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Homogeneity of Certain Invariant Distributions on the Lie Algebra of p-adic GL_n

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Abstract. Let F be a non-Archimedean local field with ring of integers R and prime ideal $\wp$. Suppose T is a $\mathrm{GL}_n(F)$-invariant distribution on $\mathfrak{g} = \mathrm{M}_n(F)$, the Lie algebra of $\mathrm{GL}_n(F)$. If T has support in the set of topologically nilpotent elements, then the restriction of T to the set of functions which are compactly supported and invariant under $\mathrm{M}_n(\wp)$ may be expressed as a linear combination of nilpotent orbital integrals restricted to the same set of functions.

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1. Introduction

Let F be a non-Archimedean local field with ring of integers R and prime ideal $\wp = \varpi R$. Suppose that $R/\wp \cong \mathbb{F}_q$, the finite field with q elements. Let $\mathrm{M}_n(F)$ denote the set of $n \times n$ matrices with entries in F. We let G be $\mathrm{GL}_n(F)$ realized as the set of elements in $\mathrm{M}_n(F)$ having nonzero determinant. We denote by $\mathfrak{g}$ the Lie algebra of G, and we will take $\mathfrak{g}$ to be $\mathrm{M}_n(F)$ with the usual bracket operation.

If ω is a compact set in $\mathfrak{g}$, then we will denote by $J(\omega)$ the set of G-invariant distributions on $\mathfrak{g}$ with support in the closure of ${}^G\omega = \{\mathrm{Ad}(g)X \mid g \in G, X \in \omega\}$. If T is a distribution in $J(\omega)$ and $\mathcal{L}$ is a lattice in $\mathfrak{g}$, then $j_{\mathcal{L}}T$ will denote the restriction of T to $C_c(\mathfrak{g}/\mathcal{L})$, the set of complex-valued, compactly supported, $\mathcal{L}$-invariant functions on $\mathfrak{g}$. It was first conjectured by Howe [5] that

$$\dim j_{\mathcal{L}} J(\omega) < \infty. \tag{1}$$

This was proved in [4] and extended to arbitrary reductive groups defined over F of characteristic zero in [3, Theorem 14.1] (see also [12]).

In [10] Waldspurger proves a more precise version of (1) for many groups with a few restrictions on F. Let $\mathfrak{k}_0 = \mathrm{M}_n(R)$ and let $\mathfrak{b}_0 = \{X \in \mathfrak{k}_0 \mid X_{ij} \in \wp \text{ if } i > j\}$. Finally, let $\mathcal{N}$ denote the set of nilpotent elements in $\mathfrak{g}$ and let $J(\mathcal{N})$ denote the set of G-invariant distributions on $\mathfrak{g}$ with support in $\mathcal{N}$. As a consequence of Waldspurger's remarkable work in [10], we have

$$j_{\mathfrak{b}_0} J(\mathfrak{k}_0) = j_{\mathfrak{b}_0} J(\mathcal{N}). \tag{2}$$

In this paper, we prove a variation of Equation (2). Namely, if $\mathfrak{k}_m = \varpi^m \cdot \mathfrak{k}_0$ for an integer m and $\mathfrak{b}_1 = \{X \in \mathfrak{k}_0 \mid X_{ij} \in \wp \text{ if i} \geqslant \text{j}\}$, then

THEOREM 1. $j_{\mathfrak{k}_1} J(\mathfrak{b}_1) = j_{\mathfrak{k}_1} J(\mathcal{N})$.

Both Equation (2) and Theorem 1, when coupled with work of Murnaghan [8] (see also [2, Theorem 3.3.2] and [1]), verify, for certain irreducible admissible representations of G, a conjecture of Hales, Moy, and Prasad [7] about where the Harish-Chandra–Howe local expansion ought to hold. In fact, in [9] Waldspurger is able to use the results of [10] to establish the Hales–Moy–Prasad conjecture for all integral depth representations of a large class of groups.

