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TWO ESTIMATES CONCERNING ASYMPTOTICS OF THE MINIMIZATIONS OF A GINZBURG-LANDAU FUNCTIONAL

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Abstract

We prove two asymptotical estimates for minimizers of a Ginzburg-Landau functional of the form

$$\int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 W(x)\right] dx.$$

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1. Introduction

Let M be a smooth Riemann surface with boundary ∂M, and let g be a smooth function; $g : \partial M \to S^1$ with a topological degree d. Let

$$H_g^{1,2}(M, \mathbb{R}^2) = \{u \in H^{1,2}(M, \mathbb{R}^2) : u|_{\partial M} = g\}.$$

For $\varepsilon > 0$, consider the Ginzburg-Landau functional

$$E_\varepsilon(u; M) = \int_M \frac{|\nabla u|^2}{2}\, dM + \int_M \frac{1}{4\varepsilon^2}(1-|u|^2)^2 w(x)\, dM \tag{1.1}$$

where w is a smooth function in $\overline{M}$ with $w > 0$ in $\overline{M}$.

It is well-known that $H_g^{1,2}(M, \mathbb{R}^2)$ is non-empty and that for $\varepsilon > 0$ the functional E_ε achieves its minimum in $H_g^{1,2}(M, \mathbb{R}^2)$, giving

$$E_\varepsilon(u_\varepsilon, M) = \inf_{u \in H_g^{1,2}} E_\varepsilon(u; M) \tag{1.2}$$

for some $u_\varepsilon \in H^{1,2}(\Omega, \mathbb{R}^2)$.

In this paper, we only discuss a Riemann surface M having the Riemann metric

$$ds^2 = h_{ij} dx^i \otimes dx^j$$

with $h_{ij} = h(x)\delta_{ij}$ on a domain Ω in $\mathbb{R}^2$ with $h > 0$. In this case, the energy $E_\varepsilon(u, M)$ has the new form on Ω:

$$E_\varepsilon(u; \Omega) = \int_\Omega \frac{|\nabla u|^2}{2} dx + \int_\Omega \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 W(x)\, dx$$

where $W(x)$ is a smooth function in $\overline{\Omega}$ such that $W > 0$ in $\overline{\Omega}$.

The minimizer u_ε then satisfies the Euler-Lagrange equation

$$(1.3) \qquad -\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2) W(x) \quad \text{in } \Omega.$$

If $W(x) \equiv 1$ in (1.1), Bethuel, Brezis and Hélein (see [1, 2 and 3]) recently proved many beautiful results for the asympotics of minimizers as $\varepsilon \to 0$. One of the main results in [3] is the following

THEOREM [BBH]. *Assume that $M = \Omega$ is a star-shaped domain in $\mathbb{R}^2$. Let $d \neq 0$ be the degree of the boundary data g. For each $\varepsilon > 0$, let u_ε be a minimizer for E_ε. For this sequence of minimizers u_ε, there exists a subsequence (u_{ε_k}) and $|d|$ points x_l, $l = 1, \dots, |d|$ such that as $\varepsilon_k \to 0$,*

$$u_{\varepsilon_k} \rightharpoonup u \text{ in } H^{1,2}_{loc}(\Omega \backslash \{x_1, \dots, x_{|d|}\}, \mathbb{R}^2)$$

where u is a harmonic map with values in S^1. Moreover u_{ε_k} converges to u weakly in $H^{1,q}$ for $q < 2$.

An extension to general domains of the above result has been obtained by Struwe (see [8, 9]). Theorem [BBH] can be extended to the above Riemann surface (see [6]).

In this paper we prove the estimate:

THEOREM A. *Let M be a Riemann surface defined before. Let u_ε be a minimizer of the functional* (1.2). *There exists a constant C independent of ε such that*

$$(1.4) \qquad \frac{1}{\varepsilon^2} \int_M (1 - |u_\varepsilon|^2)^2 W\, dM \leq C$$

uniformly in $0 < \varepsilon < \varepsilon_0$.

If $W(x) \equiv 1$ and $M = \Omega$ is star-shaped, the estimate (1.4) was first proved by Bethuel, Brezis and Hélein using the Pohozaev identity. Estimate (1.4) is one of the fundamental estimates in [3] to prove Theorem [BBH]. Srtuwe in [10] and [8] proved (1.4) for non-star-shaped domain in $\mathbb{R}^2$. We modify a method from [10], but our proof is simpler. Theorem A may allow many of the results in [3] to be extended to the case $W(x) \not\equiv 1$ (see [7]).

Finally, we give a partial answer to a problem of Bethuel, Brezis and Hélein (see open problem 7 (i) in [3]) in the following:

THEOREM B. *Let u_ε be stated as in Theorem A. Then for any $\alpha > 0$, the quantity*

$$A_\varepsilon = \int_\Omega (1 - |u_\varepsilon|)^\alpha |\nabla u_\varepsilon|^2 \, dx$$

remains bounded as $\varepsilon \to 0$.

