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BLOWING-UP AND WHITNEY (a)-REGULARITY

BY

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ABSTRACT. When a pair of adjacent C^1 strata (X, Y) is Whitney (a)-regular at a point $0 \in Y \cap \bar{X}$, there is, up to homeomorphism, just one germ at 0 of intersections $S \cap X$ where S is a Lipschitz transversal to Y at 0 (S is the graph of a Lipschitz map from $(T_0Y)^\perp$ to T_0Y, C^1 off 0). The proof uses a blowing-up construction and a strong form of Verdier's (w)-regularity.

The first author showed in 1978 ([2]) that whenever a pair of adjacent C^∞ strata (X, Y) is Whitney (a)-regular at a point $0 \in Y \cap \bar{X}$, there is, up to homeomorphism, just one germ at 0 of intersections $S \cap X$ where S is a C^∞ transversal to Y, $0 \in S$. In 1985 the second author ([3]) proved the corresponding C^1 statement: X, Y and the transversals may be of class C^1. Here we further improve the conclusion by including all Lipschitz transversals (defined as graphs of Lipschitz maps from $Y^\perp$ to Y, C^1 off 0). The proofs use a natural "blowing up" construction involving the map which defines a given transversal, a strong form of Verdier's (w)-regularity, and the Thom-Mather-Verdier isotopy theorem ([4]).

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Let $(x_1, \ldots, x_k, y_1, \ldots, y_d)$ be coordinates for $\mathbf{R}^k \times \mathbf{R}^d$. Let $\varphi : (\mathbf{R}^k, 0) \longrightarrow (\mathbf{R}^d, 0)$ be a continuous map, of class C^1 on $\mathbf{R}^k - \{0\}$, such that $\varphi_i^{-1}(0) = 0$, $1 \leqq i \leqq d$. Define

$$\pi_\varphi : \mathbf{R}^k \times \mathbf{R}^d \longrightarrow \mathbf{R}^k \times \mathbf{R}^d \text{ by } (x, u) \longmapsto (x, u_1\varphi_1(x), \ldots, u_d\varphi_d(x)),$$

which is of class C^1 on $\mathbf{R}^k \times \mathbf{R}^d - (0 \times \mathbf{R}^d)$. There is an inverse β_φ of class C^1 defined for $x \neq 0$, by

$$\beta_\varphi(x, y) = (x, y_1\varphi_1(x)^{-1}, \ldots, y_d\varphi_d(x)^{-1}).$$

We shall call π_φ the φ-blowing-up of $\mathbf{R}^k \times \mathbf{R}^d$.

EXAMPLE 1. Let $\varphi(x) = x$. Then $\pi_\varphi : \mathbf{R} \times \mathbf{R}^d \longrightarrow \mathbf{R} \times \mathbf{R}^d$ coincides with the blowing-up in the usual sense.

EXAMPLE 2. Let $k = 2$, $d = 1$, $\varphi(x_1, x_2) = (x_1^2 + x_2^2)^{1/2}$. Then $\pi_\varphi(x_1, x_2, y) = (x_1, x_2, \sqrt{x_1^2 + x_2^2}y)$ is not a blowing-up in the usual sense.

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Recall that a C^1 submanifold M in $\mathbf{R}^n$ is Whitney (a)-regular over a C^1 submanifold N at $q \in \bar{M} \cap N$ if $d(T_qN, T_pM) \to 0$ as $p \in M$ tends to q. Here, for linear subspaces A, B, letting π_B stands for the orthogonal projection onto B,

$$d(A,B) = \sup \|a - \pi_B(a)\|, a \in A, \|a\| = 1.$$

Also recall that (M, N) is Verdier (w)-regular at $q_0 \in N$ if there exists a constant $C > 0$ and a neighbourhood U of q_0 such that

$$d(T_qN, T_pM) < C\|p - q\|, \text{ for all } p \in U \cap M, q \in U \cap N.$$

DEFINITION. *The pair* $(\mathbf{M}, \mathbf{N})$ *is differentiably regular at* q_0 *if for all* $\epsilon > 0$ *there exists a neighbourhood* U *of* q_0 *such that*

$$d(T_qN, T_pM) < \epsilon\|p - q\|, \textit{ for all } p \in U \cap M, q \in U \cap N.$$

Of course, a differentiably regular pair is (w)-regular.

THEOREM 1. *Let* $X \subset \mathbf{R}^k \times \mathbf{R}^d - (0 \times \mathbf{R}^d)$ *be a* C^1 *submanifold,* $0 \in \bar{X}$, *such that* $(X, 0 \times \mathbf{R}^d)$ *is Whitney* (a)*-regular at 0. Let* $\varphi : (U, 0) \to (\mathbf{R}^d, 0)$ *be a Lipschitz map, defined on a neighbourhood* U *of* 0 *in* $\mathbf{R}^k$, *of class* C^1 *on* $U - \{0\}$, *and such that* $\varphi_i^{-1}(0) = 0$, $1 \leqq i \leqq d$. *Then* $(\beta_\varphi(X), 0 \times \mathbf{R}^d)$ *is differentiably regular (at each point of* $0 \times \mathbf{R}^d$*).*

LEMMA. *Let* $\varphi : (\mathbf{R}^k, 0) \to (\mathbf{R}^d, 0)$ *be continuous and of class* C^1 *on* $\mathbf{R}^k - \{0\}$. *Then* φ *is Lipschitz (at* 0*) if and only if* $\|\varphi(x)\| \, \|x\|^{-1}$ *and* $\|\partial\varphi_i/\partial x_j(x)\|$, $1 \leqq i \leqq d$, $1 \leqq j \leqq k$, *are bounded for* x *in some neighbourhood of* 0 *in* $\mathbf{R}^k$.

