A CONTRIBUTION TO CHRONOGEOMETRY

A. D. ALEXANDROV

To H. S. M. Coxeter on his sixtieth birthday

This paper deals with systems of pairwise equal and parallel cones in the affine n-space E^n, i.e. cones of a system are obtained one from another by means of translations. This subject is closely connected with the geometrical interpretation of the theory of relativity, so one may say that it belongs to "elementary chronogeometry." The term "chronogeometry," which is due, it seems, to A. D. Fokker, means the relativistic theory of space–time. From the most general viewpoint, chronogeometry may be defined as a theory of spaces where "a geometry" is determined by a relation of precedence of points. Every point X of such a space is the vertex of a "cone" C_X of points preceded by X. We speak of "elementary chronogeometry" if these "cones" are ordinary cones in an affine space.

1. Transformations preserving a system of cones.

1.1. Let C be a cone in the affine n-space E^n, i.e. a point set consisting of rays issuing from a given point, the vertex of C. Suppose that C has the following properties: (1) C does not lie in a plane, (2) its closed convex hull HC does not contain straight lines, (3) HC is not a Descartes product of a ray and an $(n-1)$-dimensional cone. The latter condition implies that $n > 2$.

THEOREM 1. *Let each point $X \in E^n$ be the vertex of the cone C_X equal and parallel to a given C with the above properties. Then every one-to-one mapping f of E^n onto itself that transforms every C_X into $C_{f(X)}$ is linear.*

Remark. It can be easily seen that the statement of our theorem becomes false as soon as one of the conditions (1)–(3) on C is omitted. However, if C is not convex, conditions (2) and (3) may be replaced by weaker ones; we shall discuss this below.

COROLLARY. *The mappings preserving the system of elliptic cones are Lorentz transformations.*

The essence of this statement is that even continuity of mappings is not assumed. By our theorem the mappings are linear and, therefore, according to a well-known result, they prove to be Lorentz transformations. (This corollary was proved in **(2)**.)

Received May 29, 1967.

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

1.2. As a preliminary to the proof of our theorem, we shall prove a few lemmas. In these lemmas C, C_X, f denote the cones and the mappings involved in our theorem, but the condition (3) on C may be omitted.

LEMMA 1. *The mapping f preserves the system of the convex hulls of the cones C_X.*

Proof. Put $C_X = C_X{}^0$ and define for each X and each integer $i > 0$ the set

$$C_X{}^i = \bigcup_{Y \in C_X{}^{i-1}} C_Y{}^{i-1}$$

the sum of all $C_Y{}^{i-1}$ with $Y \in C_X{}^{i-1}$.

If A, $B \in C_X{}^{i-1}$ and Y lies on the segment XA, the ray from Y parallel to XB crosses the segment AB. Therefore, if $A, B \in C_X{}^{i-1}$, the segment $AB \subset C_X{}^i$. Hence it follows that

$$C_X{}^\infty = \bigcup_{i=0}^{\infty} C_X{}^i$$

is the convex hull of C_X (not necessarily closed).

With this construction, the mapping f that preserves the system of the cones C_X preserves the system of $C_X{}^\infty$ as well.

1.3. LEMMA 2. *The mapping f is continuous.*

Proof. By Lemma 1 we can replace the cones C_X by their convex hulls. Thus we suppose that the C_X are convex. We exclude the point X from C_X.

Define for each X the set $C_X{}^-$ of all points Y for which $C_Y \supset C_X$:

$$C_X{}^- = \{Y\colon C_Y \supset C_X\}.$$

The set $C_X{}^-$ has the following properties:

(1) $C_X{}^-$ is a cone containing the cone symmetric to C_X with respect to X and contained in the closure of this cone.

(2) If P is a supporting plane of the closure $\bar{C}_X$ of C_X such that

$$P \cap \bar{C}_X \neq (X),$$

then $P \cap C_X{}^- \neq (X)$ (C_X, $C_X{}^-$ being cones, these conditions mean that P has common rays with both $\bar{C}_X$, $C_X{}^-$).