2. Some lemmas

Since $W(x)$ is smooth on $\overline{\Omega}$ and $W(x) > 0$ on $\overline{\Omega}$, there exists a constant Q such that

$$\frac{1}{Q} \leq W(x) \leq Q \quad \text{on } \overline{\Omega}, \text{ and } \quad |\nabla W(x)| \leq Q \quad \text{on } \overline{\Omega}. \tag{2.1}$$

From [3, 8 and 6] we have

LEMMA 2.1. *There exists a constant $C_1 = C_1(\Omega, g, Q)$ such that for $0 < \varepsilon \leq 1$,*

$$E_\varepsilon(u_\varepsilon, \Omega) \leq C_1(|\ln \varepsilon| + 1).$$

LEMMA 2.2. *Any critical point $u \in H^{1,2}_g(\Omega)$ of E_ε satisfies the estimate $|u| \leq 1$ a.e. on Ω. For each $\varepsilon > 0$, let u_ε be a minimizer of the functional E_ε. Then there exists a constant $C_2 = C_2(\Omega, g, Q)$ such that*

$$|\nabla u_\varepsilon| \leq C_2 \varepsilon^{-1} \quad \text{a.e. } on\ \overline{\Omega}.$$

For $\rho > 0$ let

$$f(\rho) = f(\rho, x_0, \varepsilon, u_\varepsilon) = \rho \int_{\partial B_{\rho(x_0)} \cap \Omega} \left[\frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1 - |u_\varepsilon|^2)^2}{4\varepsilon^2} W \right] do$$

with do denoting the arc-length element on ∂B_ρ.

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LEMMA 2.3. *There are constants* $\gamma = \gamma(G, g, \delta)$ *and* $\varepsilon_0 = \varepsilon_0(\Omega, g) > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

$$\inf_{B_\rho(x_0)} |u_\varepsilon| \geq \frac{1}{2}, \qquad E_\varepsilon\left(u_\varepsilon, \Omega \cap B_\rho(x_0)\right) \leq \delta,$$

whenever $\varepsilon^{1/2} \leq \rho \leq \varepsilon^{1/4}$, $\rho \leq 1/Q^2$ *and* $f(\rho) \leq \gamma$.

PROOF. If $|u(x)| \geq 1/2$ in D, it follows that $v(x) := u(x)/|u(x)| \in H^{1,2}(\partial D)$ and

$$\int_{\partial D} |\nabla v(x)|^2\, do \leq 4 \int_{\partial D} |\nabla u(x)|^2\, do.$$

We extend v to be constant on rays from 0 on $B_\rho(x_0)\backslash\Omega$. Also let $\bar{v} = e^{i\bar{\phi}}$: $B_{\rho/8}(0) \to S^1$ be the unique harmonic map such that $\bar{v}(x) = v(\rho x/|x|)$ for $x \in \partial B_{\rho/8}(0)$. Then $\bar{v} \in H^{1,2}(B_{\rho/8}(0))$ and

$$\int_{B_{\rho/8}(0)} |\nabla \bar{v}|^2\, dx \leq C\rho \int_{\partial D} |\nabla v|^2\, do \leq Cf(\rho).$$

Finally let

$$V(x) = \begin{cases} (8/7 - 8|x|/7\rho)v(\rho x/|x|) + (8|x|/7\rho - 1/7)u(\rho x/|x|), & \text{if } \rho/8 \leq |x| \leq \rho, \\ \bar{v}(x), & \text{for } 0 \leq |x| \leq \rho/8 \end{cases}$$

to see that for sufficiently small $\gamma > 0$ we have

$$E_\varepsilon(u_\varepsilon; D) \leq E_\varepsilon(V; D) \leq Cf(\rho) \leq \delta \tag{2.2}$$

as desired.

For $0 < \varepsilon < \varepsilon_0$ and minimizers u_ε of E_ε, consider the set

$$\Sigma_\varepsilon = \{x \in G : |u(x)| < 1/2, \text{ or } E_\varepsilon\left(u_\varepsilon; G \cap B_{\varepsilon^{1/2}}(x) \geq \delta\right)\}$$

and its cover $(B_{\varepsilon/5}(x))_{x\in\Sigma_\varepsilon}$ of Σ_ε. By Vitali's covering lemma we can find a disjoint collection of balls $B_{\varepsilon/5}(x_j)$, $x_j \in \Sigma_\varepsilon$, $1 \leq j \leq J$ such that $\Sigma_\varepsilon \subset \bigcup_j B_\varepsilon(x_j)$.

LEMMA 2.4. *There exists a number* $J_0 = J_0(\Omega, g) \in \mathbb{N}$ *such that for any disjoint collection of balls* $B_{\varepsilon/5}(x_j)$, $x_j \in \Omega$, $1 \leq j \leq J$ *with* $|u_\varepsilon(x_j)| < 1/2$ *we have* $J \leq J_0$.

For each $\varepsilon > 0$ and any corresponding minimizer u_ε we fix this choice of (x_j). Given $\sigma > 0$ we denote $\Omega^\sigma = \Omega\backslash \bigcup_{j=1}^J B_\sigma(x_j)$.