2. A Homogeneity Result

We will need some additional notation. Let $P_\emptyset \leqslant G$ denote the standard Borel subgroup, i.e., the set of all invertible upper triangular matrices. Let $N_\emptyset$ denote the unipotent radical of $P_\emptyset$ and let $\mathfrak{n}_\emptyset$ denote its Lie algebra. Finally, $A_\emptyset$ will denote the maximal split torus in G consisting of diagonal matrices. So, $P_\emptyset = A_\emptyset N_\emptyset$.

Suppose $1 \leqslant k < n$. Let P_k be the proper parabolic subgroup of G containing $P_\emptyset$ and having a Levi decomposition $M_k N_k$ where $M_k \cong \mathrm{GL}_k(F) \times F^\times \times F^\times \times \cdots \times F^\times$ is embedded in G in the obvious way. Let $\mathfrak{p}_k = \mathfrak{m}_k + \mathfrak{n}_k$ be the corresponding Lie algebras.

Let $K_0 = \mathfrak{k}_0^\times$ denote the standard maximal compact open subgroup of G. If $X \in \mathfrak{g}$ and $\mathcal{L}$ is a lattice in $\mathfrak{g}$, then let $[X + \mathcal{L}]$ denote the characteristic function of the coset $X + \mathcal{L}$. Let $\mathcal{O}(0)$ denote the set of nilpotent orbits in $\mathfrak{g}$ and $|\mathcal{O}(0)|$ its cardinality. Finally, for $g \in G$ and $X \in \mathfrak{g}$, let ${}^gX$ denote $\mathrm{Ad}(g)X$. We begin with a variation of [6, Lemma 2.4].

LEMMA 2. ${}^G\mathfrak{b}_1 \subset \mathfrak{k}_1 + \mathcal{N}$.

Proof. This follows from the affine Bruhat decomposition of G with respect to the standard Iwahori subgroup. It is also an easy consequence of [2, Lemma 1.6.1] (see also [2, Corollary 4.3.5]) which uses the formalism of Moy and Prasad [7]. □

LEMMA 3. $\dim\big(J(\mathfrak{b}_1)|_{C(\mathfrak{k}_0/\mathfrak{k}_1)}\big) \leqslant |\mathcal{O}(0)|$.

Proof. Fix $D \in J(\mathfrak{b}_1)$. It follows from Lemma 2 that $D|_{C(\mathfrak{k}_0/\mathfrak{k}_1)}$ is determined by its values on the functions $[n + \mathfrak{k}_1]$ with $n \in \mathcal{N} \cap \mathfrak{k}_0$. Of course, $D([n + \mathfrak{k}_1]) = D([{}^kn + \mathfrak{k}_1])$ for $n \in \mathcal{N} \cap \mathfrak{k}_0$ and $k \in K_0$. Therefore, the dimension of $J(\mathfrak{b}_1)|_{C(\mathfrak{k}_0/\mathfrak{k}_1)}$ is less than or equal to the number of nilpotent $\mathrm{GL}_n(\mathbb{F}_q)$-orbits in $\mathrm{M}_n(\mathbb{F}_q)$. But the latter number is $|\mathcal{O}(0)|$. □

LEMMA 4. *Fix a distribution $D \in J(\mathfrak{b}_1)$, a negative integer j, and $X \in \mathcal{N} \cap (\mathfrak{k}_j \setminus \mathfrak{k}_{j+1})$. If $D|_{C(\mathfrak{k}_{j+1}/\mathfrak{k}_1)} = 0$, then $D([X + \mathfrak{k}_1]) = 0$.*

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Fix j, D, and X as in the statement of Lemma 4. Before we begin the proof of this lemma, we need some additional notation and a simple result.