The first property is evident because of the convexity of C_X. In order to verify the second, we cross C_X by a plane Q such that $D_X = C_X \cap Q$ is bounded; D_X is a convex set. The section $D_Y = C_Y \cap Q$ is a set homotetic to D_X with respect to the point Z in which Q crosses the line XY. If $Y \in \partial C_X{}^-$, then $Z \in \partial D_X$.

If P is a supporting plane of $\bar{C}_X$ and $\bar{C}_X \cap P \neq (X)$, the $(n-2)$-plane $R = P \cap Q$ is a supporting plane of $\bar{D}_X = \bar{C}_X \cap Q$.

It follows that property (2) is equivalent to the following: For every supporting plane R of $\bar{D}_X$ the set $R \cap \bar{D}_X$ contains at least one point Z such that the set obtained from D_X by a homotetic enlargement with centre Z contains D_X.

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

This is evident, for if $D_X \cap R$ is empty, every $Z \in \bar{D}_X \cap R$ satisfies this condition, and otherwise we take $Z \in D_X \cap R$.

Now we construct the cone $C_X^=$ which is determined by C_Y^- in the same way as C_X^- is determined by C_Y. By the definition of the cones C_X^- and $C_X^=$, the mapping f preserves the system of these cones.

We replace C_X by $C_X^=$ and denote it by C_X; the vertex X is excluded from $C_X^=$.

Consider the sets $N_{XY} = C_X \cap C_Y^-$.

If Y lies within C_X, the set N_{XY} intersects every N_{XZ} with $Z \in C_X$. But if $Y \in \partial C_X$, this is not so, for C_X has faces (its intersections with supporting planes) which have no common points (X being excluded).

Thus the inner points of C_X differ from the boundary ones by a property determined by the sets N_{XY}. But the mapping f transforms these sets into the same sets; therefore it preserves the interior of the cones C_X.

If a given point A is the inner point of the cones C_X, C_Y^-, the intersection $C_X \cap C_Y^-$ is a neighbourhood of A. Therefore f preserves neighbourhoods, and Lemma 2 is proved.

1.4. Lemmas 1 and 2 imply that the mapping f preserves the system of the closed convex hulls HC_X of the cones C_X.

LEMMA 3. *The mapping f transforms the tangent planes of the cones HC_X into tangent planes, i.e. P being a tangent plane of HC_X, $f(P)$ is one of $HC_{f(X)}$.*

Proof. Replace the cones C_X by HC_X and denote HC_X by C_X. Take a point X and a point $Y \in \partial C_X$, $Y \neq X$. Define the set

$$T_{XY} = \cup C_Z \qquad (X \in C_Z,\ Y \in \partial C_Z),$$

the sum of all C_Z with $X \in C_Z$, $Y \in \partial C_Z$.

The set T_{XY} consists of the rays drawn from Y through all points belonging to C_X. This statement becomes evident if we draw through Y a plane Q such that $C_X \cap Q$ is bounded. Then the sections $C_Z \cap Q$ of the cones C_Z with $X \in C_Z$, $Y \in \partial C_Z$ are homotetic to $C_X \cap Q$ with respect to Y, and

$$C_Z \cap Q \supset C_X \cap Q.$$

Therefore the sum $\cup(C_Z \cap Q)$ of all these sections contains all rays drawn from Y through the points of $C_X \cap Q$. Correspondingly, $\cup C_Z$ consists of rays drawn from Y through the points of C_X.

If the closure $\bar{T}_{XY}$ of T_{XY} is a half-space, ∂T_{XY} is the tangent plane to C_X at Y.

In general, $\bar{T}_{XY}$ is a convex cone with the vertex Y and contains the line XY. Let R_{XY} be the maximal plane through Y which is contained in $\bar{T}_{XY}$. For a pair of points X', Y', $T_{X'Y'} = T_{XY}$ if and only if X', $Y' \in R_{XY}$. Thus R_{XY} is the set of all X' for which there exist Y' such that $T_{X'Y'} = T_{XY}$.