LEMMA 2.5. *There exists a constant* $C_4 = C_4(\Omega, g, Q) > 0$ *such that for any* $\sigma > 0$

$$E_\varepsilon(u_\varepsilon : \Omega_\varepsilon^\sigma) \leq \pi|d||\ln\sigma| + C_4$$

uniformly in $0 < \varepsilon < \varepsilon_0$.

3. Proof of Theorem A

Without loss of generality, we consider a point $p \in \partial\Omega$ and $B_R(p) \cap \partial\Omega$. Then after a transformation we can change the problem (1.1) from $B_R(p) \cap \partial\Omega$ into a new domain $B_R^+(0)$ where

$$B_R^+(0) := \left\{x = (x^1, x^2) \in \mathbb{R}^2 : (x^1)^2 + (x^2)^2 \le R, \quad x^2 \ge 0\right\}.$$

Then the Ginzburg-Landau equation (1.3) becomes

$$\frac{\partial}{\partial x^i}\left[\bar{h}_{ij}(x)\frac{\partial u_\varepsilon}{\partial x^j}\right] = -\bar{W}(x)u_\varepsilon(1 - |u_\varepsilon|^2) \quad \text{in } B_R^+, \tag{3.1}$$

$$u(x_1, 0) = g(x), \quad \text{on } B_R^+ \cap \{x \in \mathbb{R}^2 : x^2 = 0\}, \tag{3.2}$$

where $\bar{h}_{ij}(x)$ and $\bar{W}(x)$ are smooth functions and there exists a constant Λ such that

$$\Lambda^{-1}|\xi|^2 \le \bar{h}_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \quad \text{for } \xi = (\xi_1, \xi_2),$$
$$|\nabla\bar{h}_{ij}|(x) \le \Lambda, \qquad |\nabla^2\bar{h}_{ij}|(x) \le \Lambda, \qquad |\nabla\bar{W}|(x) \le \Lambda, \qquad \Lambda \ge \bar{W}(x) \ge \Lambda^{-1}.$$

We rescale the variable x by setting $\tilde{u}(x) = u_\varepsilon(\varepsilon x)$ and $\tilde{R} = \varepsilon^{-1}R$, changing equations (3.1)-(3.2) to the form

$$\frac{\partial}{\partial x^i}\left[\bar{h}_{ij}(x)\frac{\partial \tilde{u}}{\partial x^j}\right] = -\bar{W}(\varepsilon x)\tilde{u}(1 - |\tilde{u}|^2) \quad \text{in } B_{\tilde{R}}^+, \tag{3.3}$$

$$\tilde{u}(x_1, 0) = g(\varepsilon x), \quad \text{on } B_{\tilde{R}}^+ \cap \{x \in \mathbb{R}^2 : x^2 = 0\}. \tag{3.4}$$

Next we derive a Bochner-type formula for $\tilde{u}$ by assuming that $1/2 \le |\tilde{u}| \le 1$. For simplicities, we still denote $\bar{h}_{ij}$ by h_{ij}.

A simple calculation gives

$$\begin{aligned}
&\frac{\partial}{\partial x^i}\left(h_{ij}(\varepsilon x)\frac{\partial}{\partial x^j}\left|\frac{\partial \tilde{u}}{\partial x^1}\right|^2\right) \\
&= 2h_{ij}(\varepsilon x)\frac{\partial^2\tilde{u}}{\partial x^1\partial x^i}\frac{\partial^2\tilde{u}}{\partial x^1\partial x^j} + 2\frac{\partial\tilde{u}}{\partial x^1}\frac{\partial}{\partial_1}\left[\frac{\partial}{\partial x^j} = \left(h_{ij}\frac{\partial\tilde{u}}{\partial x^i}\right)\right] \\
&\quad - 2\varepsilon^2\frac{\partial\tilde{u}}{\partial x^1}\frac{\partial^2 h_{ij}}{\partial x^i\partial x^1}\frac{\partial\tilde{u}}{\partial x^j} - 2\varepsilon\frac{\partial h_{ij}}{\partial x^1}\frac{\partial\tilde{u}}{\partial x^1}\frac{\partial^2\tilde{u}}{\partial x^i\partial x^j} \\
&:= I_1 + I_2 - I_3 - I_4.
\end{aligned}$$

It is obvious that

$$I_1 \ge 2\Lambda^{-1}\left|\frac{\partial^2\tilde{u}}{\partial x^1\partial x^i}\right|^2$$

and

$$|I_3| = 2\varepsilon^2 \left| \frac{\partial \tilde{u}}{\partial x^1} \frac{\partial \tilde{u}}{\partial x^j} \frac{\partial^2 h_{ij}}{\partial x^1 \partial x^j} \right| \le 2\varepsilon^2 \Lambda \left| \frac{\partial \tilde{u}}{\partial x^1} \right| \left| \frac{\partial \tilde{u}}{\partial x^j} \right|.$$

