TONUITY: A NOVEL INDIVIDUAL-ORIENTED RETIREMENT PLAN

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ABSTRACT

For insurance companies in Europe, the introduction of Solvency II leads to a tightening of rules for solvency capital provision. In life insurance, this especially affects retirement products that contain a significant portion of longevity risk (e.g., conventional annuities). Insurance companies might react by price increases for those products, and, at the same time, might think of alternatives that shift longevity risk (at least partially) to policyholders. In the extreme case, this leads to so-called tontine products where the insurance company's role is merely administrative and longevity risk is shared within a pool of policyholders. From the policyholder's viewpoint, such products are, however, not desirable as they lead to a high uncertainty of retirement income at old ages. In this article, we alternatively suggest a so-called tonuity that combines the appealing features of tontine and conventional annuity. Until some fixed age (the switching time), a tonuity's payoff is tontine-like, afterwards the policyholder receives a secure payment of a (deferred) annuity. A tonuity is attractive for both the retiree (who benefits from a secure income at old ages) and the insurance company (whose capital requirements are reduced compared to conventional annuities). The tonuity is a possibility to offer tailor-made retirement products: using risk capital charges linked to Solvency II, we show that retirees with very low or very high risk aversion prefer a tontine or conventional annuity, respectively. Retirees with medium risk aversion, however, prefer a tonuity. In a utility-based framework, we therefore determine the optimal tonuity characterized by the critical switching time that maximizes the policyholder's lifetime utility.

KEYWORDS

Longevity risk, tontines, Solvency II, pooled annuities, capital requirements, lifetime utility.

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1. Introduction

Recent increases in life expectancy and the steep decline of interest rates have led to improved awareness of the risks contained in retirement products. In response, insurance regulation gets more importance as demonstrated, for example, by the introduction of Solvency II in Europe. Insurance companies are now forced to back their investment and longevity risk by more risk capital. This development incentivizes insurance companies to modify their products and shift risks partially to the customer: Return guarantees of (equity-linked) life insurance contracts are lowered or even omitted. Further, the replacement of conventional annuities by pooled annuities is discussed in the literature.

This article addresses the latter and discusses the product design of retirement products, focusing on longevity risk. In a first step, we compare conventional annuities to so-called tontines. Tontines were recently revisited[1] by several authors, for example, Sabin (2010), Milevsky and Salisbury (2015, 2016) or Gründl and Weinert (2016). The concept is simple: a group of policyholders provides premia in a common pool that is administered by an insurance company. At inception, the members specify a withdrawal plan. The money from this withdrawal plan is shared between surviving members. If a pool member dies, any claims on the common pool cease. At first glance, this product seems unattractive: the constant payout of an annuity is replaced by an uncertain cash flow that depends on the number of surviving members of the pool. Fluctuations are largest at old ages where a secure cash flow is most desirable. Milevsky and Salisbury (2015) show that the retiree's lifetime utility is higher for conventional annuities than for tontines, when comparing actuarially fairly priced products and ignoring risk charges. However, in Europe, Solvency II regulation forces insurers to provide solvency capital for longevity risk (see, e.g., EIOPA, 2014). We show that the attractiveness of conventional annuities is decreased if policyholders are charged for the cost of required longevity risk capital. We can find situations where a tontine is preferable over an annuity.

From the policyholder's viewpoint, it might, however, neither be desirable to be fully insured against longevity risk (and pay the resulting high risk capital charges) nor to be fully prone to longevity risk (and risk an uncertain retirement income, especially at old ages). This has inspired hybrid retirement products. So-called pooled annuity funds are frequently discussed in recent literature (see, e.g., Piggott *et al.*, 2005; Richter and Weber, 2011; Donnelly *et al.*, 2013; Qiao and Sherris, 2013; Donnelly *et al.*, 2014; Donnelly, 2015). Being very similar to a tontine, such a product includes a smoothing mechanism that updates the payout already if there is a change in expected future life expectancy of the underlying cohort. Secondly, and trivially, the policyholder might just split her retirement funds between annuity and tontine products (see, e.g., Menoncin, 2008; Huang and Milevsky, 2011; Gründl and Weinert, 2016). Inspired by the first 17th-century tontine schemes in France and Great Britain,[2] we propose a third approach. We suggest the policyholder to buy a term-tontine that provides a tontine-like payout at early ages of retirement.

After some initially specified date (the switching time), however, the tontine-like payments are switched to a constant (deferred) annuity payoff. In the remainder of this article, this new product is named "tonuity". To us, this product seems to be attractive as it combines the appealing features of tontine and annuity for insurers as well as policyholders: (1) Risk capital charges are significantly reduced. (2) The increased liquidity need in case of an unexpected increase in life expectancy is shared between policyholders and the insurance company. If there is a longevity shock, losses are covered not only by the insurance company, but also by the policyholder. Income fluctuations for the policyholder can, by design, only occur at early ages of retirement where they can be compensated by, for example, a part-time job. (3) At old ages, payout fluctuations are fully eliminated.