Owing to the given representation of the sets T_{XY}, the mapping f transforms them into the same sets. Therefore, it transforms the planes R_{XY} into the same

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

planes. Being continuous, it preserves their dimensionality. Hence, in particular, it transforms tangent planes $R_{XY} = \partial T_{XY}$ into tangent planes.

1.5. *The proof of Theorem* 1. From the conditions (1)–(3) imposed on the cone C its closed convex hull HC has n tangent planes P_i which bound an n-faced solid angle V; HC has more tangent planes Q, and no edge of V lies in all Q (because of the condition (3)). Take an edge e of V and a Q which does not pass through e. Let e' be another edge which does not lie in Q. By Lemma 3 the mapping f transforms the planes P_i, Q into planes and the planes parallel to them remain parallel. Therefore the 2-plane $R = (e, e')$ is transformed into a 2-plane R' and the lines parallel to e, e' and $d = R \cap Q$ remain, respectively, parallel. Hence it follows that the mapping f of R onto R' is linear. Thus f is linear on all edges e of V and on all lines parallel to them, i.e. f is linear.

2. "Spaces" and "times" determined by a system of cones.

2.1. Let C be a cone in E^n consisting of rays passing from a given point O through all points of a bounded closed domain in an $(n - 1)$-plane which does not contain O. We consider the system of cones C_X homotetic (equal and parallel) to C with the vertices X at all points $X \in E^n$. C_X^- denotes the cone symmetric to C_X with respect to X.

The system $\{C_X\}$ of the cones C_X determines in E^n a "geometry." In the study of it we follow an analogy with the relativistic theory of space–time. The surfaces ∂C_X of the cones C_X are analogous to the light cones.

Definition 1. By a *space* (with respect to $\{C_X\}$) we understand the set of all straight lines $\mathfrak{L}$ parallel to a ray passing from an X within C_X. It is an $(n - 1)$-dimensional affine space, the lines $\mathfrak{L}$ being its points.

This definition corresponds to the physical concept of space determined with respect to a given inertial system. The system is connected with a particle whose motion is depicted by the line $\mathfrak{L}_0$. A point in the space is determined by a line of events—the inertial motion of a particle which has zero velocity in the given system of reference. Such lines $\mathfrak{L}$ are parallel to $\mathfrak{L}_0$.

Definition 2. Let A, B be two points, B lying within the cone C_A. The line $\mathfrak{L}_0 = AB$ determines the space R consisting of lines $\mathfrak{L} \| \mathfrak{L}_0$. Let $S(AB)$ denote the set of all $\mathfrak{L} \in R$ which intersect the set $C_A \cap C_B^-$. We say that $S(AB)$ is a sphere in R around the point $\mathfrak{L}_0 = AB$. According to this definition the centre of the sphere is its centre of symmetry, and all spheres in the same space are homotetic to each other.

The boundary of $S(AB)$ consists of lines crossing $\partial C_A \cap \partial C_B^-$. ∂C_A is the cone depicting the propagation of light from the event A, and ∂C_B^- depicts the propagation of light to the event B. Physically speaking, the light emitted from A is reflected at the points $\mathfrak{L} \in \partial S(AB)$ and is focused at B. Thus our definition has the following physical meaning. The distances from a given point $\mathfrak{L}_0$ to other points $\mathfrak{L}$ are equal if the light emitted from $\mathfrak{L}_0$ at one

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

moment (event A), being reflected at the points $\mathfrak{L}$, returns to $\mathfrak{L}_0$ simultaneously (event B).

2.2. THEOREM 2. *All spheres in all spaces* (*i.e. for all pairs* A, B *in Definition* 2) *are convex, if and only if* C_X *are convex.*

Proof. Let N be a ray in ∂C_A at which ∂C_A is not convex, i.e. C_A has no local supporting plane along N. Then, if $B_0 \in N$, the set of lines $\mathfrak{L} \| N$ which cross $C_A \cap C_{B_0}^-$ is not convex. Therefore, as soon as B is sufficiently near to B_0, the sphere $S(AB)$ will not be convex either.

