsilon\beta\alpha^{-1} = \begin{pmatrix}2 & f\\ 0 & 2^{r+1}\end{pmatrix}$ ($f$ : a unit in $\boldsymbol{Z}_2$). Since our assertions holds, for $\alpha' = \begin{pmatrix}2 & f\\ 0 & 2^{r+1}\end{pmatrix}$, it is easy to see that $\mathfrak{O}_2\beta = \beta\alpha^{-1}\mathfrak{O}_2\alpha$. This completes our that proof there exists $\beta\in\mathfrak{g}_2$ such that $\mathfrak{O}_2\beta = \beta\mathfrak{O}_2'$ for any $\alpha\in A_2$ which satisfies $\mathfrak{O}_2' = \alpha^{-1}\mathfrak{O}_2\alpha$. For other prime $p\neq 2$, it is proved in [2] that there exists $\beta_p\in\mathfrak{g}_p$ such that $\mathfrak{O}_p\beta_p = \beta_p\mathfrak{O}_p'$, and $\beta_p$ is a unit for almost all primes $p$. Hence the $\mathfrak{g}$-ideal $\mathfrak{a} = \cap\mathfrak{g}_p\beta_p$ serves our theorem with $\beta_2 = \beta$.

**2.4.** Let $\mathfrak{g}$ and $D$ be as in 2.2, and $\mathfrak{O}$ be of type $(q_1, q_2, 2)$, then the criterion for $\mathfrak{g}_2 = \mathfrak{O}_2\cap K_2$ is described as follows.

Lemma 2. *$\mathfrak{g}_2$ is optimally embedded in $\mathfrak{O}_2$ if and only if* $\left\{\dfrac{D}{2}\right\} = 1$.

*Proof.* Suppose $\mathfrak{g}_2 = \mathfrak{O}_2\cap K_2$, put $\mathfrak{O}_2 = \begin{pmatrix}\boldsymbol{Z}_2 & \boldsymbol{Z}_2\\ (4) & \boldsymbol{Z}_2\end{pmatrix}$, $\mathfrak{g}_2 = \boldsymbol{Z}_2[1, \omega]$, and $\omega = \begin{pmatrix}0 & b\\ 4c & d\end{pmatrix}\in\mathfrak{O}_2$. Then the discriminant of $\mathfrak{g}_2$ in $\boldsymbol{Z}_2$ is $d^2+16bc$. Hence if $(d, 2) = 1$, then $d^2\equiv 1 \pmod 8$, this implies $\left\{\dfrac{D}{2}\right\} = \left(\dfrac{d^2+16bc}{2}\right) = 1$, and if $(d, 2) = 2$, then $(d^2+16bc)/4 = (d/2)^2 + 4bc\in\boldsymbol{Z}_2$ and $\equiv 0, 1 \pmod 4$, therefore $\left\{\dfrac{D}{2}\right\} = 1$. Conversely, if $\left\{\dfrac{D}{2}\right\} = 1$, we can show that $\mathfrak{g}_2 = \boldsymbol{Z}_2[1, \omega]$ is optimally embedded in an order $\mathfrak{O}_2'$ which is isomorphic to $\mathfrak{O}_2 = \begin{pmatrix} \boldsymbol{Z}_2 & \boldsymbol{Z}_2 \\ (4) & \boldsymbol{Z}_2 \end{pmatrix}$. Put namely $\omega = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$; then the discriminant of $\mathfrak{g}_2$ is $d^2 + bc$. If $(d, 2) = 1$, then $d^2 \equiv 1 \pmod 8$, hence $\sqrt{D} \in \boldsymbol{Z}_2$ and $\omega$ satisfies $\left(\omega - \dfrac{d+\sqrt{D}}{2}\right)\left(\omega - \dfrac{d-\sqrt{D}}{2}\right) = 0$. Consider $\omega'' = \begin{pmatrix} 0 & 0 \\ 4c & \sqrt{D} \end{pmatrix} \in \mathfrak{O}_2$; then $\mathfrak{g}'' = \boldsymbol{Z}_2[1, \omega'']$ is embedded in $\mathfrak{O}_2$ optimally, and $\omega''$ and $\omega' = \omega - \dfrac{d+\sqrt{D}}{2}$ satisfy the same quadratic equation hence there exists $\alpha \in A_2$ such that $\alpha\omega'\alpha^{-1} = \omega''$. So, $\boldsymbol{Z}_2[1, \omega] = \mathfrak{g}$ is embedded in $\alpha^{-1}\mathfrak{O}_2\alpha$ optimally. In the case $(d, 2) = 2$, $D/4$ should be $\equiv 0, 1 \bmod 4$ hence $bc \equiv 0 \pmod 4$, or $\equiv 1 \pmod 4$. In the former case, take $b', c' \in \boldsymbol{Z}_2$ such that $b'$ is a unit and $bc = b'c'$. Then $\omega' = \begin{pmatrix} 0 & b' \\ c' & d \end{pmatrix}$ and $\omega$ satisfy the same equation and $\boldsymbol{Z}_2[1, \omega']$ is embedded optimally in $\mathfrak{O}_2$. In the latter case, $bc \equiv 1 \pmod 4$ implies $d \equiv 0 \pmod 4$. Put $\omega' = \begin{pmatrix} a' & b' \\ 4 & e' \end{pmatrix}$ and take $a', b', e'$ such that $\omega'$ and $\omega$ satisfy the same equation, namely, $a', e' = d/2 \pm \sqrt{(d/2)^2 + bc - 4b'}$. Then, since $(d/2)^2 + bc \equiv 1 \pmod 4$ we can take $b' \in \boldsymbol{Z}_2$ such that $(d/2)^2 + bc - 4b' \equiv 1 \pmod 8$, hence we see $a', e' \in \boldsymbol{Z}_2$. Therefore $\mathfrak{g} = \boldsymbol{Z}_2[1, \omega']$ is optimally embedded in $\mathfrak{O}_2$. This completes the proof of Lemma 2.