In this article, we make the following contributions to the literature. We follow Milevsky and Salisbury (2015) and determine the optimal payout function of a tonuity maximizing the expected lifetime utility for a retiree with constant relative risk aversion (CRRA) and no bequest motives. Instead of using fixed risk loadings, we suggest linking risk loadings to the Solvency II risk margin (RM) (following the calculation of solvency capital requirements (SCRs) as mandated by the Solvency II framework, for example, in Directive 2009/138/EC, 2009; Olivieri and Pitacco, 2009; Börger, 2010; EIOPA, 2014). This allows us to quantify the tradeoff between longevity risk charges and utility losses due to longevity risk. We distinguish tonuity products by their switching time, the extreme cases of a zero switching time being a conventional annuity and an infinite switching time being the tontine product. Choosing the switching time optimally allows us to design a tailor-made retirement product for the policyholder. Our numerical results suggest that tonuities are preferred by retirees with medium risk aversion, while policyholders with low or high risk aversion would buy a tontine or annuity, respectively.

The remainder of the article is organized as follows. In Section 2, we describe the payoffs of annuities and tontines and determine the optimal annuity and tontine payoff if policyholders apply a CRRA utility function. Then, we show how to compute risk capital charges under European Solvency II regulation. In Section 3, we discuss (dis)advantages of annuity and tontine. To combine the appealing features of the two products, Section 4 introduces the novel tonuity product. We derive the tonuity payoff (including the optimal switching time) maximizing the policyholder's expected lifetime utility. A numerical analysis and comparison of annuity, tontine and tonuity is carried out in Section 5. Finally, we provide some concluding remarks in Section 6 and detailed proofs in Appendix A.

2. Tontines and annuities: payoff and optimality

In this section, we first review two retirement products, namely tontines and annuities. Both products are bought at the time of retirement by retirees aged x. At contract initiation, the retirees provide a single premium P_0 to the insurance

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company. Depending on their residual lifetime, they receive payments during their retirement phase (see Section 2.1). While in an annuity product, mortality risk is fully borne by the insurance company, in a tontine product, mortality risk of n policyholders is pooled and (mainly) borne by the policyholders. We make the assumption that policyholders within one tontine pool are of the same age x. Further, they have, conditional on the evolution of the underlying mortality law, identical and independent distributions of remaining lifetimes and identical longevity risk preferences. Following Solvency II regulation, the insurance company needs to provide risk capital in order to cover the products' longevity risk. To cover at least the cost of capital (CoC) provision, the insurer may demand a single charge F_0 at contract initiation, in addition to the net premium P_0 (see Section 2.4).

With respect to mortality, we commonly distinguish two different types of risk: firstly, there is idiosyncratic (or unsystematic) mortality risk. We understand this as the risk stemming from the fact that the individual's lifetime is uncertain but follows some population-wide mortality law. As such, this risk can be diversified by choosing the pool size n large enough. Secondly, there is aggregate (or systematic) mortality risk. This risk comes from the fact that we cannot determine the true underlying mortality law with certainty: this might, for example, be caused by unexpected medical progress or changes in lifestyle. As this type of risk changes the mortality law for the population as a whole, it cannot be diversified by simply choosing the pool size n large enough. If aggregate mortality risk increases the expected lifetime, this is also referred to as longevity risk. Typical patterns of improvement of human lifetime are named by terms like compression, extension or rectangularization. This describes that people get on average older; however, these improvements cannot help individuals to exceed some (unknown) maximum age. Then, less people die at young ages (extension) and deaths accumulate at old ages (compression). This leads to a rectangular shape of the survival curve (see, e.g., Fries, 1980).

We use ${}_tp_x$ to denote the best-estimate t-year survival probability of an x-year-old policyholder, according to some continuous time mortality law, obtained from, for example, publicly available life tables. To account for uncertainty in this mortality law (i.e., systematic or aggregate mortality risk), we are inspired by, for example, Lin and Cox (2005) and apply a mortality shock ϵ to our best-estimate survival curve ${}_tp_x$ to obtain the shocked survival curve ${}_tp_x^{1-\epsilon}$. This way, we provide consistency in our survival probabilities: we can still guarantee that (a) ${}_tp_x^{1-\epsilon} \in (0, 1]$ for all $t \geq 0$ and (b) for any $0 \leq s < t$, survival probabilities fulfill the property ${}_tp_x^{1-\epsilon} = {}_sp_x^{1-\epsilon} \cdot {}_{t-s}p_{x+s}^{1-\epsilon}$, where ${}_{t-s}p_{x+s}^{1-\epsilon}$ denotes the survival curve of a person aged $x+s$. From this decomposition, we see that the shock occurs immediately and applies to all future points in time. In general, we will assume that ϵ is a (continuous) random variable with density $f_\epsilon(\varphi)$ and support on $(-\infty, 1)$. If $\epsilon < 0$, this leads to survival probabilities that are lower than the best-estimate ${}_tp_x$. If $\epsilon \in (0, 1)$, people live on average longer than expected. We later calibrate the shock distribution ϵ such