From equation (3.3) we obtain

$$\begin{aligned} I_2 &= 2 \frac{\partial \tilde{u}}{\partial x^1} \frac{\partial}{\partial x^1} \left[\bar{W}(\varepsilon x) \tilde{u}(|\tilde{u}|^2 - 1) \right] \\ &= 2\varepsilon \frac{\partial \tilde{u}}{\partial x^1} \frac{\partial \bar{W}}{\partial x^i} \tilde{u}(|\tilde{u}|^2 - 1) + 2 \left| \frac{\partial \tilde{u}}{\partial x^1} \right|^2 \bar{W}(\varepsilon x)(|\tilde{u}|^2 - 1) + \bar{W} \left| \frac{\partial |\tilde{u}|^2}{\partial x^1} \right|^2 \\ &\ge \varepsilon \Lambda^{-1} \left| \frac{\partial \tilde{u}}{\partial x^1} \right| (|\tilde{u}|^2 - 1) - 2\Lambda^{-1} \left| \frac{\partial \tilde{u}}{\partial x^1} \right|^2 (|\tilde{u}|^2 - 1). \end{aligned}$$

Note that

$$|I_4| = \left| 2\varepsilon \frac{\partial h_{ij}}{\partial x^1} \frac{\partial \tilde{u}}{\partial x^1} \frac{\partial^2 \tilde{u}}{\partial x^i \partial x^j} \right| \le 2\varepsilon \Lambda \left| \frac{\partial \tilde{u}}{\partial x^1} \right| \left(\left| \frac{\partial^2 \tilde{u}}{\partial^2 x^1} \right|^2 + 2 \left| \frac{\partial^2 \tilde{u}}{\partial x^1 \partial x^2} \right|^2 + \left| \frac{\partial^2 \tilde{u}}{\partial^2 x^2} \right|^2 \right).$$

By equation (3.3) we get

$$\begin{aligned} \frac{\partial^2 \tilde{u}}{\partial^2 x^2} &= \frac{1}{h_{22}} \frac{\partial}{\partial x^2} \left(h_{22} \frac{\partial \tilde{u}}{\partial x^2} \right) - \frac{1}{h_{22}} \varepsilon \frac{\partial h_{22}}{\partial x^2} \frac{\partial \tilde{u}}{\partial x^2} \\ &= \frac{1}{h_{22}} \left[\bar{W}(|\tilde{u}|^2 - 1)\tilde{u} - \frac{\partial}{\partial x^1} \left(h_{1j} \frac{\partial \tilde{u}}{\partial x^j} \right) - \frac{\partial}{\partial x^2} \left(h_{21} \frac{\partial \tilde{u}}{\partial x^1} \right) - \varepsilon \frac{\partial h_{22}}{\partial x^2} \frac{\partial \tilde{u}}{\partial x^2} \right]. \end{aligned}$$

Then

$$|I_4| \le C\varepsilon^2 \left| \frac{\partial \tilde{u}}{\partial x^1} \right| \left| \frac{\partial \tilde{u}}{\partial x^j} \right| + C\varepsilon \left| \frac{\partial \tilde{u}}{\partial x^1} \right| \left| \frac{\partial^2 \tilde{u}}{\partial x^1 \partial x^j} \right| + C\varepsilon \left| \frac{\partial \tilde{u}}{\partial x^1} \right| (1 - |\tilde{u}|^2).$$

On the other hand, applying equation (3.3) we obtain

$$\begin{aligned} \frac{\partial}{\partial x^i} \left[h_{ij}(x) \frac{\partial}{\partial x^j} (1 - |\tilde{u}|^2)^2 \right] &= (1 - |\tilde{u}|^2) h_{ij} \frac{\partial \tilde{u}}{\partial x^i} \frac{\partial \tilde{u}}{\partial x^j} + h_{ij} \frac{\partial |\tilde{u}|^2}{\partial x^i} \frac{\partial |\tilde{u}|^2}{\partial x^j} \\ &\ge \frac{\Lambda^{-1}}{4} (1 - |\tilde{u}|^2)^2 - C|\nabla \tilde{u}|^4. \end{aligned}$$

From the above argument, we obtain the following Bochner-type inequality

$$\text{(3.5)} \quad \frac{\partial}{\partial x^i} \left[h_{ij}(x) \frac{\partial}{\partial x^j} \left(\left| \frac{\partial \tilde{u}}{\partial x^1} \right|^2 + \frac{1}{8} (1 - |\tilde{u}|^2)^2 \right) \right] \ge -C \left(\varepsilon^2 + |\nabla \tilde{u}|^2 \right) |\nabla \tilde{u}|^2$$

where the constant C is independent of ε and $\tilde{R}$. Moreover we have

$$|\frac{\partial \tilde{u}}{\partial x^1}|^2 = \varepsilon^2 |\frac{\partial g}{\partial x_1}|^2 \le C\varepsilon^2 \quad \text{on } B^+_{\tilde{R}} \cap \{x \in \mathbb{R}^2 : x^2 = 0\}. \tag{3.6}$$