**2.5.** Let $G$ be the group of integral two-sided $\mathfrak{O}_2$ ideals modulo scalar matrix which is calculated in Lemma 1, and $\mathfrak{g}_2 = \mathfrak{O}_2 \cap K_2$ as in Lemma 2, and let $H(\mathfrak{g}_2)$ be the subgroup which is defined as follows $H(\mathfrak{g}_2) = \{\mathfrak{M} \in G \mid \mathfrak{M} = \mathfrak{O}_2\beta,\ \beta \in \mathfrak{g}_2\}$. Namely, $H(\mathfrak{g}_2)$ is the subgroup consists of all two sided ideals generated by $\mathfrak{g}$-ideals.

LEMMA 3. *Let $D$ be the discriminant of $\mathfrak{g}$ and define $\delta(D) = \delta(\mathfrak{g}_2) = [G : H(\mathfrak{g}_2)]$, then*

$$\delta(D) = \begin{cases} 2, & \text{if } D/4 \in \boldsymbol{Z} \text{ and } D/4 \equiv 5 \pmod 8, \\ 3, & \text{otherwise.} \end{cases}$$

*Proof.* Put $\mathfrak{O}_2 = \begin{pmatrix} \boldsymbol{Z}_2 & \boldsymbol{Z}_2 \\ (4) & \boldsymbol{Z}_2 \end{pmatrix}$, $\mathfrak{g}_2 = \boldsymbol{Z}_2[1, \omega]$, and $\omega = \begin{pmatrix} 0 & b \\ 4c & d \end{pmatrix}$. Then, by Lemma 1, $\delta(D) = 2$ if $H(\mathfrak{g}_2) = \{\iota, \xi\pi, \pi\xi\}$ and $\delta(D) = 3$, otherwise. Hence if $\delta(D) = 2$, $U_2\omega U_2 = U_2\pi\xi U_2$ or $U_2(\omega - d)U_2 = U_2\xi\pi U_2$, therefore $(b, 2) = (c, 2) = 1$ and $(d, 4) = 2$. As the discriminant of $\mathfrak{g}_2$ is $d^2 + 16bc$, we obtain $D/4 \in \boldsymbol{Z}$ and $D/4 \equiv 5 \bmod 8$ since $(d/2)^2 \equiv 1 \bmod 8$. Conversely, if $D/4 \equiv 5 \bmod 8$ it is easy to see $(d, 4) = 2$ and $(b, 2) = (c, 2) = 1$, hence $U_2\omega U_2 = U_2\pi\xi U_2$ therefore $\delta(D) = 2$. Thus we obtain our Lemma 3.