If $W \in \mathfrak{k}_0$, then $\overline{W}$ denotes the image of W in $\mathrm{M}_n(\mathbb{F}_q) = \mathfrak{k}_0/\mathfrak{k}_1$. If $W \in \mathfrak{k}_j$, then we define $\overline{\mathrm{rank}}(W) = \mathrm{rank}_{\mathbb{F}_q}(\overline{\varpi^{-j}W}|_{\mathbb{F}_q^n})$. Let $m = \overline{\mathrm{rank}}(X)$. Note that, by hypothesis, $0 < m < n$. An element $Y \in \mathfrak{g}$ will be called *good* if

(1) $Y \in \mathfrak{k}_j \cap \mathcal{N}$, $\overline{\varpi^{-j}Y} \in \overline{\mathfrak{n}_\emptyset \cap \mathfrak{k}_0}$, and $\overline{\mathrm{rank}}(Y) = m$,
(2) there exists a set $S_Y \subset \{2, 3, \ldots n\}$ with cardinality m such that if $k \notin S_Y$, then the kth column of $\overline{\varpi^{-j}Y}$ is zero, and
(3) if $\delta(Y)$ denotes the greatest element of the set S_Y, then there exists $k < \delta(Y)$ such that $Y \in \mathfrak{p}_k + \mathfrak{k}_1$.

LEMMA 5. *Suppose that $Y \in \mathfrak{k}_j \cap \mathcal{N}$, $\overline{\varpi^{-j}Y} \in \overline{\mathfrak{n}_\emptyset \cap \mathfrak{k}_0}$, and $\overline{\mathrm{rank}}(Y) = m$. Let d be the greatest integer such that the* dth *row of $\overline{\varpi^{-j}Y}$ is nonzero. If $Y \in \mathfrak{p}_d + \mathfrak{k}_1$, then there exists a $u \in K_0 \cap N_\emptyset$ such that ${}^uY$ is good and $\delta({}^uY) > d$.*

Proof. For $\alpha \in R$ and $s < t$, we will let $e_{st}(\alpha) \in N_\emptyset \cap K_0$ denote the matrix

$$\left(e_{st}(\alpha)\right)_{cd} = \begin{cases} 1 & \text{if } c = d, \\ \alpha & \text{if } c = s \text{ and } d = t, \\ 0 & \text{otherwise.} \end{cases}$$

Note that conjugating a matrix in $\mathfrak{g}$ by $e_{st}(\alpha)$

(1) adds α times the tth row to the sth row, and
(2) adds $-\alpha$ times the sth column to the tth column.

Since $\overline{\mathrm{rank}}(Y) = m$, the linear span of the columns of $\overline{\varpi^{-j}Y}$ has dimension m. Therefore, by conjugating Y by elements of the form $e_{st}(\alpha)$ with $s < t$ and $\alpha \in R$, we can obtain an element with the desired properties. □

Proof of Lemma 4. We begin with a warning about notation. Since $\mathfrak{k}_1$ is K_0-invariant, we have $D([X + \mathfrak{k}_1]) = D([{}^kX + \mathfrak{k}_1])$ for all $k \in K_0$. Therefore, we will often ignore $\mathfrak{k}_1$ when conjugating by elements of K_0 and deal only with X.

Since G has an Iwasawa decomposition ($G = P_\emptyset K_0$), we may assume that $X \in \mathfrak{n}_\emptyset$. Since $X \in \mathfrak{n}_\emptyset$, we may assume that X is good from Lemma 5. Note that in all that follows, we use only the fact that X is good.

The proof is by induction on $m (= \overline{\mathrm{rank}}(X))$. Here is the plan. We will produce a finite collection of $X_i \in \mathcal{N} \cap \mathfrak{k}_j$ such that

$$D([X + \mathfrak{k}_1]) = \sum_i c_i \cdot D([X_i + \mathfrak{k}_1])$$

for constants $c_i \in \mathbb{Q}$ and either

(1) $\overline{\mathrm{rank}}(X_i) < m$ for all i or

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(2) X_i is good and $\delta(X_i) > \delta(X)$ for all i.