Similarly we have

$$\frac{\partial}{\partial x^i}\left[h_{ij}(x)\frac{\partial}{\partial x^j}\left(\left|\frac{\partial \tilde{u}}{\partial x^2}\right|^2 + \frac{1}{8}(1-|\tilde{u}|^2)^2\right)\right] \ge -C\left(\varepsilon^2 + |\nabla \tilde{u}|^2\right)|\nabla \tilde{u}|^2. \tag{3.7}$$

Using equation (3.3), we have

$$\begin{aligned}\frac{\partial}{\partial x^2}\left|\frac{\partial \tilde{u}}{\partial x^2}\right|^2 = &-\frac{2}{h_{22}}\left(h_{11}\frac{\partial^2 \tilde{u}}{\partial^2 x^1} + \varepsilon \sum_{i,j=1}^{2} \frac{\partial h_{ij}}{\partial x^i}\frac{\partial \tilde{u}}{\partial x^j}\right)\frac{\partial \tilde{u}}{\partial x^2}\\ &- \frac{2}{h_{22}}(h_{12}+h_{21})\frac{\partial \tilde{u}}{\partial x^2}\frac{\partial^2 \tilde{u}}{\partial x^1 \partial x^2}\end{aligned}$$

on $B^+_{\tilde{R}} \cap \{x \in \mathbb{R}^2 : x^2 = 0\}$. Thus we have an oblique derivative condition for $|\partial \tilde{u}/\partial x^2|^2$ on the flat boundary; that is,

$$\left|\left[\frac{\partial}{\partial x^2} + \frac{(h_{12}+h_{21})}{h_{22}}\frac{\partial}{\partial x^1}\right]\left|\frac{\partial \tilde{u}}{\partial x^2}\right|^2\right| \le C(\varepsilon^2|\nabla \tilde{u}| + \varepsilon|\nabla \tilde{u}|^2) \tag{3.8}$$

on $B^+_{\tilde{R}} \cap \{x \in \mathbb{R}^2 : x^2 = 0\}$.

THEOREM 3.1. (ε_0*-regularity*) *Let* $\tilde{u}$ *be a solution of equation* (3.3) *in* $B^+_{\tilde{R}}$ *with boundary condition* (3.4). *Assume that* $1/2 \le |\tilde{u}| \le 1$ *in* $B^+_{\tilde{R}}$ *and* $\varepsilon^{-3/4} \le R \le c\varepsilon^{-1/2}$. *Then there exists* $\eta > 0$ *such that if* $E(\tilde{u}, B^+_{\tilde{R}}) < \eta$, *then*

$$|\nabla \tilde{u}|^2 + \frac{1}{4}(1-|\tilde{u}|^2)^2 \le \frac{C}{\tilde{R}^2} \quad \text{on } B^+_{\tilde{R}}$$

where C *is a constant independent of* $\tilde{R}$.

PROOF. Set $e(\tilde{u}) = |\nabla \tilde{u}|^2 + (1-|\tilde{u}|^2)^2/4$. Choose $r_0 < \tilde{R}$ such that

$$(R - r_0)^2 \sup_{B^+_{r_0}} e(\tilde{u}) = \max_{0 \le r \le \tilde{R}}\{(\tilde{R}-1)^2 \sup_{B^+_r} e(\tilde{u})\}$$

and let $x_0 \in \bar{B}^+_{r_0}$ be determined so that

$$e_0 := e(\tilde{u})(x_0) = \sup_{B^+_{r_0}} e(\tilde{u}).$$

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Next we are going to prove $e_0 \le 4(\tilde{R} - r_0)^{-2}$. Let $\rho_0 = e_0^{-1/2}$. We suppose $\rho_0 \le (\tilde{R} - r_0)/2$.

We rescale: $v(x) = \tilde{u}(x_0 + \rho_0 x)$, $\nabla v = \rho_0 \nabla \tilde{u}$. Denote

$$D_{\rho_0} = \{x \in \mathbb{R}^2 : x_0 + \rho_0 x \in B^+_{\tilde{R}}\},$$
$$\partial^+ D_{\rho_0} = \{x \in R^2 : x_0 + \rho_0 x \in B^+_R \text{ and } x_0^1 + \rho_0 x^1 = 0\},$$
$$e_{\rho_0}(v)(x) = |\nabla v|^2 + \frac{\rho_0^2}{2}(1 - |v|^2)^2 = \rho_0^2 e(\tilde{u})(x_0 + \rho_0 x).$$

Then

$$1 = e_{\rho_0}(v)(0) = \sup_{B_1 \cap D_{\rho_0}} e_{\rho_0}(v) = \rho_0^2 \sup_{B^+_{\rho_0}} e(\tilde{u}) \le \rho_0^2 \sup_{B^+_{(\tilde{R}+r_0)/2}} e(\tilde{u}) \le 4.$$

Set

$$e^{(1)}_{\rho_0}(x) = \left|\frac{\partial v}{\partial x^1}\right|^2 + \frac{\rho_0^2}{4}(1 - |v|^2)^2 = \rho_0^2 \left[\left|\frac{\partial \tilde{u}}{\partial x^1}\right|^2 + \frac{1}{4}(1 - |\tilde{u}|^2)^2\right]$$

and

$$e^{(2)}_{\rho_0}(x) = \left|\frac{\partial v}{\partial x^2}\right|^2 + \frac{\rho_0^2}{8}(1 - |v|^2)^2 = \rho_0^2 \left[\left|\frac{\partial \tilde{u}}{\partial x^2}\right|^2 + \frac{1}{4}(1 - |\tilde{u}|^2)^2\right].$$