**2.6.** We remark the following two lemmas which are special cases of [6, § 3.10, § 3.11], and these lemmas are necessary to prove the theorem 2.

LEMMA 4. *The class number $h$ of an order of type $(q_1, q_2, q_3)$ is the same the class number of a maximal order in $A$. Hence if $A$ is indefinite, $h = 1$.*

LEMMA 5. *Let $\mathfrak{O}$ be as in Lemma 4 and $\mathfrak{M}$ an integral two-sided $\mathfrak{O}$-ideal. Let $b \in \boldsymbol{Z}$ and $\alpha \in \mathfrak{O}$ such that $N\alpha \equiv b \bmod^* (\mathfrak{M} \cap \boldsymbol{Z})$. Then there exists an element $\beta \in \mathfrak{O}$ such that $\beta \equiv \alpha \bmod \mathfrak{M}$ and $N\beta = b$. Here mod* means the multiplicative congruence.*

By Lemma 5 we note that $\mathfrak{O}$ contains an element of norm $-1$.

Now we assume $A$ is indefinite $\mathfrak{g}$ and is an order in an imaginary quadratic subfield $K$ of $A$ optimally embedded in $\mathfrak{O}$. Then for a unit $\varepsilon \in \mathfrak{O}$ $\varepsilon^{-1}\mathfrak{g}\varepsilon$ is also optimally embeeddd in $\mathfrak{O}$. Let us denote the set of orders $\{\varepsilon^{-1}\mathfrak{g}\varepsilon;$ norm $(\varepsilon) = 1\}$ by simply $(\mathfrak{g})$, and call it the proper classes of orders. Then we obtain

THEOREM 2. *The number of proper classes of orders $(\mathfrak{g})$ which is optimally embedded in an order $\mathfrak{O}$ of type $(q_1, q_2, 2)$ and is isomorphic to a given order $\mathfrak{g}_1$ in $K$ is equal to*

$$\frac{\delta(D_1)}{2}\left(1+\left\{\frac{D_1}{2}\right\}\right)\left\{\frac{D_1}{2}\right\}\prod_{p|q_1}\left(1-\left\{\frac{D_1}{p}\right\}\right)\prod_{q|q_1}\left(1+\left\{\frac{D_1}{p}\right\}\right)h(D_1)$$

*where $D_1$ denotes the discriminant of $\mathfrak{g}_1$, and $h(D_1)$ the class number of $\mathfrak{g}_1$-ideals, and $\delta(D_1)$ is defined in* LEMMA 3.

*Proof.* This theorem is proved by the same method as in [2, Satz 7] by virtue of Lemma 2 and 3. So we only sketch the proof. Namely, let $\mathfrak{g}$ be an order, isomorphic to $\mathfrak{g}_1$ and optimally embedded in $\mathfrak{O}$. Since the class number of $\mathfrak{O}$-ideals is 1 by lemma 4, there exists $\alpha \in A$ such that $\mathfrak{g} = \alpha\mathfrak{g}_1\alpha^{-1}$. Then $\mathfrak{g}$ is optimally embedded in $\mathfrak{O}$ and in $\alpha^{-1}\mathfrak{O}\alpha$, hence there exists $\mathfrak{g}_1$-ideal such that $\mathfrak{O}\mathfrak{a} = \mathfrak{a}\alpha^{-1}\mathfrak{O}\mathfrak{a}$. Therefore $\mathfrak{M} = \mathfrak{O}\mathfrak{a}\alpha^{-1}$ is a two-sided $\mathfrak{O}$-ideal. We make correspond to every pair of class of orders $((\mathfrak{g}), (\mathfrak{g}_1))$ the pair $((\mathfrak{M}), (\mathfrak{a}))$, where $(\mathfrak{M})$ is the class of $\mathfrak{M}$ the group of two sided $\mathfrak{O}$-ideals modulo two sided $\mathfrak{O}$-ideals which is generated by $\mathfrak{g}$-ideals, and $(\mathfrak{a})$ is the ideal class of $\mathfrak{a}$. This correspondence is one to one if and only if $\mathfrak{O}$ contains a unit with norm $-1$, and this is so our case by Lemma 5. Hence the classes of orders $(\mathfrak{g})$ which are optimally embedded in $\mathfrak{O}$ and isomorphic to $(\mathfrak{g}_1)$ is equal to the number of pairs $((\mathfrak{M}), (\mathfrak{a}))$. Combining lemma 2 and 3 with Eichler's result

