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Biprojectivity and Biflatness for Convolution Algebras of Nuclear Operators

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Abstract. For a locally compact group G, the convolution product on the space $\mathcal{N}(L^p(G))$ of nuclear operators was defined by Neufang [11]. We study homological properties of the convolution algebra $\mathcal{N}(L^p(G))$ and relate them to some properties of the group G, such as compactness, finiteness, discreteness, and amenability.

1 Introduction

Let G be a locally compact group and let $H = L^2(G)$. In [11], M. Neufang defined a new product on the space $\mathcal{N}(H)$ of nuclear operators on H making it into a Banach algebra. While the usual product on $\mathcal{N}(H)$ can be viewed as a noncommutative version of the pointwise product on ℓ^1, this new product is an analogue of the convolution product on $L^1(G)$. The resulting Banach algebra $\mathcal{S}_1(G) = (\mathcal{N}(H), *)$ shares many properties with $L^1(G)$ [11]. In particular, the group G can be completely reconstructed from $\mathcal{S}_1(G)$ (compare with the classical result of Wendel [16] about $L^1(G)$). Some important theorems of harmonic analysis (*e.g.*, a theorem of Hewitt and Ross [4, 35.13] characterizing multipliers on $L^\infty(G)$ for a compact group G) also have their counterparts for $\mathcal{S}_1(G)$. Further, $\mathcal{S}_1(G)$ (like $L^1(G)$) has a right identity if and only if G is discrete, it always has a right b.a.i., and it is a right ideal in its bidual if and only if G is compact. A closed subspace of $\mathcal{S}_1(G)$ is a right ideal if and only if it is invariant with respect to a certain natural action of G. Other examples showing that $\mathcal{S}_1(G)$ behaves in much the same way as $L^1(G)$ can be found in [11].

The aim of this paper is to study some homological properties of $\mathcal{S}_1(G)$, specifically, *biprojectivity* and *biflatness*. (For a detailed exposition of the homology theory for Banach algebras we refer to [7]; some facts can also be found in [8], [1], and [12]). Recall that $L^1(G)$ is biprojective if and only if G is compact [5, 6], and is biflat if and only if G is amenable [9, 6]. Thus it is natural to ask whether similar results hold for $\mathcal{S}_1(G)$. We show (Theorem 4.3) that the latter result concerning the biflatness of $L^1(G)$ is also true for $\mathcal{S}_1(G)$. On the other hand, it turns out (Theorem 3.7) that $\mathcal{S}_1(G)$ is biprojective if and only if G is finite. We also show that properties of G such as discreteness and compactness are equivalent to projectivity of certain $\mathcal{S}_1(G)$-modules.

Remark 1.1 Perhaps it is appropriate to note that $\mathcal{S}_1(G)$ is never amenable (except for the trivial case $G = \{e\}$) because it has a nontrivial right annihilator (see [11] or Section 2 below).

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2 Preliminaries

Let G be a locally compact group equipped with a left Haar measure, and let $1 < p < \infty$. Given $t \in G$, denote by $L_t\colon L^p(G) \to L^p(G)$ the left translation operator defined by $(L_t f)(s) = f(ts)$. For a function f on G we set, as usual, $\check{f}(t) = f(t^{-1})$ and $\tilde{f}(t) = \Delta(t^{-1}) f(t^{-1})$, where Δ is the modular function on G. For each $h \in L^\infty(G)$ we denote by $M_h\colon L^p(G) \to L^p(G)$ the multiplication operator $f \mapsto hf$. The trace duality between the space of bounded operators, $\mathcal{B}(L^p(G))$, and the space of nuclear operators, $\mathcal{N}(L^p(G))$, will be denoted by the brackets $\langle \cdot , \cdot \rangle$.

The *convolution product* $*$ on $\mathcal{N}(L^p(G))$ introduced by Neufang [11] is defined as follows. First consider the bilinear map

$$\mathcal{B}(L^p(G)) \times \mathcal{N}(L^p(G)) \to L^\infty(G), \qquad (T, \rho) \mapsto \bigl(t \mapsto \langle \rho, L_t T L_{t^{-1}} \rangle\bigr) \,. \tag{1}$$

Next consider the representation

$$L^\infty(G) \to \mathcal{B}(L^p(G)), \qquad h \mapsto M_h \,. \tag{2}$$

Composing (1) and (2), we obtain a bilinear map

$$\mathcal{B}(L^p(G)) \times \mathcal{N}(L^p(G)) \xrightarrow{\odot} \mathcal{B}(L^p(G)), \qquad (T, \rho) \mapsto T \odot \rho \,.$$

For every $\rho \in \mathcal{N}(L^p(G))$ the map $T \mapsto T \odot \rho$ is weak* continuous [11]. Therefore we have a well-defined bilinear map

$$\mathcal{N}(L^p(G)) \times \mathcal{N}(L^p(G)) \xrightarrow{*} \mathcal{N}(L^p(G)),$$
$$\langle T, \rho * \tau \rangle = \langle T \odot \rho, \tau \rangle \ \text{ for each } \ T \in \mathcal{B}(L^p(G)).$$

Neufang [11, Satz 5.2.1 and Prop. 5.4.1], proved that $(\mathcal{N}(L^p(G)), *)$ is an associative Banach algebra with a right b.a.i. We shall denote this algebra by $\mathcal{N}^p(G)$. (Note that the algebra $\mathcal{N}^2(G)$ is denoted by $\mathcal{S}_1(G)$ in [11], and the dual module $\mathcal{S}_1^*(G) = \mathcal{B}(L^2(G))$ is denoted by $\mathcal{S}_\infty(G)$ in order to emphasize a relation with the Schatten classes.)