Then from equations (3.5)–(3.6), we have

$$\mathscr{L} e^{(1)}_{\rho_0} \ge -C(e^{(1)}_{\rho_0} + \rho_0^2 \varepsilon^2) \quad \text{in } B_1 \cap D_{\rho_0},$$
$$|e_1(v)| \le C\rho_0^2 \varepsilon^2 \quad \text{on } B_1 \cap \partial^+ D_{\rho_0},$$

where $\mathscr{L} := \partial(h_{ij}(x_0 + \rho_0 \varepsilon x)\partial/\partial x^j)/\partial x^i$ is a uniform elliptic operator. Using Moser's subsolution-estimate (see [5, Theorems 8.17 and 9.20]) we then have

$$e^{(1)}_{\rho_0}(0) \le \int_{B_1 \cap D_{\rho_0}} e_{\rho_0}(v)\,dx + C\rho_0 \varepsilon \le \int_{B^+_{\rho_0}} e(\tilde{u})\,dx + CR\varepsilon.$$

Similarly we have from equations (3.7) – (3.8)

$$\mathscr{L} e^{(2)}_{\rho_0} \ge -C(e^{(2)}_{\rho_0} + \rho_0^2 \varepsilon^2) \quad \text{in } B_1 \cap D_{\rho_0},$$
$$|\partial_\gamma e^{(2)}_{\rho_0}(v)| \le CR\varepsilon \quad \text{on } B_1 \cap \partial^+ D_{\rho_0},$$

where γ is a oblique vector on $B_1 \cap \partial^+ D_{\rho_0}$, τ is a tangent of $B_1 \cap \partial D^+_{\rho_0}$ and there exists a constant c such that $c^{-1} \le |\gamma \cdot \tau| \le c$. Then using a variation of Moser's sup-estimate (see [10]) we have

$$(*) \qquad e^{(2)}_{\rho_0}(0) \le C\left(\int_{B^+_{\rho_0}} e(\tilde{u})\,dx\right)^{2/p} + CR\varepsilon,$$

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for some $p > 2$. For completeness, we repeat a proof of the estimate ($*$) from [10]. We may assume that the oblique vector γ is the outer unit normal vector n of $B_1 \cap \partial^+ D_{\rho_0}$ by changing the variable x. Let f solve $\mathscr{L} f = -Ce^{(2)}_{\rho_0} + C\varepsilon^2\rho_0^2$, with $f = 0$ on $\partial B_1 \cap D_{\rho_0}$, $\partial_n f = \partial_n e^{(2)}_{\rho_0}$ on $B_1 \cap \partial^+ D_{\rho_0}$ and suitatble boundary conditions on the remaining parts of the boundary. Then applying Sobolev inequality, we obtain

$$\|f\|_{L^\infty} \le C\|f\|_{W^{2,p/2}} \le C(\|e^{(2)}_{\rho_0}\|_{L^{p/2}} + \rho_0^2\varepsilon^2) \le C\left[\left(\int_{B_1\cap D_\rho} e_{\rho_0}(\tilde{u})dx\right)^{2/p} + \rho_0^2\varepsilon^2\right].$$

Moreover, the function $\bar{f} = e^{(2)}_{\rho_0} - f$ solves

$$\mathscr{L}\bar{f} \ge 0 \quad \text{in } B_1 \cap D_{\rho_0},$$
$$\partial_n \bar{f} = 0 \quad \text{on } B_1 \cap \partial^+ D_{\rho_0}.$$

Extending $\bar{f}$ to B_1 by reflection in the flat boundary $\partial^+ D_\rho \cap B_1$, and applying Moser's estimate to $\bar{f}$, we obtain

$$\begin{aligned}\bar{f}(0) &\le C\int_{B_1} \bar{f}\,dx \le C\int_{B_1\cap D_{\rho_0}} e^{(2)}_{\rho_0}\,dx + C\int_{B_1\cap D_{\rho_0}} |f|\,dx \\ &\le C\left(\int_{B^+_{\rho_0}} e(\tilde{u})\,dx\right)^{2/p} + CR\varepsilon.\end{aligned}$$

Hence the desired estimate ($*$) follows.

Therefore for $\varepsilon < \varepsilon_0$, and choosing ε_0 and η small enough,

$$1 = e^{(1)}_{\rho_0}(0) + e^{(2)}_{\rho_0}(0) = e_{\rho_0}(0) \le C\left(\int_{B^+_{\tilde{R}}} e(\tilde{u})\,dx\right)^{2/p} + CR\varepsilon \le C\eta^{2/p} + C\varepsilon^{1/2} < 1.$$

This proves $e_0 \le 4(\tilde{R} - r_0)^{-2}$. Then we have $|\nabla\tilde{u}|^2 + (1 - |\tilde{u}|^2)^2/4 \le 16\tilde{R}^{-2}$. This proves Theorem 3.1.