https://doi.org/10.1017/S0027763000014410 Published online by Cambridge University Press

for the local behaviours of $\mathfrak{O}_p$ at $p|q_1q_2$, this number is given by

$$\frac{\delta(D_1)}{2}\left(1+\left\{\frac{D_1}{2}\right\}\right)\left\{\frac{D_1}{2}\right\}\cdot\prod_{p|q_1}\left(1-\left\{\frac{D_1}{p}\right\}\right)\cdot\prod_{p|q_2}\left(1+\left\{\frac{D_1}{p}\right\}\right)h(D_1).$$

This completes the proof.

## 3. The trace of Hecke operators for $\Gamma_0^{q_1}(4q_2)$

**3.1.** In this paragraph we assume that $A$ is indefinite. We regard $\Gamma_0^{q_1}(4q_2)$ as a subgroup of $SL_2(\boldsymbol{R})$ after a fixed isomorphism $A\underset{Q}{\otimes}\boldsymbol{R}\cong M_2(\boldsymbol{R})$, and we define a linear transformation $T(\Gamma\alpha\Gamma)$ in $S_k(\Gamma)$, where $S_k(\Gamma)$ is the complex vector space of cusp forms of weight $k$ with respect to the group $\Gamma=\Gamma_0^{q_1}(q_2)$. Let namely $\Gamma\alpha\Gamma=\bigcup_{\nu=1}^{d}\Gamma\alpha_\nu$ be a disjoint sum; then, for $f\in S_k(\Gamma)$ we set

$$(T(\Gamma\alpha\Gamma)f)(z)=(N\alpha)^{\frac{k}{2}}\sum_{\nu=1}^{d}j(\alpha_\nu,\ z)^{-k}f(\alpha_\nu(z))$$

where $\alpha_\nu(z)=\dfrac{a_\nu z+b_\nu}{c_\nu z+d_\nu}$, for $\alpha_\nu=\begin{pmatrix}a_\nu & b_\nu\\ c_\nu & d_\nu\end{pmatrix}$, $z\in H$ and $j(\alpha_\nu,\ z)=(c_\nu z+d_\nu)$.

We shall give the trace of $T(\Gamma\alpha\Gamma)$ following Shimizu's treatment [3] and Eichler's [1] in the representation space $S_k(\Gamma)$ fo $\Gamma=\Gamma_0^{q_1}(4q_2)$, in the case $n=N\alpha$ is prime to $4q_2$.

**3.2.** If $k$ is even and greater than 2, $trT(\Gamma\alpha\Gamma)$ is obtained in [4, Theorem 1], which is as follows:

$$trT(\Gamma\alpha\Gamma)=t_0+t_1+t_2+t_3,$$

$$t_0=\frac{k-1}{4\pi}\cdot\ \mathrm{vol}\ (\mathfrak{F}).\ \varepsilon(\sqrt{n})$$

$$t_1=-\sum_{\alpha_1\in C_1}\frac{1}{(\Gamma(\alpha_1):\{\pm 1\})}\cdot\frac{\rho_{\alpha_1}^{k-1}-\rho'^{k-1}_{\alpha_1}}{\rho_{\alpha_1}-\rho'_{\alpha_1}}\cdot N(\alpha_1)^{1-\frac{k}{2}},$$

$$t_2=-\sum_{\alpha_1\in C_2}\frac{2}{(\Gamma(\alpha_1):\{\pm 1\})}\cdot\frac{\mathrm{Min}\{|\rho_{\alpha_1}|,|\rho'_{\alpha_1}|\}^{k-1}}{|\rho_{\alpha_1}-\rho'_{\alpha_1}|}\cdot N(\alpha_1)^{1-\frac{k}{2}},$$

$$t_3=-\lim_{s\to 0}\frac{s}{2}\sum_{\alpha_1\in C_3}\left(\frac{d(\alpha_1)}{m(\alpha_1)}\right)^{1+s}$$

where $C_1$(resp. : $C_2$, $C_3$) is a complete system of inequivalent elliptic elements (resp. : hyperbolic elements leaving a parabolic point of $\Gamma$ fixed, parabolic elements) in $\Gamma\alpha\Gamma$ with respect to the equivalence relation

$$\alpha \sim \alpha' \Longleftrightarrow \alpha' = \pm \gamma \alpha \gamma^{-1} \text{ for } \gamma \in \Gamma.$$