The algebra $\mathcal{N}^p(G)$ can be considered as an extension of $L^1(G)$ in the following way. Consider the map $\iota^1\colon L^\infty(G) \to \mathcal{B}(L^p(G))$ defined by the rule $\iota^1(h) = M_{\check{h}}$. This map is continuous with respect to the weak* topologies determined by the dualities $\bigl\langle L^\infty(G), L^1(G) \bigr\rangle$ and $\bigl\langle \mathcal{B}(L^p(G)), \mathcal{N}(L^p(G)) \bigr\rangle$. Hence there exists the predual map $\sigma\colon \mathcal{N}(L^p(G)) \to L^1(G)$. Neufang [11, Satz 5.3.1], proved that σ is a Banach algebra homomorphism from $\mathcal{N}^p(G)$ onto $L^1(G)$. Furthermore,

$$\operatorname{Ker} \sigma = \{\rho \in \mathcal{N}^p(G) \,;\, \langle \rho, M_h \rangle = 0 \quad \forall\, h \in L^\infty(G)\} \,. \tag{3}$$

Therefore (see [11, Satz 5.3.4]) we have an extension

$$0 \to I \to \mathcal{N}^p(G) \xrightarrow{\sigma} L^1(G) \to 0 \tag{4}$$

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of Banach algebras. Note also that the definition of the product in $\mathcal{N}^p(G)$ together with (3) implies that $\mathcal{N}^p(G)I = 0$.

We shall need a more explicit description of σ. Take $q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and recall that there exists an isometric isomorphism

$$L^p(G) \mathbin{\widehat{\otimes}} L^q(G) \xrightarrow{\sim} \mathcal{N}(L^p(G)), \qquad f \otimes g \mapsto (h \mapsto \langle g, h \rangle f). \tag{5}$$

Here the brackets $\langle \cdot , \cdot \rangle$ denote the usual L^p-L^q duality. Identifying an elementary tensor $f \otimes g \in L^p(G) \mathbin{\widehat{\otimes}} L^q(G)$ with the corresponding rank-one operator, we see that

$$\begin{aligned}\langle \sigma(f \otimes g), h \rangle = \langle f \otimes g, M_{\check{h}} \rangle = \langle M_{\check{h}}(f), g \rangle = \langle \check{h} f, g \rangle \\ \int_G h(t^{-1}) f(t) g(t)\, dt = \int_G \Delta(t^{-1}) h(t) f(t^{-1}) g(t^{-1})\, dt = \langle (fg)^{\sim}, h \rangle\end{aligned}$$

for every $h \in L^\infty(G)$. Therefore,

$$\sigma(f \otimes g) = (fg)^{\sim} \qquad (f \in L^p(G),\ g \in L^q(G))\,. \tag{6}$$

Consider the algebra homomorphism $\varepsilon \colon L^1(G) \to \mathbb{C}$ given by $\varepsilon(f) = \int_G f\, d\mu$. It is clear from (6) that $\varepsilon\sigma = \mathrm{Tr}$. Thus $\mathbb{C}$ can be viewed as a $L^1(G)$-module via ε and as a $\mathcal{N}^p(G)$-module via Tr. We shall denote these modules by $\mathbb{C}_\varepsilon$ and $\mathbb{C}_{\mathrm{Tr}}$, respectively.

Recall some notation and some definitions from the homology theory of Banach algebras (for details, see [7, 8]). Let A be a Banach algebra. The category of left (resp. right) Banach A-modules is denoted by A-**mod** (resp. **mod**-A). If B is another Banach algebra, then A-**mod**-B stands for the category of Banach A-B-bimodules. Spaces of morphisms in the above categories are denoted by ${}_A\mathbf{h}(X, Y)$, $\mathbf{h}_A(X, Y)$, and ${}_A\mathbf{h}_B(X, Y)$, respectively. The space of continuous linear operators between Banach spaces X and Y is denoted by $\mathcal{B}(X, Y)$. For each left Banach A-module X denote by $A \cdot X \subset X$ the closed linear span of $\{a \cdot x;\, a \in A,\ x \in X\}$. If $A \cdot X = X$, then X is said to be *essential*.

A sequence $X_\bullet = (0 \to X_1 \to X_2 \to X_3 \to 0)$ of left Banach A-modules is called *admissible* (resp. *weakly admissible*) if it splits as a sequence of Banach spaces (resp. if the dual sequence $X_\bullet^* = (0 \to X_3^* \to X_2^* \to X_1^* \to 0)$ splits as a sequence of Banach spaces). A left Banach A-module Y is said to be *projective* (resp. *injective*) if for each admissible sequence $X_\bullet$ in A-**mod** the induced sequence ${}_A\mathbf{h}(Y, X_\bullet)$ (resp. ${}_A\mathbf{h}(X_\bullet, Y)$) is exact. A left Banach A-module Y is called *flat* if for each admissible sequence $X_\bullet$ in **mod**-A the induced sequence $X_\bullet \mathbin{\widehat{\otimes}}_A Y$ is exact. Recall that each projective module is flat, and that $Y \in A$-**mod** is flat iff the dual module, Y^*, is injective in **mod**-A [8, 7.1], [7, VII.1]. If the *canonical morphism* $\pi \colon A \mathbin{\widehat{\otimes}} Y \to Y$, $a \otimes y \mapsto a \cdot y$ is a retraction in A-**mod** (*i.e.*, if there exists an A-module morphism $\rho \colon Y \to A \mathbin{\widehat{\otimes}} Y$ such that $\pi\rho = \mathbf{1}_Y$), then Y is projective. The converse is true provided Y is essential [7, IV.I].