REMARK. Theorem 3.1 holds true also for interior points of Ω_ε.

PROOF OF THEOREM A. Denote

$$\Omega^{(k)}_\varepsilon := \Omega \setminus \cup_{i=1}^J B_{\varepsilon^{1/4} - k\varepsilon^{1/2}}(x_i); \quad k = 1, 2.$$

Applying Lemma 2.3 and Theorem 3.1, we obtain

$$1 - |u_\varepsilon(x)|^2 \le C\varepsilon^{1/2} \tag{3.9}$$

uniformly for $x \in \Omega_\varepsilon^{(1)}$ and $\varepsilon \le \varepsilon_0$ where C is a constant independent of ε. By equation (1.3), we have

$$\Delta\left(\frac{1-|u_\varepsilon|^2}{2}\right) + |\nabla u_\varepsilon|^2 = \frac{1}{\varepsilon^2}|u_\varepsilon|^2(1-|u_\varepsilon|^2)W(x) \tag{3.10}$$
$$\ge \frac{1}{4Q\varepsilon^2}(1-|u_\varepsilon|^2) \quad \text{on } \Omega_\varepsilon^{(2)}.$$

Let $\phi \in C^\infty(\overline{\Omega})$ be a cut-off function satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ on $\Omega_\varepsilon^{(1)}$, $\phi \equiv 0$ on $\Omega \backslash \Omega_\varepsilon^{(2)}$, $|\nabla^k \phi| \le 2\varepsilon^{-k/2}$, $k = 1, 2$. Multiplying (3.10) by $(1-|u_\varepsilon|^2)\phi^2$ and integrating by parts gives

(3.11)
$$\begin{aligned}
&\frac{1}{4Q\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2\phi^2\,dx + \frac{1}{2}\int_\Omega |\nabla(1-|u_\varepsilon|^2)|^2\phi^2\,dx \\
&\quad\le \int_\Omega |\nabla u_\varepsilon|^2(1-|u_\varepsilon|^2)\phi^2\,dx + \frac{1}{4}\int_\Omega (1-|u_\varepsilon|^2)^2 \,\Delta\, \phi^2\,dx \\
&\quad\le \sup_{\Omega_\varepsilon^{(2)}}(1-|u_\varepsilon|^2)\int_\Omega |\nabla u_\varepsilon|^2\,dx + 4\varepsilon^{-1}\int_\Omega (1-|u_\varepsilon|^2)^2\,dx.
\end{aligned}$$

Applying Lemma 2.1 and the estimate (3.9) yields

$$\frac{1}{4\varepsilon^2}\int_{\Omega_\varepsilon^{(1)}} (1-|u_\varepsilon|^2)^2\,dx + \frac{1}{2}\int_{\Omega_\varepsilon^{(1)}} |\nabla(1-|u_\varepsilon|^2)|^2\,dx \le C$$

for $\varepsilon \le \varepsilon_0$.

From [10, Lemma 3.1] or [6, Lemma 4] we obtain

$$\int_{B_{\varepsilon^{1/4}}(x_i)\cap\Omega} \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\,dx \le C.$$

Combining this estimate with (3.11) gives $\varepsilon^{-2}\int_\Omega (1-|u_\varepsilon|^2)^2 W\,dx \le C$ uniformly in $0 < \varepsilon < \varepsilon_0$, as desired.

4. Proof of Theorem B

Let x_i $(i = 1, \ldots, J)$ be singularties as stated in Lemma 2.5. By Lemmas 2.3 and 2.5, and Theorem 1.2, we have the following properties:

$$0 < 1/2 \le |u_\varepsilon| \le 1 \quad \text{in} \quad \Omega\backslash \cup_{i=1}^J B_\varepsilon(x_j), \tag{4.1}$$

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2\,dx \le K \tag{4.2}$$

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where K is a uniform constant for $\varepsilon \le \varepsilon_0$ and $J \le J_0$. Moreover, by Lemma 2.7 we have

$$\int_{\Omega^R} |\nabla u_\varepsilon|^2\, dx \le 2\pi |d| |\ln R| + C_5. \tag{4.3}$$

Using Lemma 2.2, we have $|\nabla u_\varepsilon| \le C_2/\varepsilon$. Then

$$\sum_{j=1}^{J} \int_{B_\varepsilon(x_j)} |\nabla u_\varepsilon|^2\, dx \le J_0 \pi C_2^2. \tag{4.4}$$

For a fixed $R > 0$, denote

$$\Omega_\varepsilon^R := \bigcup_{j=1}^{J} B_R(x_j) \backslash B_\varepsilon(x_j).$$

Using (4.1), (4.3) and (4.4), it suffices to prove that the quantity $\int_{\Omega_\varepsilon^R} (1-|u_\varepsilon|)^\alpha |\nabla u_\varepsilon|^2\, dx$ remains bounded as $\varepsilon \to 0$.