$\Gamma(\alpha_1)$ is the group of all $\gamma \in \Gamma$ such that $\alpha_1 = \pm \gamma \alpha_1 \gamma^{-1}$, and $\rho_{\alpha_1}$, $\rho'_{\alpha_1}$ are characteristic roots of $\alpha_1$. Furthermore, $d(\alpha_1)$, $m(\alpha_1)$ are defined as follows; for the fixed point $x$ of $\alpha_1 \in C_3$ we can find $g \in GL_2(\boldsymbol{R})$ such that $gx = \infty$; then every element $\beta$ leaving $x$ fixed is written in the form $g\beta g^{-1}(z) = z \pm m\beta$ with a non negativ number $m(\beta)$, and $d(\alpha)$ is the least positive value of $m(\beta)$ when $\beta$ runs over $\Gamma(\alpha)$. Lastly, vol $(\mathfrak{F})$ denotes the volume of the fundamental domain for the group $\Gamma_0^{q_1}(4q_2)$, which is easily obtained by the group index relation; $[\Gamma_0^{q_1}(1) : \Gamma_0^{q_1}(4q_2)] = 6(q_2+1)$ and the volume of the fundamental domain for the group $\Gamma^{q_1}(1)$, namely

$$\text{vol } (\mathfrak{F}) = 2\pi \prod_{p \mid q_1} (p-1) \cdot \prod_{p \mid q_2} (p+1),$$

and $\varepsilon(\sqrt{n}) = 1$ or $0$ according as $\sqrt{n} \in \boldsymbol{Z}$ or not.

**3.3.** First we shall determine $C_1$. For an equivalence class $\alpha_1 \in C_1$, let $K_{\alpha_1}$ be the imaginary quadratic field generated by the eigen-value of $\alpha_1$ over $\boldsymbol{Q}$, and put $\mathfrak{g} = K_{\alpha_1} \cap \mathfrak{O}$. Then $\mathfrak{g}$ is an order of $K_{\alpha_1}$, which is optimally embedded in $\mathfrak{O}$. We know that there is an one to one correspondence between the equivalence classes $\{\alpha_1\}$ of $C_1$ and the proper classes of orders $(\mathfrak{g})$, which are optimally embedded in $\mathfrak{O}$ and contain an elliptic element with norm $n = N\alpha$. By virtue of theorem 2, we see

$$\begin{aligned}
&\sum_{\alpha_1 \in C_1} \frac{1}{[\Gamma(\alpha_1) : \{\pm 1\}]} \cdot \frac{\rho_{\alpha_1}^{k-1} - \rho'^{k-1}_{\alpha_1}}{\rho_{\alpha_1} - \rho'_{\alpha_1}} (N\alpha)^{1-\frac{k}{2}} \\
&= \sum{}' \frac{1}{2[E(\mathfrak{g}) : \{\pm 1\}]} h(D) \frac{\delta(D)}{2} \cdot \left(1 + \left\{\frac{D}{2}\right\}\right) \left\{\frac{D}{2}\right\} \prod_{p \mid q_1} \left(1 - \left\{\frac{D}{p}\right\}\right) \\
&\times \prod_{p \mid q_2} \left(1 + \left\{\frac{D}{p}\right\}\right) \cdot \frac{\rho^{k-1} - \rho'^{k-1}}{\rho - \rho'} (N\alpha)^{1-\frac{k}{2}},
\end{aligned}$$

where the sum $\sum'$ runs over all orders $\mathfrak{g}$ which contain an elliptic element $\nu$ with norm $N\alpha = n$, and $D$ is the discriminant of $\mathfrak{g}$, $\rho$, $\rho'$ are eigenvalues of $\nu$, and $E(\mathfrak{g})$ denotes the group of units in $\mathfrak{g}$. We remark that $[\Gamma(\alpha_1) : E(\mathfrak{g})] = 2$ since $\mathfrak{O}$ contains an element with norm $-1$ [see 3, 4.3].

