Proceedings of the Edinburgh Mathematical Society (1981) **24**, 217–219 ©

LIE ALGEBRAS ALL OF WHOSE MAXIMAL SUBALGEBRAS HAVE CODIMENSION ONE

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(Received 28th March 1980)

Let $\mathfrak{X}$ denote the class of finite-dimensional Lie algebras L (over a fixed, but arbitrary, field F) all of whose maximal subalgebras have codimension 1 in L. In (**2**) Barnes proved that the solvable algebras in $\mathfrak{X}$ are precisely the supersolvable ones. The purpose of this paper is to extend this result and to give a characterisation of all of the algebras in $\mathfrak{X}$. Throughout we shall place no restrictions on the underlying field of the Lie algebra.

Precisely, the result we shall prove is

Theorem 1. *The Lie algebra $L \in \mathfrak{X}$ if and only if $L/\phi(L) = S \oplus R$, where $\phi(L)$ is the Frattini ideal of L, S is a 3-dimensional simple ideal of $L/\phi(L)$ isomorphic to $L_1(0)$ (see below), or is $\{0\}$, and R is a supersolvable ideal of $L/\phi(L)$ (possibly $\{0\}$).*

If U is a subalgebra of L we denote by U_L (the *core* of U) the largest ideal of L contained in U; if $U_L = 0$ we say that U is *core-free.* We shall need the following classification of Lie algebras with core-free subalgebras of codimension 1 which is given by Amayo in (**1**).

Theorem 2. (Amayo (**1**), Theorem 3.1). *Let L have a core-free subalgebra of codimension* 1. *Then either* (i) $\dim L \leqq 2$, *or else* (ii) $L \cong L_m(\Gamma)$ *for some m and Γ satisfying certain conditions (see* (**1**) *for details).*

We shall also need the following properties of $L_m(\Gamma)$ which are given in (**1**).

Theorem 3. (Amayo (**1**)). (i) *If $m > 1$ and m is odd, then $L_m(\Gamma)$ has only one subalgebra of codimension* 1.

(ii) *If $m > 1$ and m is even, then $L_m(\Gamma)$ has precisely two subalgebras of codimension* 1.

(iii) *$L_1(\Gamma)$ has a basis $\{u_{-1}, u_0, u_1\}$ with multiplication $u_{-1}u_0 = u_{-1} + \gamma_0 u_1$ ($\gamma_0 \in F$,* $\gamma_0 = 0$ if $\Gamma = \{0\}$), *$u_{-1}u_1 = u_0$, $u_0 u_1 = u_1$.*

(iv) *If F has characteristic different from* 2 *then $L_1(\Gamma) \cong L_1(0)$.*

(v) *If F has characteristic* 2 *then $L_1(\Gamma) \cong L_1(0)$ if and only if γ_0 is a square in F.*

Using the above we can deduce

Lemma 4. *Let $L \in \mathfrak{X}$ and suppose that M is a maximal subalgebra of L. Then either* (i) dim $L/M_L \leqq 2$, *or else* (ii) $L/M_L \cong L_1(0)$.

Proof. Clearly M/M_L is a core-free subalgebra of L/M_L of codimension 1. Suppose that dim $L/M_L > 2$. Then, by Theorem 2, $\bar{L} = L/M_L \cong L_m(\Gamma)$. Furthermore, it follows easily from Theorem 3(i) and (ii) that $m = 1$. Suppose that $\bar{L} \not\cong L_1(0)$. Theorem 3(iii), (iv) and (v) implies that F has characteristic 2 and $\bar{L}$ has a basis $\{u_{-1}, u_0, u_1\}$ with $u_{-1}u_0 = u_{-1} + \gamma_0 u_1$, $u_{-1}u_1 = u_0$, $u_0 u_1 = u_1$, where γ_0 is not a square in F. But a simple calculation how verifies that the subalgebra spanned by u_{-1} is maximal, contradicting the fact that $L \in \mathfrak{X}$. The result follows.

One more lemma is needed; namely

Lemma 5. *Suppose that $L = S_1 \oplus S_2$ where S_1 and S_2 are 3-dimensional simple ideals of L, each isomorphic to $L_1(0)$. Then $L \notin \mathfrak{X}$.*

Proof. Pick a basis $u_{i0}, u_{i1}, u_{i(-1)}$ for S_i $(i = 1, 2)$ such that $u_{i(-1)}u_{i0} = u_{i(-1)}$, $u_{i0}u_{i1} = u_{i1}$, $u_{i(-1)}u_{i1} = u_{i0}$. It is easily checked that the subalgebra of L spanned by $u_{10} + u_{20}$, $u_{11} + u_{21}$, $u_{1(-1)} + u_{2(-1)}$ is maximal.