At the end of this proof, it will be clear that if $\delta(X) = n$, then the first outcome must occur. Therefore, repeating the steps below a finite number of times will produce a finite collection of $X_i \in \mathcal{N} \cap \mathfrak{k}_j$ and $c_i \in \mathbb{Q}$ such that

$$D([X + \mathfrak{k}_1]) = \sum_i c_i \cdot D([X_i + \mathfrak{k}_1])$$

and $\overline{\mathrm{rank}}(X_i) < m$ for all i. The lemma follows.

Step I. We have that

$$\ker(\overline{\varpi^{-j}X}|_{\mathbb{F}_q^n}) = \{(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{F}_q^n \mid \xi_\beta = 0 \text{ if } \beta \in \mathrm{S_X}\}.$$

Let L be the lift of this kernel in R^n, that is,

$$L = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n | \alpha_\beta \in \wp \text{ if } \beta \in \mathrm{S_X}\}.$$

Let

$$\mathfrak{C} = \{Z \in \mathfrak{k}_0 \mid Z \cdot L \subset \varpi \cdot R^n = (\varpi \cdot R)^n \text{ and } Z \cdot R^n \subset L\}.$$

If $\mathfrak{c} \in \mathfrak{C}$, then

$$\mathfrak{c}_{rs} \in \begin{cases} R & \text{if } r \notin S_X \text{ and } s \in S_X, \text{ and} \\ \wp & \text{otherwise.} \end{cases}$$

From [10, Lemme II.4.2] we have the following lemma.

LEMMA 6. *Choose* $\mathfrak{c} \in \mathfrak{C}$. *There exists* $Z \in \varpi^{-j}\mathfrak{k}_0$ *such that*

$$^{(1+Z)}X \equiv X + \mathfrak{c} \bmod \mathfrak{k}_1.$$

Proof (Waldspurger). Fix $\mathfrak{c} \in \mathfrak{C}$. Let $X' = \overline{\varpi^{-j}X}$ and $\mathfrak{c}' = \bar{\mathfrak{c}}$. Let $r_0 = \ker(X')$. Then $\mathfrak{c}' \cdot r_0 = \{0\}$ and $\mathfrak{c}' \cdot \mathbb{F}_q^n \subset r_0$. Therefore, there exists a $Z' \in \mathrm{M}_n(\mathbb{F}_q)$ such that $\mathfrak{c}' = Z' \cdot X'$ and $\mathrm{im}\,(Z') = \mathrm{im}\,(\mathfrak{c}')$. Choose $Z \in \mathfrak{k}_{-j}$ so that $Z' = \overline{\varpi^j Z}$. The lemma follows. □

From this lemma it follows that

$$D([X + \mathfrak{k}_1]) = \mathrm{const} \cdot D([X + \mathfrak{C}]).$$

Step II. Let $a \in G$ be the diagonal matrix

$$a_{rs} = \begin{cases} 1 & \text{if } r = s \notin S_X, \\ \varpi^{-1} & \text{if } r = s \in S_X, \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

Then we write ${}^{a}(X+\mathfrak{C})$ as a finite disjoint union $\bigcup_{\alpha}(X'+\alpha+\mathfrak{k}_1)$, where $X'\in\mathfrak{k}_j$ with

$$X'_{cd}=\begin{cases}\varpi X_{cd} & \text{if } c\notin S_X \text{ and } d\in S_X,\\ \varpi^{-1}X_{cd} & \text{if } c\in S_X \text{ and } d\notin S_X, \text{ and}\\ X_{cd} & \text{otherwise,}\end{cases}$$

and $\alpha\in\mathfrak{k}_0$ with $\alpha_{cd}\in\wp$ unless $c\in S_X$ and $d\notin S_X$.

We now have

$$\begin{aligned}D([X+\mathfrak{k}_1])&=\text{const}\cdot D([X+\mathfrak{C}])\\&=\text{const}\cdot\sum_{\alpha}D([X'+\alpha+\mathfrak{k}_1])\end{aligned}$$

where the sum is over those α as above such that $X'+\alpha\in\mathfrak{k}_1+\mathcal{N}$ (because the support of D is contained in $\mathfrak{k}_1+\mathcal{N}$). If no such α exist, then $D([X+\mathfrak{k}_1])=0$.