Remark 2.1 Let $A \to B$ be a Banach algebra homomorphism with dense range. Assume $Y \in B$-**mod** is projective (resp. injective, resp. flat) in A-**mod**. Then Y is

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projective (resp. injective, resp. flat) in B-**mod**. To see this, it suffices to observe that ${}_A\mathbf{h}(X,Y) = {}_B\mathbf{h}(X,Y)$ for each $X \in B$-**mod**, and that $X \widehat{\otimes}_A Y \cong X \widehat{\otimes}_B Y$ for each $X \in$ **mod**-B. See also [7, IV.I].

Projective and flat right A-modules and A-bimodules are defined similarly.

A Banach algebra A is called *biprojective* (resp. *biflat*) if A is a projective (resp. flat) Banach A-bimodule. Recall that A is biprojective (resp. biflat) iff the product map $\pi_A \colon A \widehat{\otimes} A \to A,\ a \otimes b \mapsto ab$ is a retraction in A-**mod**-A (resp. iff the dual map $\pi_A^* \colon A^* \to (A \widehat{\otimes} A)^*$ is a coretraction in A-**mod**-A); see [7, IV.5 and VII.2].

A Banach algebra A is said to be *contractible* (resp. *amenable*) if the first Hochschild cohomology group, $\mathcal{H}^1(A,X)$, is trivial for each (resp. for each dual) Banach A-bimodule X. Recall that A is contractible iff it is biprojective and unital, and is amenable iff it is biflat and has a b.a.i. [8, 7.1].

Following Selivanov [15], we say that a Banach algebra A is *superbiprojective* (resp. *superbiflat*) if it is biprojective (resp. biflat) and $\mathcal{H}^2(A,X) = 0$ for each (resp. for each dual) Banach A-bimodule X. Selivanov proved that A is superbiflat iff it is biflat and has a one-sided b.a.i. On the other hand, if A is biprojective and has a one-sided identity, then it is superbiprojective [15].

3 Biprojectivity

Lemma 3.1 *Let*

$$0 \to I \to \mathfrak{A} \xrightarrow{\sigma} A \to 0 \tag{7}$$

be an extension of Banach algebras such that $\mathfrak{A}I = 0$. Assume there exists an antihomomorphism $\beta \colon A \to A$ and a linear continuous map $\alpha \colon \mathfrak{A} \to \mathfrak{A}$ such that $\beta^2 = \mathbf{1}_A$ and $\beta\sigma = \sigma\alpha$. Suppose also that A is essential and projective as a right $\mathfrak{A}$-module via σ. Then (7) *is admissible.*

Proof Denote by $\pi \colon A \widehat{\otimes} \mathfrak{A} \to A$ the right action of $\mathfrak{A}$ on A determined by σ, i.e., $\pi(a \otimes u) = a\sigma(u)$. Since A is essential and projective in **mod**-$\mathfrak{A}$, there exists a right $\mathfrak{A}$-module morphism $\rho \colon A \to A \widehat{\otimes} \mathfrak{A}$ such that $\pi\rho = \mathbf{1}_A$ [7, IV.1]. Next observe that the condition $\mathfrak{A}I = 0$ implies that $\mathfrak{A}$ has a natural structure of right Banach A-module. Indeed, the product map $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ vanishes on $\mathfrak{A} \times I$ and hence determines a right action $\mathfrak{A} \times A = \mathfrak{A} \times (\mathfrak{A}/I) \to \mathfrak{A}$ by the rule $(u,a) \mapsto u \cdot a = uv$ where $v \in \sigma^{-1}(a)$. The corresponding linear map $\mathfrak{A} \widehat{\otimes} A \to \mathfrak{A},\ u \otimes a \mapsto u \cdot a$ will be denoted by ϕ.

Define $\tilde{\varkappa} \colon A \to \mathfrak{A}$ as the composition

$$A \xrightarrow{\rho} A \widehat{\otimes} \mathfrak{A} \xrightarrow{\tau} \mathfrak{A} \widehat{\otimes} A \xrightarrow{\alpha \otimes \beta} \mathfrak{A} \widehat{\otimes} A \xrightarrow{\phi} \mathfrak{A}$$

where τ stands for the flip $a \otimes u \mapsto u \otimes a$.