Hence we obtain

$$\begin{aligned}
t_1 = \frac{1}{2} \sum\nolimits_0 &\frac{\delta(D)}{2} \cdot \left(1 + \left\{\frac{D}{2}\right\}\right) \left\{\frac{D}{2}\right\} \prod_{p \mid q_1} \left(1 + \left\{\frac{D}{p}\right\}\right) \\
&\times \prod_{p \mid q_2} \left(1 + \left\{\frac{D}{p}\right\}\right) \cdot \frac{\rho^{k-1} - \rho'^{k-1}}{\rho - \rho'} h(D) \cdot n^{1-\frac{k}{2}}
\end{aligned}$$

where $\sum_0$ runs over all $s$, $f$ with $|s| < 2\sqrt{n}$ and with $D = (s^2 - 4n)f^{-2} \equiv 0, 1$ mod 4 $(f > 0)$, and $\rho$, $\rho'$ are the roots of the equation $x^2 - sx + n = 0$.

**3.4.** $t_2$, $t_3$ appear only if $A = M_2(\boldsymbol{Q})$. In this case, if $r$ runs through all divisors of $4q_2$ other than itself, then the est of all $r^{-1}$, together with $\infty$ forms a complete system of $\Gamma$-inequivalent parabolic points. Let $C_{2\infty}$ (resp. $C_{3\infty}$) be an equivalent class in $C_2$ (resp. $C_3$) which fixes the point $\infty$. Let $r$ be a divisor of $4q_2$ and put $\sigma_r = \begin{pmatrix} r & b \\ 4q_2 & rd \end{pmatrix}$ or $\begin{pmatrix} r & 0 \\ 4q_2 & r \end{pmatrix}$ according as $(r, 4q_2r^{-1}) = 1$ or not, where $rd - 4q_2r^{-1}b = 1$. Then we see $C_\lambda = \cup \sigma C_{\lambda\infty}\sigma_r^{-1}$ $(\lambda = 2, 3)$. By [3, Lemma 4.2, 4.3], we can take for $C_{2\infty}$ the set of all $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$, $0 < a < d$, $0 \leq b \leq \dfrac{d-a}{2}$. In this case $[\Gamma(\alpha) : \{\pm 1\}] = 2$ or $1$ according as $2b \equiv 0 \bmod (a-d)$ or not. We note that $t_3$ appears only if $n = N\alpha$ is a square integer; in this case we can take as $C_{3\infty}$ the set of $\alpha_1$ all such that $\alpha_1 = \begin{pmatrix} \sqrt{n} & b \\ 0 & \sqrt{n} \end{pmatrix}$ $b > 0$, $b \in \boldsymbol{Z}$. Furthermore $d(\alpha_1) \cdot m(\alpha_1)^{-1} = b^{-1}\sqrt{n}$ for all $\alpha_1$. Hence we obtain

$$t_2 = -3 \cdot 2^t \cdot n^{1-\frac{k}{2}} \cdot \sum_{\substack{ad=n \\ 0<a<\sqrt{n} \\ 0 \leq b < d-a}} \frac{a^{k-1}}{d-a} = -3 \cdot 2^t \cdot n^{1-\frac{k}{2}} \sum_{\substack{a|n \\ 0<a<\sqrt{n}}} a^{k-1},$$

and if $\sqrt{n} \in \boldsymbol{Z}$,

$$t_3 = -3 \cdot 2^t \cdot \lim_{s \to 0} \frac{s}{2} \sum_{b>0} \left(\frac{\sqrt{n}}{b}\right)^{1+s} = -3 \cdot 2^{t-1} \cdot \sqrt{n},$$

where $t$ denotes the number of prime factors of $q_2$.