But, according to a well-known theorem, a closed domain which has local supporting planes at every boundary point is convex; cf. **(4)**. Hence it follows that if C_X are not convex, then non-convex spheres exist.

On the other hand, if C_X are convex, all spheres are, obviously, convex. Thus our theorem is proved.

If we accept a given $S(AB)$ for the unit sphere, we may define the distance in the space R by means of translation and homotetic transformation of $S(AB)$.

The distance satisfies the ordinary conditions imposed on a metric if and only if the spheres are convex and symmetrical with respect to their centres. The latter condition being fulfilled, our Theorem 2 proves to be equivalent to the following:

THEOREM 2a. *The distances determined in spaces by means of our spheres are the distances in the ordinary sense* (*i.e. they satisfy the triangle inequality*) *for every space if and only if the cones* C_X *are convex.*

2.3. Now we introduce the concept of "time."

Definition 3. Let R be a space (Definition 1) and $X \in E^n$. We say that a point $Y \in E^n$ is simultaneous with X with respect to R: $Y \operatorname{sim} X$ (R), if there exist two points A, B, $B \in C_A$, such that X is the middle of the segment AB, the line AB belongs to R, and $Y \in \partial C_A \cap \partial C_B^-$.

This definition is nothing but the well-known Einstein definition of the simultaneity of events. The surface ∂C_A of the cone C_A depicts the light propagating from A, and ∂C_B^- depicts the light that, being reflected, returns to the same spatial point $\mathfrak{L}$ at the moment B. The event Y of reflection is, by definition, simultaneous with the event X that is the middle of the segment of events between A and B.

The set $T(XR)$ of all Y: $Y \operatorname{sim} X$ (R) is a cone with the vertex X. Cones with different X are equal and parallel to each other.

The relation $Y \operatorname{sim} X$ (R) is symmetric and transitive if and only if the cones $T(XR)$ are planes. This follows from the observation that the symmetry and transitivity of this relation are equivalent to the statement that

$$T(XR) = T(YR)$$

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

for every $Y \in T(XR)$, i.e. every point $Y \in T(XR)$ is the vertex of this cone $T(XR)$ and $T(XR)$ is a plane.

2.4. THEOREM 3. *The relation* $Y \operatorname{sim} X\ (R)$ *is symmetric and transitive if and only if there exists a plane P such that the section $P \cap C_A$ has a centre of symmetry, which lies on the line $\mathfrak{L} \in R$ passing through A. If M is this centre,* $T(MR) = P$. (All planes parallel to P have the same property with respect to all cones C_A such that $P \cap C_A$ has inner points.)

Proof. Suppose that a plane P with the indicated property exists. Let M be the centre of symmetry of $P \cap C_A$. Then, B being symmetrical to A with respect to M, $\partial C_A \cap \partial C_B^-$ is contained in P. Hence $T(MR) = P$ and, therefore, according to the above observation, the relation "sim" (with respect to the space R determined by the line AB) is symmetric and transitive.

Now, suppose that the relation "sim" is symmetric and transitive. Then, as is shown above, the cones $T(XR)$ are planes.

Take a line $\mathfrak{L} \in R$ and two points $A, B \in \mathfrak{L}$, $B \in C_A$, and let M be the middle of the segment AB. Then, by the very definition of the set $T(MR)$, the set $\partial C_A \cap \partial C_B^-$ is contained in the plane $T = T(MR)$. This set bounds in T the set $\mathfrak{D}$ consisting of points in which the lines $\mathfrak{L} \in S(AB)$, i.e. those crossing $C_A \cap C_B^-$, cross the plane T. Hence it follows that $\mathfrak{D} \subset \mathfrak{D}_A = T \cap C_A$, $\mathfrak{D} \subset \mathfrak{D}_B = T \cap C_B^-$.