Let us compute $\sigma\tilde{\varkappa}$. For every $u \in \mathfrak{A},\ a \in A$, and $v \in \sigma^{-1}(a)$ we have

$$(\sigma\phi)(u \otimes a) = \sigma(uv) = \sigma(u)\sigma(v) = \sigma(u)a\,.$$

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In other words, $\sigma\phi = \pi^{\mathrm{op}}$, where $\pi^{\mathrm{op}} \colon \mathfrak{A} \widehat{\otimes} A \to A$ is the left action of $\mathfrak{A}$ on A determined by σ. Next,

$$\big(\pi^{\mathrm{op}} \circ (\alpha \otimes \beta)\big)(u \otimes a) = \pi^{\mathrm{op}}\big(\alpha(u) \otimes \beta(a)\big) = \sigma(\alpha(u))\beta(a)$$
$$= \beta(\sigma(u))\beta(a) = \beta(a\sigma(u)) = (\beta\pi)(a \otimes u) = (\beta\pi\tau)(u \otimes a)\,.$$

Hence $\pi^{\mathrm{op}} \circ (\alpha \otimes \beta) = \beta\pi\tau$. Finally,

$$\sigma\tilde{\varkappa} = \pi^{\mathrm{op}} \circ (\alpha \otimes \beta)\tau\rho = \beta\pi\tau\tau\rho = \beta\pi\rho = \beta\,.$$

Since $\beta^2 = \mathbf{1}_A$, we conclude that the map $\varkappa = \tilde{\varkappa}\beta$ satisfies $\sigma\varkappa = \mathbf{1}_A$. Therefore (7) is admissible. ■

Recall (see, *e.g.*, [3, 10]) that a Banach space E is said to have the *Radon-Nikodým property* (RNP for short) if for each finite measure space (X, μ) every μ-continuous E-valued measure of finite variation is differentiable with respect to μ. For our purposes, the following properties of the RNP will be important.

(a) The RNP is inherited by closed subspaces.
(b) If E and F are reflexive Banach spaces one of which has the approximation property, then the space $\mathcal{N}(E^*, F)$ of nuclear operators from E^* to F has the RNP [2]. In particular, $\mathcal{N}(L^p(X, \mu))$ has the RNP for every measure space (X, μ).
(c) If (X, μ) is a measure space, then $L^1(X, \mu)$ has the RNP if and only if μ is purely atomic [3, III.1].

Lemma 3.2 *Let G be a locally compact group and let $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *Extension* (4) *is admissible;*
(ii) *Extension* (4) *splits;*
(iii) *G is discrete.*

Proof (ii)⇒(i): obvious.

(i)⇒(iii). If G is nondiscrete, then $L^1(G)$ does not have the RNP (see (c) above). Since $\mathcal{N}^p(G) = \mathcal{N}(L^p(G))$ has the RNP (see (b) above), we conclude that $L^1(G)$ is not isomorphic to a subspace of $\mathcal{N}^p(G)$. Hence extension (4) is not admissible.

(iii)⇒(ii). If G is discrete, then the map

$$\varkappa \colon \ell^1(G) \to \mathcal{N}^p(G), \qquad f \mapsto M_{\check{f}}$$

is a continuous right inverse to $\sigma \colon \mathcal{N}^p(G) \to \ell^1(G)$ (see [11, Satz 5.3.7]). It is easy to check that $\varkappa$ is an algebra homomorphism. To see this, for each $t \in G$ let δ_t denote the function which equals 1 at t and 0 elsewhere. We claim that $M_{\delta_s} * M_{\delta_t} = M_{\delta_{ts}}$ for each $s, t \in G$. Indeed, for each $T \in \mathcal{B}(\ell^p(G))$ we have

$$\langle T, M_{\delta_s} * M_{\delta_t}\rangle = \langle T \odot M_{\delta_s}, \delta_t \otimes \delta_t\rangle = \langle (T \odot M_{\delta_s})(\delta_t), \delta_t\rangle = \langle \delta_s \otimes \delta_s, L_t T L_{t^{-1}}\rangle$$
$$= \langle T L_{t^{-1}}(\delta_s), L_{t^{-1}}(\delta_s)\rangle = \langle T(\delta_{ts}), \delta_{ts}\rangle = \langle T, \delta_{ts} \otimes \delta_{ts}\rangle = \langle T, M_{\delta_{ts}}\rangle\,.$$

Therefore $M_{\delta_s} * M_{\delta_t} = M_{\delta_{ts}}$ for each $s, t \in G$, and so

$$\varkappa(\delta_s * \delta_t) = \varkappa(\delta_{st}) = M_{\delta_{t^{-1}s^{-1}}} = M_{\delta_{s^{-1}}} * M_{\delta_{t^{-1}}} = \varkappa(\delta_s) * \varkappa(\delta_t)\,.$$

Thus we see that $\varkappa$ is an algebra homomorphism. Next,

$$\sigma\varkappa(\delta_s) = \sigma(\delta_{s^{-1}} \otimes \delta_{s^{-1}}) = \delta_s,$$

i.e., $\sigma\varkappa = \mathbf{1}_{\ell^1(G)}$ (see also [11, Satz 5.3.7]). Therefore extension (4) splits. ∎

Lemma 3.3 *Let* (7) *be an extension of Banach algebras such that* $\mathfrak{A}I = 0$*. Assume that* (7) *splits and* A *has a left identity. Then* A *is projective in* **mod**-$\mathfrak{A}$.