**3.5.** If $k = 2$, regarding $T(\Gamma\alpha\Gamma)$ as a modular correspondence of the Riemann surface $\mathfrak{R} = \mathfrak{F} \cup \{\text{cusps}\}$, $T(\Gamma\alpha\Gamma)$ induces an endomorphism of the $i$-th Betti group $B^i(\mathfrak{R})$ of $\mathfrak{R}$ $(i = 0, 1, 2)$. Then the trace of the representation of $\Gamma\alpha\Gamma$ by the Betti group of $\mathfrak{R}$ is $tr\, T(\Gamma\alpha\Gamma) = tr^0 T(\Gamma\alpha\Gamma) - tr^1 T(\Gamma\alpha\Gamma) + tr^2 T(\Gamma\alpha\Gamma)$, where $tr^i T(\Gamma\alpha\Gamma)$ is the trace of the endomorphism induced by $T(\Gamma\alpha\Gamma)$ on $B^i(\mathfrak{R})$. We see $tr^0 T(\Gamma\alpha\Gamma) = tr^2 T(\Gamma\alpha\Gamma) =$ number of left representatives of $\Gamma\alpha\Gamma$, and $tr^1 T(\Gamma\alpha\Gamma)$ is calculated by the same method as in owing to the explicit determination of $C_1$, $C_2$, $C_3$ given in 3.3, 3.4. We thus find for $n = N\alpha$ $((n, 4q_2) = 1)$,

$$tr^1 T(\Gamma\alpha\Gamma) = \sum\nolimits_0 \frac{\delta(D)}{2}\left(1 + \left\{\frac{D}{2}\right\}\right)\left\{\frac{D}{2}\right\} \prod_{p|q_1}\left(1 - \left\{\frac{D}{p}\right\}\right) \prod_{p|q_2}\left(\frac{D}{p}\right)\Big\} h(D)$$
$$-\varepsilon(\sqrt{n}) \cdot 2 \cdot \mathrm{vol}\,(\mathfrak{F})$$

$$+\alpha(q_1)\cdot 3\cdot 2^{t+1} \sum_{\substack{d|n \\ 0<d\leqq\sqrt{n}}}' d,$$

where $\sum_0$ is the same as in 3.3, $\varepsilon(\sqrt{n}) = 1$ or $0$ according as $\sqrt{n}\in\boldsymbol{Z}$ or not, $\alpha(q_1) = 1$ or $0$ according as $q_1 = 1$ or not, and $t$ is defined in **3.4.** Since we consider the trace in the space $S_2(\Gamma)$ or in other words, in the space of differential forms of the first kind on $\mathfrak{R}$, the trace which is obtained by the above method should be multiplied by $\frac{1}{2}$ with the reason in [1, p. 156]. Hence, summing up we obtain

THEOREM 3. *Assume $A$ is indefinite and $S_k(\Gamma_0^{q_1}(4q_2))$ denotes the space of cusp forms of weight $k$ with respect to $\Gamma_0^{q_1}(4q_2)$. Then the trace $tr(T_n)$ $((n, 4q_2) = 1$ of Hecke operator acting on $S_k(\Gamma_0^{q_1}(4q_2))$ is given as follows*

$$tr(T_n) = d_k - \frac{1}{2}\sum_0 \frac{\delta(D)}{2}\Big(1+\Big\{\frac{D}{2}\Big\}\Big)\Big\{\frac{D}{2}\Big\}\prod_{p|q_1}\Big(1-\Big\{\frac{D}{p}\Big\}\Big)\prod_{p|q_2}\Big(1+\Big\{\frac{D}{p}\Big\}\Big)$$

$$\times \frac{\rho^{k-1}-\rho'^{k-1}}{\rho-\rho'}\cdot n^{1-\frac{k}{2}}\cdot h(D) + \varepsilon(\sqrt{n})\cdot\frac{1}{2}\cdot\prod_{p|q_1}(p-1)\cdot\prod_{p|q_2}(p+1)$$

$$-\alpha(q_1)\cdot 3\cdot 2^t\cdot n^{1-\frac{k}{2}}\cdot \sum_{\substack{d|n \\ 0<d\leqq\sqrt{n}}}' d^{k-1}.$$