The "if" part is a version of the Mean Value Theorem as in the account of [1]; "only if" follows from the definition of derivative.

PROOF OF THEOREM 1. The entries of the Jacobian matrix, B, of β_φ are given by

$$b_{ij} = \delta_{ij}, \text{ if } 1 \leqq i,j \leqq k; b_{ij} = 0 \text{ if } 1 \leqq i \leqq k, k+1 \leqq j \leqq k+d;$$

$$b_{k+i,j} = -y_i \frac{\partial\varphi_i}{\partial x_j}\varphi_i(x)^{-2} \text{ if } 1 \leqq i \leqq d, 1 \leqq j \leqq k; b_{k+i,k+j} = \delta_{ij}\varphi_i(x)^{-1}, \text{ if } 1 \leqq i,j \leqq d.$$

By the Lemma, $\|\varphi(x)\| \, \|x\|^{-1}$ and $\|\partial\varphi_i/\partial x_j\|$, $1 \leqq i \leqq d$, $1 \leqq j \leqq k$, are bounded by a constant M in a neighbourhood U of 0 in $\mathbf{R}^k$. By Whitney (a)-regularity of $(X, 0 \times \mathbf{R}^d)$ at 0, for each point (x, y) in the intersection of X and some neighbourhood of 0, the tangent space to X contains d vectors of the form

$$w_i = \frac{\partial}{\partial y_i} + \sum_{j=1}^{k} e_{ij}(x,y)\left[\frac{\partial}{\partial x_j} + \sum_{s \neq i} \frac{y_s}{\varphi_s}\frac{\partial\varphi_s}{\partial x_j}\frac{\partial}{\partial y_s}\right], 1 \leqq i \leqq d,$$

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where $\|(e_{i1},\ldots,e_{ik})\| \longrightarrow 0$ as $(x,y) \longrightarrow (0,0)$.

Now, take a fixed $u_0 \in \mathbf{R}^d$. A quick calculation shows that

$$d\beta_\varphi(w_i) = \sum_{j=1}^{k} e_{ij}\frac{\partial}{\partial x_j} + \frac{1}{\lambda_i(x,y)}\frac{\partial}{\partial y_i}$$

where

$$\lambda_i(x,y)^{-1} = -\varphi_i(x)^{-2}y_i\sum_{j=1}^{k} e_{ij}\frac{\partial\varphi_i}{\partial x_j} + \varphi_i(x)^{-1}.$$

Then

$$d\beta_\varphi(\lambda_i w_i) - \frac{\partial}{\partial y_i} = \lambda_i\sum_{j=1}^{k} e_{ij}\frac{\partial}{\partial x_j}.$$

To prove Theorem 1 it remains to show that whenever (x,y) and $(y_1\varphi_1(x)^{-1},\ldots, y_d\varphi_d(x)^{-1})$ are sufficiently close to $(0,0)$ and u_0 respectively, we have

$$|\lambda_i(x,y)| < 2M\|x\|.$$

But

$$|\lambda_i(x,y)|\|x\|^{-1} = |\varphi_i(x)|\|x\|^{-1}\left[1 - \varphi_i(x)^{-1}y_i\sum_{j=1}^{k} e_{ij}\frac{\partial\varphi_i}{\partial x_j}\right]$$

and

$$|\varphi_i(x)|\,\|x\|^{-1} < M, \left|\frac{\partial\varphi_i}{\partial x_i}\right| < M, \text{ for } x \in U.$$

Moreover, $|\varphi_i(x)^{-1}y_i|$ is close to $|(u_0)_i|$. Hence Theorem 1 follows.

NOTATION. Write X_φ for the germ at 0 of $X \cap \{(x,\varphi(x)) : x \in \mathbf{R}^k\}$, and X_0 for the germ at 0 of $X \cap \{(x,0) : x \in \mathbf{R}^k\}$, for maps $\varphi : (\mathbf{R}^k,0) \longrightarrow (\mathbf{R}^d,0)$.