Proof As in Lemma 3.1, denote by $\pi\colon A \widehat{\otimes} \mathfrak{A} \to A$ the right action of $\mathfrak{A}$ on A given by $\pi(a \otimes u) = a\sigma(u)$. Let $\varkappa\colon A \to \mathfrak{A}$ be a homomorphism such that $\sigma\varkappa = \mathbf{1}_A$. Take a left identity e of A and define the map $\rho\colon A \to A \widehat{\otimes} \mathfrak{A}$ by the rule $\rho(a) = e \otimes \varkappa(a)$. Evidently, $\pi\rho = \mathbf{1}_A$. Since $\mathfrak{A}I = 0$, we see that $vu = v\varkappa(\sigma(u))$ for each $u, v \in \mathfrak{A}$. Hence

$$\rho(a \cdot u) = \rho(a\sigma(u)) = e \otimes \varkappa(a\sigma(u)) = e \otimes \varkappa(a)\varkappa(\sigma(u)) = e \otimes \varkappa(a)u = \rho(a) \cdot u$$

for each $a \in A$ and each $u \in \mathfrak{A}$. This means that ρ is a morphism in **mod**-$\mathfrak{A}$. Since $\pi\rho = \mathbf{1}_A$, we conclude that A is projective in **mod**-$\mathfrak{A}$. ∎

Theorem 3.4 *Let* G *be a locally compact group and let* $1 < p < \infty$*. Then the following conditions are equivalent:*

(i) $L^1(G)$ *is projective in* **mod**-$\mathcal{N}^p(G)$*;*
(ii) G *is discrete.*

Proof (i)⇒(ii). For each $1 \le r \le \infty$ define the map $\alpha_r\colon L^r(G) \to L^r(G)$ by $\alpha_r(f)(t) = \Delta(t^{-1})^{1/r} f(t^{-1})$. It is easy to check that α_r is an isometry and $\alpha_r^2 = \mathbf{1}_{L^r(G)}$. Now let $\beta = \alpha_1\colon L^1(G) \to L^1(G)$ and $\alpha = \alpha_p \otimes \alpha_q\colon \mathcal{N}^p(G) \to \mathcal{N}^p(G)$. Evidently, β is an antihomomorphism. For each $f \in L^p(G)$ and each $g \in L^q(G)$ we have

$$\begin{aligned}\sigma(\alpha(f \otimes g))(t) &= \sigma(\alpha_p(f) \otimes \alpha_q(g))(t) = \Delta(t^{-1})\alpha_p(f)(t^{-1})\alpha_q(g)(t^{-1})\\ &= \Delta(t)^{-1}\Delta(t)^{1/p} f(t)\Delta(t)^{1/q} g(t) = f(t)g(t)\,.\end{aligned}$$

On the other hand,

$$\beta(\sigma(f \otimes g))(t) = \Delta(t^{-1})\sigma(f \otimes g)(t^{-1}) = \Delta(t)^{-1}\Delta(t) f(t)g(t) = f(t)g(t)\,.$$

Hence $\beta\sigma = \sigma\alpha$. Finally, $L^1(G)$ is an essential $\mathcal{N}^p(G)$-module since σ is surjective and $L^1(G)$ has a b.a.i. By Lemma 3.1, extension (4) is admissible. Now Lemma 3.2 shows that G is discrete.

(ii)⇒(i). If G is discrete, then extension (4) splits by Lemma 3.2. Since $L^1(G)$ is unital in this case, Lemma 3.3 implies that $L^1(G)$ is projective in **mod**-$\mathcal{N}^p(G)$. ∎

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Lemma 3.5 *Let G be a compact group and let 1 denote the function that is identically 1 on G. Then $(1 \otimes 1) * a = \operatorname{Tr} a \cdot 1 \otimes 1$ for each $a \in \mathcal{N}^p(G)$.*

Proof For each $T \in \mathcal{B}(L^p(G))$ we have $T \odot (1 \otimes 1) = M_h$, where

$$h(t) = \langle 1 \otimes 1, L_t T L_{t^{-1}} \rangle = \langle L_t T L_{t^{-1}}(1), 1 \rangle = \langle T L_{t^{-1}}(1), L_{t^{-1}}(1) \rangle = \langle T(1), 1 \rangle$$

for each $t \in G$. Hence $M_h = \langle T(1), 1 \rangle \mathbf{1}_{L^p(G)}$, and for each $a \in \mathcal{N}^p(G)$ we have

$$\begin{aligned} \langle T, (1 \otimes 1) * a \rangle &= \langle T \odot (1 \otimes 1), a \rangle = \langle T(1), 1 \rangle \langle \mathbf{1}_{L^p(G)}, a \rangle \\ &= \langle T(1), 1 \rangle \operatorname{Tr} a = \langle T, \operatorname{Tr} a \cdot 1 \otimes 1 \rangle, \end{aligned}$$

as required. ∎

Theorem 3.6 *Let G be a locally compact group and let $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *$\mathbb{C}_{\mathrm{Tr}}$ is projective in* **mod**-*$\mathcal{N}^p(G)$;*
(ii) *G is compact.*

Proof (i)⇒(ii). If $\mathbb{C}_{\mathrm{Tr}}$ is projective in **mod**-$\mathcal{N}^p(G)$, then $\mathbb{C}_\varepsilon$ is projective in **mod**-$L^1(G)$ (see Remark 2.1). This means exactly that G is compact (see [6] or [7, IV.5]).