On the other hand, $\partial\mathfrak{D}_A = T \cap \partial C_A$, $\partial\mathfrak{D}_B = T \cap \partial C_B$, and, therefore, $\partial\mathfrak{D}$ is a part of $\partial\mathfrak{D}_A$ and $\partial\mathfrak{D}_B$. But $\mathfrak{D}_A$, $\mathfrak{D}_B$ being closed domains, i.e. connected sets, the inclusions $\partial\mathfrak{D} \subset \partial\mathfrak{D}_A$, $\partial\mathfrak{D} \subset \partial\mathfrak{D}_B$ imply that $\mathfrak{D} \supset \mathfrak{D}_A$, $\mathfrak{D} \supset \mathfrak{D}_B$.

With the above inclusions we get that $\mathfrak{D} = \mathfrak{D}_A = \mathfrak{D}_B$. The point M being the centre of $\mathfrak{D}$, it proves to be the centre of $\mathfrak{D}_A = T \cap C_A$, i.e. the cone C_A has the property indicated in the theorem.

2.5. THEOREM 4. *The relation* "sim" *is symmetric and transitive with respect to every space if and only if the cones C_X are elliptic.*

By Theorem 3, this statement is equivalent to the following:

THEOREM 4a. *In order that a cone C be elliptic, it is necessary and sufficient that every inner point of it be the centre of symmetry of a plane section of C.*

In other words, this property means that for every line $\mathfrak{L}$ passing through X within C_X there exists a plane P such that the point $P \cap \mathfrak{L}$ is the centre of symmetry of $P \cap C_X$.

If we consider a plane section of the cone C, Theorem 4a reduces to the following:

THEOREM 4b. *In order that a bounded closed domain be an ellipsoid, it is necessary and sufficient that every inner point of it be its projective centre of symmetry.* (A point O is the projective centre of symmetry of a set M if there exists a projective transformation of M onto itself, which maps every straight line through O onto itself, changing its orientation.)

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

Proof. The necessity is obvious.

Let a domain $\mathfrak{D}$ have the indicated property. Let $A \in \partial\mathfrak{D}$ be a point at which $\mathfrak{D}$ has a supporting plane containing no other points of $\partial\mathfrak{D}$. Draw from A a ray N passing through inner points of $\mathfrak{D}$. Every such point being the projective centre of symmetry of $\mathfrak{D}$, the ray N crosses $\partial\mathfrak{D}$ at a point B which has the same property as the point A. This observation leads to the conclusion that $\mathfrak{D}$ is convex, and that every point $A \in \partial\mathfrak{D}$ can be transformed into any other $B \in \partial\mathfrak{D}$.

Now we observe that every point of any plane section of $\mathfrak{D}$ is the projective centre of this section. Hence it follows that it is sufficient to prove our theorem for a convex $\mathfrak{D}$ in a 2-plane. In this case the proof is simple and we omit it.

2.6. It is possible to introduce other definitions of "times" and "spaces."

Let P be a plane having no common points with every $C_A\backslash(A)$ if $A \in P$. P can be considered as a set of simultaneous points. Correspondingly, the set of all planes parallel to P represents an "a priori time."

The intersections $C_X \cap P$, $C_Y^- \cap P$ may be called the "expanding" and "contracting" spheres in P; they consist of points simultaneously reached by light emitted from X and focused at Y.

If these spheres coincide, i.e. if every expanding sphere is a contracting one, they have centres of symmetry. Then M being the centre of symmetry of $C_X \cap P$, the lines parallel to XM determine a space R, and P proves to be the set $T(MR)$ of points simultaneous with M in the sense of Definition 3. Thus the coincidence of the expanding and contracting spheres is equivalent to the conditions of Theorem 3, and their coincidence for every plane P is equivalent to the conditions of Theorem 4.