As in [3], the estimate (4.1) implies that $d_j = \deg(u_\varepsilon, \partial B_\varepsilon(x_j))$ is well-defined and we consider a reference map

$$u_0(z) = \left(\frac{z-p_1}{|z-p_1|}\right)^{d_1} \left(\frac{z-p_2}{|z-p_2|}\right)^{d_2} \cdots \left(\frac{z-p_J}{|z-p_J|}\right)^{d_J}$$

where $z = x^1 + ix^2$, $p_j = x_j^1 + ix_j^2$, $j = 1, \dots, J$.

Set $\rho = |u_\varepsilon|$; we may write, locally in Ω_ε^R, $u_\varepsilon = \rho e^{i\phi}$. Similarly, we may write, locally in Ω_ε^R, $u_0 = e^{i\phi_0}$, with $|\nabla u| = |\nabla \phi_0|$ and $\nabla\phi_0(z) = \sum_j d_j V_j(z)/|z-p_j|$, where $V_j(z)$ is the unit vector tangent to the circle of radius $|z-p_j|$, centred at p_j:

$$V_j(z) = \left(-\frac{y-p_j}{|z-p_j|}, \frac{x-p_j}{|z-p_j|}\right).$$

There is a well-defined function $\psi : \Omega_\varepsilon^R \to \mathbb{R}$ such that $u_\varepsilon = \rho u_0 e^{i\psi}$ in Ω_ε. Then we have $|\nabla u_\varepsilon|^2 = |\nabla\rho|^2 + \rho^2 |\nabla\phi_0 + \nabla\psi|^2$. From [4] and [3], we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon^R} |\nabla u_\varepsilon|^2\, dx &\ge \int_{\Omega_\varepsilon^R} |\nabla\rho|^2 + \int_{\Omega_\varepsilon^R} |\nabla u_0|^2 + \frac{1}{8}\int_{\Omega_\varepsilon^R} |\nabla\psi|^2 - C \\ &\ge 2\pi |d| \ln R/\varepsilon + \int_{\Omega_\varepsilon^R} (|\nabla\rho|^2 + \frac{1}{8}|\nabla\psi|^2)\, dx - C.\end{aligned}$$

Combining this with Lemma 2.1 gives

$$\int_{\Omega_\varepsilon^R} (|\nabla\rho|^2 + |\nabla\psi|^2)\, dx \le C_6 \tag{4.5}$$

where C_6 is a constant depending on d and J_0. Therefore

$$\begin{aligned}\int_{\Omega_\varepsilon^R}&(1-|u_{|\varepsilon}|)^\alpha|\nabla u_\varepsilon|^2\,dx\\&=\int_{\Omega_\varepsilon^R}(1-|u_\varepsilon|)^\alpha(|\nabla\rho|^2+\rho^2|\nabla\phi_0|^2+2\rho^2\nabla\phi_0\cdot\nabla\psi+\rho^2|\nabla\psi|^2)\,dx\\&\le\int_{\Omega_\varepsilon^R}(1-|u_\varepsilon|)^\alpha(|\nabla\rho|^2+4|\nabla\psi|^2+4|\nabla\phi_0|^2)\,dx.\end{aligned}$$

Since $|\nabla\phi_0|=|\nabla u_0|\le\|d\|\sum_j 1/|z-p_j|$,

$$\begin{aligned}\left(\int_{\Omega_\varepsilon^R}|\nabla u_0|^{2q}\,dx\right)^{1/q}&\le C\|d\|\sum_{j=1}^{J}\left(\int_{\Omega_\varepsilon^R}\frac{1}{|z-p_i|^{2q}}\,dx\right)^{1/q}\\&\le\|d\|\sum_{j=1}^{J}\left(\int_\varepsilon^L\frac{1}{r^{2q-1}}\,dr\right)^{1/q}\le C(\varepsilon^{-2q+2})^{1/q}\end{aligned}$$

for any $q>1$ where $L:=\max_{y_1,y_2\in G}\operatorname{dist}(y_1,y_2)$.

Choose p and q such that $p=2/\alpha$ and $1/p+1/q=1$. Then by Hölder's inequality, we get

(4.6)
$$\begin{aligned}\int_{\Omega_\varepsilon^R}(1-|u|^2)^\alpha|\nabla u_0|^2\,dx&\le\left(\int_{\Omega_\varepsilon^R}(1-|u|^2)^{\alpha p}\,dx\right)^{1/p}\left(\int_{\Omega_\varepsilon^R}|\nabla u_0|^{2q}\,dx\right)^{1/q}\\&\le C\left(\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^R}(1-|u_\varepsilon|^2)^2\,dx\right)^{1/p}\left(\varepsilon^{2q/p}\varepsilon^{-2q+2}\right)^{1/q}\\&\le C.\end{aligned}$$

Combining (4.5) with (4.6) we obtain $\int_{\Omega_\varepsilon^R}(1-|u_\varepsilon|^2)^\alpha|\nabla u_\varepsilon|^2\,dx\le C_7$ where C_7 is a uniform constant for $\varepsilon<\varepsilon_0$. This proves Theorem B.

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