*where*

$$d_k = \begin{cases} \sum_{d|n} d, & \text{if } k=2, \\ 0, & \text{if } k>2, \end{cases} \qquad \varepsilon(\sqrt{n}) = \begin{cases} 1, & \text{if } \sqrt{n}\in\boldsymbol{Z}, \\ 0, & \text{if } \sqrt{n}\notin\boldsymbol{Z}, \end{cases}$$

$$\alpha(q_1) = \begin{cases} 1, & \text{if } q_1 = 1, \\ 0, & \text{if } q_1 > 1, \end{cases}$$

*the sum $\sum_0$ runs over all $s$, $f$ with $|s| < 2\sqrt{n}$, $f>0$ and $D=(s^2-4p)f^{-2}\equiv 0, 1$ (mod 4), and $\rho$, $\rho'$ are the roots of the equation $x^2-sx+n=0$. Furthermore, $\delta(D) = 2$ or $3$ accoding as $D/4\equiv 5$ (mod 8) or not, $h(D)$ is the class number of an order with discriminant $D$. $\sum'$ denotes the sum with a multiplicity $1/2$ for $d=\sqrt{n}$, cnd $t$ the number of prime factors of $q_2$.*

**3.6.** In this section we consider the elliptic modular group $\Gamma_0(4N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z})\,|\, c\equiv 0 \bmod (4N)\right\}$ where $N=\prod_{i=1}^{t} N_i$ is a product of distinct odd prime $N_i (1\leq i\leq t)$. Let $\chi_i$ be a character of the multiplicative group $(\boldsymbol{Z}/$

$N_iZ)^\times$ and put $\chi=\prod_{i=1}^{t}\chi_i$ then $\chi$ is a character of $(Z/NZ)^\times$ in a natural way, and we suppose $\chi$ is not a trivial character. We denote by $S_k(\Gamma_0(4N), \chi)$ the complex vector space of modular cusp forms satisfying

$$f\left(\frac{az+b}{cz+d}\right)=\chi(d)(cz+d)^k f(z) \qquad \text{for every } \begin{pmatrix} a & c \\ b & d\end{pmatrix}\in\Gamma_0(4N).$$

By an obvious reason we assume $\chi(-1)=(-1)^k$. The Hecke operators $T_n^\chi$ $((n, 4N)=1)$ acting on the space $S_k((\Gamma_0(4N), \chi)$ is defined by

$$(T_n^\chi\cdot f)(z)=n^{\frac{k}{2}}\sum_{\substack{ad=n\\ d>0\\ 0\leqq b<d}}\chi(a)f\left(\frac{az+b}{d}\right)d^{-k}$$

The trace $tr(T_n^\chi)$ in the representation space $S_k(\Gamma_0(4N), \chi)$ is calculated by the same method discussed in the preceeding sections combining with Shimizu's arguments [4] and we easily find the following

THEOREM 3'. *The trace $tr(T_n^\chi)$ ($n$ is prime to $4N$) in the representation space $S_k(\Gamma_0(4N), \chi)$ is given as follows*

$$tr(T_n^\chi)=-\frac{1}{2}\sum{}_0\frac{\delta(D)}{2}\cdot\left(1+\left\{\frac{D}{2}\right\}\right)\left\{\frac{D}{2}\right\}\prod_{p|q_1}\left(1-\left\{\frac{D}{p}\right\}\right)\prod_{p|q_2}\left(1+\left\{\frac{D}{p}\right\}\right)$$

$$\times\frac{\rho^{k-1}-\rho'^{k-1}}{\rho-\rho'}\cdot n^{1-\frac{k}{2}}h(D)\cdot\chi(s, n)+\varepsilon(\sqrt{n})\cdot\frac{1}{2}\prod_{p|q_1}(p-1)\cdot\prod_{p|q_2}(p+1)\cdot\chi(\sqrt{n})$$

$$-3\cdot n^{1-\frac{k}{2}}\cdot\sum_{\substack{d|n\\ 0<d\leqq\sqrt{n}}}{}' d^{k-1}\cdot\prod_{i=1}^{t}\left(\chi_i(d)+\chi_i\left(\frac{n}{d}\right)\right),$$

*where $\chi(s, n)$ is defined by*

$$\chi(s, n)=2^{-t}\prod_{i=1}\sum_{\alpha^2-s\alpha+n\equiv 0 \bmod (N_i)}\chi_i(\alpha),$$

*and other notataions are the same as in* Theorem 3.

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