Proof of theorem 1. (a) Suppose first that $L \in \mathfrak{X}$. If L is solvable, it is supersolvable ((**2**), Theorem 7), so suppose further that L is not solvable. Factor out $\phi(L)$, so we may assume that $\phi(L) = 0$. There is a maximal subalgebra M of L such that L/M_L is not solvable (since otherwise $(L^2)^2 = L^{(3)} \subset \phi(L) = 0$ and L is solvable). By Lemma 4, $L/M_L \cong L_1(0)$.

Let K be any maximal subalgebra of L and suppose that $M_L \not\subset K$. Then $L = M_L + K$. Put $B = M_L + K_L$. Since L/M_L is simple, $B = M_L$ or $B = L$. The former implies that $M_L = K_L \subset K$, a contradiction; so $L = B = M_L + K_L$. Now $L/(M_L \cap K_L) \cong (M_L/(M_L \cap K_L)) \oplus (K_L/(M_L \cap K_L)) \cong (L/K_L) \oplus (L/M_L)$, so $L/K_L \not\cong L_1(0)$ (by Lemma 5). Hence dim $L/K_L \leqq 2$, and so $L^{(3)} \subset K_L \subset K$. We have proved that either $M_L \subset K$ or else $L^{(3)} \subset K$. Thus, $M_L \cap L^{(3)} \subset K$ for all maximal subalgebras K of L. It follows that $M_L \cap L^{(3)} \subset \phi(L) = 0$; in particular, $M_L^{(3)} = 0$ and M_L is solvable.

If $M_L = 0$ we are done. If $M_L \neq 0$, $M_L \not\subset \phi(L)$, and so there is a maximal subalgebra N of L such that $L = M_L + N$. As above, $L = M_L + N_L$. Put $D = N_L^{(3)} + M_L$. Then $N_L^{(3)} \cong D/M_L$ which is an ideal of L/M_L, and so $N_L^{(3)} = 0$ or else $N_L^{(3)} \cong L_1(0)$. The former is impossible since this would imply that L were solvable. Hence $D = L$ and $L = S + R$, where $R = M_L$ is solvable, $S = N_L^{(3)} \cong L_1(0)$ and $L^{(3)} \cap R = 0$. Furthermore, $SR = S^2R \subset S(SR) = S^2(SR) \subset L^{(3)} \cap R = 0$, giving $L = S \oplus R$. Finally, R is supersolvable by Theorem 7 of (**2**).

(b) Now suppose that $\bar{L} = L/\phi(L) = S \oplus R$. By Theorem 7.3 of (**3**), $\bar{L} = (A + B) \oplus S$ where $A = A_1 \oplus \ldots \oplus A_n$ is the sum of the minimal abelian ideals of $\bar{L}$ and B is abelian. Since $R = A + B$ is supersolvable, dim $A_i = 1$ for $1 \leqq i \leqq n$.

Let M be a maximal subalgebra of $\bar{L}$. If $A \not\subset M$, then there is an A_i $(1 \leqq i \leqq n)$ such that $A_i \not\subset M$. But then $\bar{L} = A_i + M$ and M has codimension 1 in $\bar{L}$. So assume that $A \subset M$.

Suppose that $B \not\subset M$. Then there is an element $b \in B$ such that $b \notin M$. But $B\bar{L} \subset A \subset M$, so $L = M + U$ where U is spanned by b. Thus, again, M has codimension 1 in $\bar{L}$.

Finally, if $R \subset M$, it is clear that M has codimension 1 in $\bar{L}$.

Remark. It is clear from the proof of Theorem 1 that for any $L \in \mathfrak{X}$ we can pick a basis $\{a_1, \ldots, a_n, b_1, \ldots, b_m, u_{-1}, u_0, u_1\}$ such that

$$u_{-1}u_0 = u_{-1}, \quad u_0u_1 = u_1, \quad u_{-1}u_1 = u_0$$

and

$$a_ib_j = \lambda_{ij}a_i \quad \text{for some} \quad \lambda_{ij} \in F \quad (1 \leqq i \leqq n, 1 \leqq j \leqq m),$$

all other products being zero.

Acknowledgement. The author would like to thank the University of California at Berkeley for their hospitality while this work was being done.

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