(ii)⇒(i). First observe that the canonical morphism

$$\pi \colon \mathbb{C}_{\mathrm{Tr}} \widehat{\otimes} \mathcal{N}^p(G) \to \mathbb{C}_{\mathrm{Tr}}, \quad \lambda \otimes u \mapsto \lambda \cdot u,$$

is identified with $\operatorname{Tr} \colon \mathcal{N}^p(G) \to \mathbb{C}_{\mathrm{Tr}}$. Define $\rho \colon \mathbb{C}_{\mathrm{Tr}} \to \mathcal{N}^p(G)$ by $\rho(\lambda) = \lambda \cdot 1 \otimes 1$. It is clear that $\pi\rho = \mathbf{1}_{\mathbb{C}}$, and Lemma 3.5 implies that ρ is a right $\mathcal{N}^p(G)$-module morphism. Hence $\mathbb{C}_{\mathrm{Tr}}$ is projective in **mod**-$\mathcal{N}^p(G)$. ∎

Theorem 3.7 *Let G be a locally compact group and let $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *$\mathcal{N}^p(G)$ is biprojective;*
(ii) *$\mathcal{N}^p(G)$ is superbiprojective;*
(iii) *G is finite.*

Proof (ii)⇒(i). Obvious.

(iii)⇒(ii). If G is finite, then $L^1(G) = \mathbb{C}[G]$ is contractible. Furthermore, all the algebras in the extension (4) are finite-dimensional. Hence (4) splits, and so $\mathcal{N}^p(G)$ is isomorphic to the semidirect product $L^1(G) \oplus I$. Since $L^1(G) \cdot I = 0$, we conclude that $\mathcal{N}^p(G)$ is superbiprojective by [15] (see also Lemma 4.2 below).

(i)⇒(iii). If $\mathcal{N}^p(G)$ is biprojective, then both $L^1(G)$ and $\mathbb{C}_{\mathrm{Tr}}$ are projective in **mod**-$\mathcal{N}^p(G)$ since they are essential and $\mathcal{N}^p(G)$ has a right b.a.i. (see [13] or [8, 7.1.60]). Now it remains to apply Theorems 3.4 and 3.6. ∎

4 Biflatness

Lemma 4.1 *Let*

$$0 \to I \to \mathfrak{A} \xrightarrow{\sigma} A \to 0 \tag{8}$$

be a weakly admissible extension of Banach algebras such that $\mathfrak{A}I = 0$ *and* $\mathfrak{A}^2 = \mathfrak{A}$. *Then the following conditions are equivalent:*

(i) $\mathfrak{A}$ *is biflat and has a right b.a.i.;*
(ii) A *is biflat and has a right b.a.i.*

Proof (i)⇒(ii). Since $\mathfrak{A}$ has a right b.a.i., we have $I\mathfrak{A} = I$. Hence $A = \mathfrak{A}/(I\mathfrak{A})$ is biflat by [14]. It is also clear that A has a right b.a.i.

(ii)⇒(i). As in Lemma 3.1, the condition $\mathfrak{A}I = 0$ implies that $\mathfrak{A}$ is a right Banach A-module in a natural way. First we note that $\mathfrak{A}$ has a right b.a.i. Indeed, let $\{e_\nu\}$ be a bounded net in $\mathfrak{A}$ such that $\{\sigma(e_\nu)\}$ is a right b.a.i. in A. Then for every $u, v \in \mathfrak{A}$ we have

$$\|uve_\nu - uv\| = \|u \cdot \sigma(ve_\nu) - u \cdot \sigma(v)\| \le \|u\| \|\sigma(v)\sigma(e_\nu) - \sigma(v)\| \to 0\,,$$

because $\{\sigma(e_\nu)\}$ is a right b.a.i. in A. Since $\{e_\nu\}$ is bounded, and since $\mathfrak{A}^2 = \mathfrak{A}$, we conclude that $\{e_\nu\}$ is a right b.a.i. in $\mathfrak{A}$.

Let $\pi_A : A \mathbin{\widehat{\otimes}} A \to A$ be the product map. Since A is biflat, the dual map π_A^* is a coretraction in A-**mod**-A. Applying the functor $\mathbf{h}_A(\mathfrak{A}, ?)$, we see that

$$\mathbf{h}_A(\mathfrak{A}, \pi_A^*) : \mathbf{h}_A(\mathfrak{A}, A^*) \to \mathbf{h}_A(\mathfrak{A}, (A \mathbin{\widehat{\otimes}} A)^*)$$

is a coretraction in A-**mod**-$\mathfrak{A}$. Using the adjoint associativity isomorphisms [7, II.5], we can identify the latter map with $(\mathbf{1}_{\mathfrak{A}} \otimes \pi_A)^* : (\mathfrak{A} \mathbin{\widehat{\otimes}}_A A)^* \to (\mathfrak{A} \mathbin{\widehat{\otimes}}_A (A \mathbin{\widehat{\otimes}} A))^*$. Therefore $(\mathbf{1}_{\mathfrak{A}} \otimes \pi_A)^*$ is a coretraction in A-**mod**-$\mathfrak{A}$.