Step III. Note that $\overline{\text{rank}}(X')=\overline{\text{rank}}(X'+\alpha)$ and $\overline{\text{rank}}(X')\leqslant\overline{\text{rank}}(X)=m$ because $\overline{\varpi^{-j}X'}$ has at most m rows with nonzero entries. (If the ith row of $\overline{\varpi^{-j}X'}$ has nonzero entries, then $i\in S_X$.) If $\overline{\text{rank}}(X')<m$, then we are done. If $\delta(X)=n$, then $\overline{\text{rank}}(X')<m$ since the bottom row of $\overline{\varpi^{-j}X'}$ has no nonzero entries.

Otherwise, let us assume that $\overline{\text{rank}}(X')=m$. This implies that $\delta(X)<n$. Fix an α as in step (II) for which $X'+\alpha\in\mathcal{N}+\mathfrak{k}_1$. Let $W=X'+\alpha$. Since $\overline{\text{rank}}(X')=m$, the matrix $\overline{\varpi^{-j}W}$ has nonzero entries in the $\delta(X)$th row. Since X was good, we have $W\in\mathfrak{p}_{\delta(X)}+\mathfrak{k}_1$ and $\overline{\varpi^{-j}W}\in\mathfrak{p}_{(\delta(X)-1)}(\mathbb{F}_q)$. Thus, since $\overline{\varpi^{-j}W}$ is nilpotent, there exists a $g\in M_{(\delta(X)-1)}(R)$ such that the element $\overline{\varpi^{-j}(\text{Ad}(g)W)}$ of $\text{M}_n(\mathbb{F}_q)$ lives in $\overline{\mathfrak{n}_{\emptyset}\cap\mathfrak{k}_0}$ and has nonzero entries in its $\delta(X)$th row. We will write W for ${}^{g}W$, and since we are only concerned with W modulo $\mathfrak{k}_1$, we will assume that $W\in\mathcal{N}$. We have

(1) $W\in\mathfrak{k}_j\cap\mathcal{N}$, $\overline{\varpi^{-j}W}\in\overline{\mathfrak{n}_{\emptyset}\cap\mathfrak{k}_0}$, and $\overline{\text{rank}}(W)=m$,
(2) the $\delta(X)$th row of $\overline{\varpi^{-j}W}$ is nonzero, and
(3) $W\in\mathfrak{p}_{\delta(X)}+\mathfrak{k}_1$

From Lemma 5 there exists a $u\in K_0\cap N_{\emptyset}$ such that ${}^{u}W$ is good and $\delta({}^{u}W)>\delta(X)$. □

Proof of Theorem 1. Lemmas 2 and 4 imply that

$$\dim\big(j_{\mathfrak{k}_1}J(\mathfrak{b}_1)\big)=\dim\big(J(\mathfrak{b}_1)|_{C(\mathfrak{k}_0/\mathfrak{k}_1)}\big).$$

Since $j_{\mathfrak{k}_1}J(\mathcal{N})\subset j_{\mathfrak{k}_1}J(\mathfrak{b}_1)$, the theorem follows from Lemma 3. □

We conclude with a corollary about Shalika germ expansions of certain orbital integrals. Similar results on G are extremely difficult to obtain (see [11]). We adopt the notation of [3, §8].

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COROLLARY 7. *For all $f \in C_c(\mathfrak{g}/\mathfrak{k}_1)$ and all regular, semisimple, topologically nilpotent elements H of $\mathfrak{g}$ we have*

$$\phi_f(H) = \sum_{\mathcal{O}\in\mathcal{O}(0)} \mu_{\mathcal{O}}(f) \cdot \Gamma_{\mathcal{O}}(H).$$

Proof. This follows from Theorem 1 and the proof of [3, Lemma 8.2]. □

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