Thus we have

THEOREM 5. *The sets $C_X \cap P$ prove to be the sets $C_Y^- \cap P$ for every plane $P(P \cap C_A = (A), A \in P)$ if and only if the cones C_X are elliptic.*

3. Some other results and observations.

3.1. In our Theorem 1, the mappings f are determined by the condition $f(C_X) = C_{f(X)}$. But one can consider more general mappings defined by the condition $f(C_X) = C_{X'}$, i.e. the image of every cone C_X is a cone $C_{X'}$ parallel and equal to C_X but it is not presupposed that $X' = f(X)$. We cannot prove that such mappings are necessarily linear under conditions as general as those of Theorem 1. Still, we can prove their linearity under somewhat different conditions.

Let C be a cone subject to the following conditions in which $\mathfrak{L}$, $\mathfrak{L}'$ denote rays contained in C issuing from the vertex: (1) There exist $\mathfrak{L}$ such that for every $\mathfrak{L}'$ the plane angle $\mathfrak{L}\mathfrak{L}'$ is contained in C. (2) All these $\mathfrak{L}$ do not lie in an $(n-1)$-dimensional plane. (3) C satisfies the conditions of Theorem 1.

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

THEOREM 6. *Let every point $X \in E^n$ be the vertex of the cone C_X equal and parallel to a given C with the indicated properties. Then every one-to-one mapping f of E^n onto itself, which transforms every C_X into some other one, $f(C_X) = C_{X'}$, is linear.*

Proof. Define for every $X \in E^n$ the set $C_X^* = \bigcap_{X \in C_Y} C_Y$. It is not difficult to verify that C_X^* is a cone which satisfies the conditions (1) and (2) imposed on the cones in Theorem 1; it contains the rays parallel to the rays $\mathfrak{L}$ with the above-indicated properties; without the condition concerning those rays, C_X^* could consist of one point X.

Every mapping f which preserves the system of the cones C_X preserves that of C_X^* and $f(C_X^*) = C^*_{f(X)}$. Thus Lemma 2 of §1 applies and gives that f is continuous. Therefore, f preserves the system of cones C_X and maps their vertices into vertices (for C_X and C_X^* have the same vertex). Theorem 1 being applied, we see that f is linear.

3.2. The conditions of Theorems 1 and 6 that the closed convex hull HC of the cone C is not a Descartes product and does not contain a straight line can be replaced by weaker ones. For instance, we may observe that by Lemma 1 $f(HC_X) = HC_{f(X)}$ and, therefore, if we put $C_X^* = HC_X \backslash C_X$, we have

$$f(C_X^*) = C^*_{f(X)}.$$

Hence, if the cones C_X^* satisfy the conditions of Theorems 1 and 6, f is linear, although HC_X might be a Descartes product or contain straight lines.

We shall not look here for the necessary and sufficient conditions to be imposed on the cones C_X so that the mappings f will be linear. The wording of these conditions and the proof of the corresponding theorem are somewhat tiresome. We observe, without proof, only the following simplest result.

THEOREM 7. *The conclusions of Theorems* 1 *and* 6 *remain true if the closed convex hull HC of the cone C is a Descartes product, but ∂C is not contained in ∂HC.*

3.3. It is possible to consider more general point sets instead of cones.

Suppose that to every $X \in E^n$ there corresponds a set $\mathfrak{M}_X$, every $\mathfrak{M}_X$ being obtained from any other $\mathfrak{M}_Y$ by means of a translation $Y \to X$. Let the cone of rays issuing from X through all points of $\mathfrak{M}_X$ be subject to the conditions (1)–(3) of Theorem 1.