Now let $\pi_{\mathfrak{A},A} : \mathfrak{A} \mathbin{\widehat{\otimes}} A \to \mathfrak{A}$, $u \otimes a \mapsto u \cdot a$ denote the right action on A on $\mathfrak{A}$. Evidently, $\pi_{\mathfrak{A},A}$ induces the morphism $\varkappa : \mathfrak{A} \mathbin{\widehat{\otimes}}_A A \to \mathfrak{A}$, $u \otimes a \mapsto u \cdot a$. We have the following commutative diagram in $\mathfrak{A}$-**mod**-A:

$$\begin{array}{ccc} \mathfrak{A} \mathbin{\widehat{\otimes}}_A A \mathbin{\widehat{\otimes}} A & \xrightarrow{\mathbf{1}_{\mathfrak{A}} \otimes \pi_A} & \mathfrak{A} \mathbin{\widehat{\otimes}}_A A \\ {\scriptstyle \varkappa \otimes \mathbf{1}_A} \downarrow & & \downarrow {\scriptstyle \varkappa} \\ \mathfrak{A} \mathbin{\widehat{\otimes}} A & \xrightarrow{\pi_{\mathfrak{A},A}} & \mathfrak{A} \end{array} \tag{9}$$

The vertical arrows in the diagram are isomorphisms in $\mathfrak{A}$-**mod**-A, because $\mathfrak{A}^2 = \mathfrak{A}$ and A has a right b.a.i. [7, II.3]. Consider now the dual diagram. We already know that $(\mathbf{1}_{\mathfrak{A}} \otimes \pi_A)^*$ is a coretraction in A-**mod**-$\mathfrak{A}$. Hence so is $\pi_{\mathfrak{A},A}^*$.

Since (8) is weakly admissible, σ^* is an admissible monomorphism in A-**mod**. On the other hand, since A is biflat, it follows that A is flat in **mod**-A, and hence A^* is injective in A-**mod** (see, *e.g.*, [7, VII.1]). Therefore σ^* is a coretraction in A-**mod**, and so $\mathcal{B}(\mathfrak{A}, \sigma^*) : \mathcal{B}(\mathfrak{A}, A^*) \to \mathcal{B}(\mathfrak{A}, \mathfrak{A}^*)$ is a coretraction in A-**mod**-$\mathfrak{A}$.

Identifying $\mathcal{B}(\mathfrak{A}, A^*)$ with $(\mathfrak{A} \widehat{\otimes} A)^*$ and $\mathcal{B}(\mathfrak{A}, \mathfrak{A}^*)$ with $(\mathfrak{A} \widehat{\otimes} \mathfrak{A})^*$, we conclude that $(\mathbf{1}_{\mathfrak{A}} \otimes \sigma)^* \colon (\mathfrak{A} \widehat{\otimes} A)^* \to (\mathfrak{A} \widehat{\otimes} \mathfrak{A})^*$ is a coretraction in A-**mod**-$\mathfrak{A}$.

It is easy to see that the product map $\pi_{\mathfrak{A}} \colon \mathfrak{A} \widehat{\otimes} \mathfrak{A} \to \mathfrak{A}$ decomposes as $\pi_{\mathfrak{A}} = \pi_{\mathfrak{A},A} \circ (\mathbf{1}_{\mathfrak{A}} \otimes \sigma)$. Hence $\pi_{\mathfrak{A}}^* = (\mathbf{1}_{\mathfrak{A}} \otimes \sigma)^* \circ \pi_{\mathfrak{A},A}^*$. But we already know that both $(\mathbf{1}_{\mathfrak{A}} \otimes \sigma)^*$ and $\pi_{\mathfrak{A},A}^*$ are coretractions in A-**mod**-$\mathfrak{A}$. Hence so is $\pi_{\mathfrak{A}}^*$.

To complete the proof, it remains to show that $\pi_{\mathfrak{A}}^*$ is actually a coretraction in $\mathfrak{A}$-**mod**-$\mathfrak{A}$. To this end, note that for each $f \in \mathfrak{A}^*$ and each $u, v \in \mathfrak{A}$ we have

$$\langle u \cdot f, v \rangle = \langle f, vu \rangle = \langle f, v \cdot \sigma(u) \rangle = \langle \sigma(u) \cdot f, v \rangle ,$$

i.e., $u \cdot f = \sigma(u) \cdot f$ for each $f \in \mathfrak{A}^*$ and each $u \in \mathfrak{A}$. Similarly, $u \cdot g = \sigma(u) \cdot g$ for each $g \in (\mathfrak{A} \widehat{\otimes} \mathfrak{A})^*$. This implies that every left A-module morphism between $(\mathfrak{A} \widehat{\otimes} \mathfrak{A})^*$ and $\mathfrak{A}^*$ is in fact a left $\mathfrak{A}$-module morphism. In particular, a left inverse of $\pi_{\mathfrak{A}}^*$ in A-**mod**-$\mathfrak{A}$ is a morphism in $\mathfrak{A}$-**mod**-$\mathfrak{A}$. Therefore $\pi_{\mathfrak{A}}^*$ is a coretraction in $\mathfrak{A}$-**mod**-$\mathfrak{A}$. This completes the proof. ■

Lemma 4.1 has the following "predual" counterpart, which is a slight generalization of an unpublished result of Selivanov [15].

Lemma 4.2 *Let*

$$0 \to I \to \mathfrak{A} \xrightarrow{\sigma} A \to 0 \tag{10}$$

be an admissible extension of Banach algebras such that $\mathfrak{A}I = 0$ and $\mathfrak{A}^2 = \mathfrak{A}$. Then the following conditions are equivalent:

(i) *$\mathfrak{A}$ is biprojective and has a right b.a.i. (resp. a right identity);*
(ii) *A is biprojective and has a right b.a.i. (resp. a right identity).*

As a corollary, $\mathfrak{A}$ is superbiprojective provided A is contractible.