COROLLARY. *Let $X \subset \mathbf{R}^{k+d} - (0 \times \mathbf{R}^d)$ be a C^1 submanifold such that $0 \times \mathbf{R}^d = \bar{X} - X$ near 0, and $(X, 0 \times \mathbf{R}^d)$ is (a)-regular at 0. Let $\varphi : (\mathbf{R}^k,0) \longrightarrow (\mathbf{R}^d,0)$ be C^1 off 0 and Lipschitz near 0 and such that for each i, $1 \leqq i \leqq d$, $\varphi_i^{-1}(0) = 0$. Then X_0 and X_φ are homeomorphic.*

PROOF. By Theorem 1, $(\beta_\varphi(X), 0 \times \mathbf{R}^d)$ is differentiably regular. After applying a C^1 diffeomorphism to $\mathbf{R}^{k+d}$, preserving $\mathbf{R}^k \times 0$ and tangent to the identity at $0 \times \mathbf{R}^d$, we may assume that X is a C^∞ submanifold (use the techniques of Smoothing Theorem 3.2 in [3]). Then $\beta_\varphi(X)$ will also be of class C^∞ and we may apply Verdier's isotopy theorem ([4]) to show that the germ at $(0,0)$ of $\beta_\varphi(X) \cap (\mathbf{R}^k \times 0)$ is homeomorphic with the germ at $(0,1,\ldots,1)$ of $\beta_\varphi(X) \cap (\mathbf{R}^k \times (1,1,\ldots,1))$. But these germs are respectively homeomorphic with $X \cap (\mathbf{R}^k \times 0) = X_0$ and $X \cap \{(x,\varphi(x))\} = X_\varphi$ by applying π_φ. This proves the Corollary.

DEFINITION. *The graph of a map* $\varphi : (\mathbf{R}^k, 0) \longrightarrow (\mathbf{R}^d, 0)$ *which is* C^1 *on* $\mathbf{R}^k - \{0\}$ *and Lipschitz near* 0 *will be called a* Lipschitz transversal *to* $0 \times \mathbf{R}^d$ *at* 0. *We say the pair* $(X, 0 \times \mathbf{R}^d)$ *has homeomorphic Lipschitz transversals at* 0, *and write* (h^{Lip}), *when for each pair of Lipschitz transversals to* $0 \times \mathbf{R}^d$ *at* 0, *defined by maps* φ *and* ψ, *the germs* X_φ *and* X_ψ *are homeomorphic.*

THEOREM 2. *Let* $X \subset \mathbf{R}^{k+d} - 0 \times \mathbf{R}^d$ *be a* C^1 *submanifold such that* $0 \times \mathbf{R}^d = \bar{X} - X$ *near* 0, *and* $(X, 0 \times \mathbf{R}^d)$ *is* (a)*-regular at* 0. *Then* $(X, 0 \times \mathbf{R}^d)$ *verifies* (h^{Lip}) *at* 0.

PROOF. Let $\varphi_1 : (\mathbf{R}^k, 0) \longrightarrow (\mathbf{R}^d, 0)$ be of class C^1 on $\mathbf{R}^k - \{0\}$ and Lipschitz near 0. Then there exists $\kappa > 0$ such that

$$\|\varphi_1(x)\| < \kappa\|x\| \text{ for all } x \neq 0.$$

Let $\psi_\kappa(x, y) = (x, y - \kappa\|x\|)$. Then $\psi_\kappa : (\mathbf{R}^{k+d}, 0) \longrightarrow (\mathbf{R}^{k+d}, 0)$ is a bilipschitz homeomorphism. It is easy to check that, when X is (a)-regular over $0 \times \mathbf{R}^d$ at 0, $\psi_\kappa(X)$ is also (a)-regular over $0 \times \mathbf{R}^d$ at 0, by calculating the effect of $d\psi_\kappa$ on vectors $\Sigma\alpha_i(x, y)(\partial/\partial x_i) + \Sigma\beta_j(x, y)(\partial/\partial y_j)$ on $T_{(x,y)}\mathbf{R}^{k+d}$ such that $\lim \|\alpha\|\,\|\beta\|^{-1} = 0$, as $(x, y) \longrightarrow (0, 0)$. Now the map φ defined by

$$\varphi(x) = \varphi_1(x) - \kappa\|x\|$$

is C^1 off 0 and Lipschitz with constant 2κ, and moreover the graph of φ is disjoint from $\mathbf{R}^k \times 0$. Thus φ satisfies the hypothesis of the Corollary, and the germs $(\psi_\kappa(X))_\varphi$ and $(\psi_\kappa(X))_0$ are homeomorphic.

Let $\omega_\kappa(x) = (\kappa\|x\|, \dots, \kappa\|x\|) \in \mathbf{R}^d$. Then $X_{\omega_\kappa} = \psi_\kappa^{-1}((\psi_\kappa(X))_0)$, so that $(\psi_\kappa(X))_0$ and X_{ω_κ} are homeomorphic.

Apply the Corollary to ω_κ. Then the germs X_{ω_κ} and X_0 are homeomorphic. Now use the fact that $\psi_\kappa(X_{\varphi_1}) = (\psi_\kappa(X))_\varphi$ to deduce that the germs X_{φ_1} and $(\psi_\kappa(X))_\varphi$ are homeomorphic. We obtain finally that X_{φ_1} and X_0 are homeomorphic, so proving Theorem 2.

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