Put $\mathfrak{M}_X = \mathfrak{M}_X{}^0$ and define the sets

$$(*) \qquad \mathfrak{M}_X{}^i = \bigcup_{Y \in \mathfrak{M}_X{}^{i-1}} \mathfrak{M}_Y{}^{i-1}, \qquad \mathfrak{M}_X{}^\infty = \bigcup_{i=0}^{\infty} \mathfrak{M}_X{}^i.$$

Then, if $\mathfrak{M}_X{}^\infty$ is a cone, we get the system of cones $C_X = \mathfrak{M}_X{}^\infty$ with the properties supposed in Theorem 1. The simplest condition which guarantees that $\mathfrak{M}_X{}^\infty$ is a cone is the following: (4) $\mathfrak{M}_X$ consists of segments and rays issuing from X. Thus we have

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

THEOREM 8. *If the sets $\mathfrak{M}_X$ satisfy all indicated conditions, the one-to-one mapping f of E^n onto itself, which maps every $\mathfrak{M}_X$ onto $\mathfrak{M}_{f(X)}$, is linear.*

This is a particular case of the following general problem. What are the necessary and sufficient conditions on the sets $\mathfrak{M}_X \subset E^n$, which are pairwise equal and parallel, for the mappings preserving the system of these sets to be linear? An even more general problem is as follows. Let R be a space with a transitive group G and $\mathfrak{M}$ a point set in R. Let $\{\mathfrak{M}_X\}$ be the system of sets derived from $\mathfrak{M}$ by means of all transformations $g \in G$. What are the conditions which guarantee that the mappings preserving this system are automorphisms of the space (R, G)? In the case studied here, (R, G) is a free abelian locally bicompact group.

3.4. Our simple considerations are connected with philosophical problems concerning space and time. The structure of the four-dimensional world, in relativitiy theory, may be considered as determined by the propagation of light or, in the language of geometry, by the system of the light cones **(5)**. A. Robb has observed that the same structure may be considered as determined by the relation of precedence of events **(6)**.

An event is a point-phenomenon and the world may be considered as a set of events. There exists a fundamental relation between events: one event acts upon another. The acting of one upon another may be defined physically as a transition of energy and impulse, in particular, by means of light.

According to the corollory of Theorem 1, Lorentz transformations are fully determined by the light cones, for even the continuity of the transformation preserving these cones is not presupposed there. Hence it follows that the following definition of space–time may be given:

Space–time is a set of all events with abstraction made of all properties with the exception of those defined by the relation of acting upon.

This definition may be made the foundation of the relativistic theory of space–time **(1)**. The spaces and times are determined with respect to an inertial system and their metric properties are determined by the same relation of acting upon, i.e. by the system of cones C_X consisting of events Y acted upon by the event X.

If $\mathfrak{M}_X$ is the set of events upon which the event X acts directly, the set $\mathfrak{M}_X^\infty$ defined by (*) consists of all events upon which X acts directly or indirectly. The relation of direct or indirect acting upon is transitive. Abstracted from its physical properties, it becomes a relation of precedence.

The convexity of the cones C_X is equivalent to the transitivity of this relation as defined by means of the relation $Y \in C_X$. According to Theorem 2, the triangle inequality for the spatial distances determined by the cones C_X is equivalent to the convexity of these cones, and therefore it is equivalent to the transitivity of the relation of precedence.

As an event is an elementary phenomenon, so acting upon is an elementary cause–effect relation. Therefore, the spatial–temporal structure of the world is

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press

determined by its cause–effect structure. The metric relations in the space are determined by the same structure.

References

1. A. D. Alexandrov, *The space–time of the theory of relativity*, Jubilee of Relativity Theory, Proceedings (Basel, 1956).
2. A. D. Alexandrov and V. V. Ovchinnikova, *Notes on the foundations of relativity theory*, Vestnik Leningrad. Univ., *11* (1953), 95.
3. S. Bonnesen and W. Fenchel, *Theorie der konvexen Körper* (Berlin, 1934), §§1, 4.
4. H. S. M. Coxeter, *The real projective plane* (Cambridge, 1955).
5. H. Minkowski, *Raum und Zeit*, *Phys. Z.*, *10* (1909), 104.
6. A. A. Robb, *A theory of time and space* (Cambridge, 1914).

Institute of Mathematics,
Novosibirsk, U.S.S.R.

https://doi.org/10.4153/CJM-1967-102-6 Published online by Cambridge University Press