Proof The proof is similar to that of Lemma 4.1.

(i)⇒(ii). Since $\mathfrak{A}$ has a right b.a.i., we have $I\mathfrak{A} = I$. Hence $A = \mathfrak{A}/(I\mathfrak{A})$ is biprojective by [13]. Evidently, if $\mathfrak{A}$ has a right b.a.i. (resp. a right identity), then so does A.

(ii)⇒(i). As in Lemma 3.1, the condition $\mathfrak{A}I = 0$ implies that $\mathfrak{A}$ is a right Banach A-module in a natural way. Arguing as in the proof of Lemma 4.1, we conclude that any bounded preimage of a right b.a.i. (resp. of a right identity) in A is a right b.a.i. (resp. a right identity) in $\mathfrak{A}$.

Since A is biprojective, the product map $\pi_A \colon A \widehat{\otimes} A \to A$ is a retraction in A-**mod**-A. Hence $\mathbf{1}_{\mathfrak{A}} \otimes \pi_A \colon \mathfrak{A} \widehat{\otimes}_A (A \widehat{\otimes} A) \to \mathfrak{A} \widehat{\otimes}_A A$ is a retraction in $\mathfrak{A}$-**mod**-A. As in Lemma 4.1, we can identify the latter morphism with $\pi_{\mathfrak{A},A} \colon \mathfrak{A} \widehat{\otimes} A \to \mathfrak{A}$ (see diagram (9)). Therefore $\pi_{\mathfrak{A},A}$ is also a retraction in $\mathfrak{A}$-**mod**-A.

Since (10) is admissible, σ is an admissible epimorphism in **mod**-A. On the other hand, since A is biprojective, it follows that A is projective in **mod**-A. Hence σ is a retraction in **mod**-A, and so $\mathbf{1}_{\mathfrak{A}} \otimes \sigma \colon \mathfrak{A} \widehat{\otimes} \mathfrak{A} \to \mathfrak{A} \widehat{\otimes} A$ is a retraction in $\mathfrak{A}$-**mod**-A.

It is easy to see that the product map $\pi_{\mathfrak{A}}\colon \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ decomposes as $\pi_{\mathfrak{A}} = \pi_{\mathfrak{A},A} \circ (\mathbf{1}_{\mathfrak{A}} \otimes \sigma)$. But we already know that both $\mathbf{1}_{\mathfrak{A}} \otimes \sigma$ and $\pi_{\mathfrak{A},A}$ are retractions in $\mathfrak{A}$-**mod**-A. Hence so is $\pi_{\mathfrak{A}}$.

Finally, the condition $\mathfrak{A}I = 0$ implies that $uv = u \cdot \sigma(v)$ and $w \cdot v = w \cdot \sigma(v)$ for each $u, v \in \mathfrak{A}$ and each $w \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$. Hence every right A-module morphism between $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ and $\mathfrak{A}$ is in fact a right $\mathfrak{A}$-module morphism. In particular, a right inverse of $\pi_{\mathfrak{A}}$ in $\mathfrak{A}$-**mod**-A is a morphism in $\mathfrak{A}$-**mod**-$\mathfrak{A}$. Therefore $\pi_{\mathfrak{A}}$ is a retraction in $\mathfrak{A}$-**mod**-$\mathfrak{A}$, i.e., $\mathfrak{A}$ is biprojective. This completes the proof. ■

Theorem 4.3 *Let G be a locally compact group and let $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *$\mathcal{N}^p(G)$ is biflat;*
(ii) *$\mathcal{N}^p(G)$ is superbiflat;*
(iii) *$\mathbb{C}_{\mathrm{Tr}}$ is flat in* **mod**-*$\mathcal{N}^p(G)$;*
(iv) *G is amenable.*

Proof (i) $\Longleftrightarrow$ (ii). Clear, because $\mathcal{N}^p(G)$ has a right b.a.i.

(i)$\Rightarrow$(iii). If $\mathcal{N}^p(G)$ is biflat, then $\mathbb{C}_{\mathrm{Tr}}$ is flat in **mod**-$\mathcal{N}^p(G)$ since $\mathbb{C}_{\mathrm{Tr}}$ is essential and $\mathcal{N}^p(G)$ has a right b.a.i. (see [8, 7.1.60]).

(iii)$\Rightarrow$(iv). If $\mathbb{C}_{\mathrm{Tr}}$ is flat in **mod**-$\mathcal{N}^p(G)$, then $\mathbb{C}_{\varepsilon}$ is flat in **mod**-$L^1(G)$ (see Remark 2.1). By [6] (see also [7, VII.2]), this happens if and only if G is amenable.

(iv)$\Rightarrow$(i). Recall that G is amenable if and only if $L^1(G)$ is amenable (see, *e.g.*, [9, 6, 7]). Since $L^\infty(G) = L^1(G)^*$ is an injective Banach space (see, *e.g.*, [17]), we see that extension (4) is weakly admissible. Therefore $\mathcal{N}^p(G)$ is biflat by Lemma 4.1. ■

Remark 4.1 M. Neufang has kindly informed the author that he has also proved the equivalence of conditions (i), (ii) and (iv) of Theorem 4.3.

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