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that (on average) we obtain our best-estimate curve, that is, $\mathbb{E}[{}_tp_x^{1-\epsilon}] \approx {}_tp_x$. The moment-generating function of the mortality shock ϵ is for $s \geq 0$ denoted by

$$m_\epsilon(s) := \mathbb{E}\,[e^{s\epsilon}]. \tag{2.1}$$

Now, given some realization φ of the longevity shock ϵ, we denote the policyholder's remaining lifetime by ζ_φ. The indicator $\mathbb{1}_{\{\zeta_\varphi > t\}}$ that is 1 if the policyholder survives time t (and 0 otherwise) is then assumed to follow a Bernoulli $({}_tp_x^{1-\varphi})$ distribution. Exploiting the (conditional) independence of the pool member's remaining lifetime, the number of pool members at time t is, given some realization φ of the mortality shock ϵ, predicted to be binomially distributed, that is, $N_\varphi(t) \sim \text{Bin}\,(n, {}_tp_x^{1-\varphi})$. Given the realization of the mortality shock φ, we are only left with unsystematic mortality risk that can be fully removed by a suitably large pool size. The unconditional remaining lifetime is in the following denoted by ζ_ϵ, and the unconditional number of pool members by $N_\epsilon(t)$.

Throughout this paper, quantities with $^{[\infty]}$ refer to tontines whereas $^{[0]}$ refer to annuities. For universal expressions that apply to both types of contracts, these symbols are dropped.

2.1. Payoff to policyholders

In an annuity contract, any policyholder continuously receives an annuity payment $c(t)$ until death. This payment stream can be written as:

$$b_{[0]}(t) := \mathbb{1}_{\{\zeta_\epsilon > t\}}\, c(t). \tag{2.2}$$

In a tontine contract, mortality risk (both systematic and unsystematic) is shared between a pool of $n \geq 1$ policyholders, while in an annuity it is borne by the insurance company.[3] If, at time $t > 0$, the number of policyholders is $N_\epsilon(t)$, each policyholder receives $n/N_\epsilon(t)$ multiplied by an initially specified payment stream $d(t)$. Following Milevsky and Salisbury (2015), this leads to the following continuous payment stream:

$$b_{[\infty]}(t) := \begin{cases} \mathbb{1}_{\{\zeta_\epsilon > t\}} \dfrac{nd(t)}{N_\epsilon(t)} & \text{if } N_\epsilon(t) > 0, \\ 0, & \text{else} \end{cases}. \tag{2.3}$$

Note that, in contrast to the annuity payment (2.2), the tontine payment depends substantially on the number of policyholders $N_\epsilon(t)$. In case the size of the pool is one ($n = 1$) and $c(t) = d(t)$, the tontine payoff (2.3) equals the payoff of an annuity (2.2).

2.2. Net Premium

First, ignoring the risk capital charge F_0, we determine the net premium of both contracts. To focus on the effects of mortality risk, we ignore financial risks. Therefore, we do not model the assets of the insurance company and simply assume that the premium earns a constant, continuously compounded, risk-free rate r. In this section, we first compute a fair (net) premium P_0 that is computed without any longevity risk loadings and is used to determine the optimal annuity and tontine payout functions. For an annuity, this net premium is given by

$$P_0 = P_0^{[0]} := \mathbb{E}\left[\int_0^\infty e^{-rt} b_{[0]}(t)\,\mathrm{d}t\right] = \int_0^\infty e^{-rt}\mathbb{E}\left[\mathbb{1}_{\{\zeta_\epsilon > t\}}\right] c(t)\,\mathrm{d}t$$
$$= \int_0^\infty e^{-rt} c(t) \int_{-\infty}^1 {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}t = \int_0^\infty e^{-rt} c(t) {}_tp_x \cdot m_\epsilon(-\log {}_tp_x)\,\mathrm{d}t. \tag{2.4}$$

Similarly, the time $t = 0$ net premium of the tontine contract is given by (see the following Remark 1)

$$P_0 = P_0^{[\infty]} := \mathbb{E}\left[\int_0^\infty e^{-rt} b_{[\infty]}(t)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty e^{-rt}\mathbb{1}_{\{\zeta_\epsilon > t\}}\frac{nd(t)}{N_\epsilon(t)}\,\mathrm{d}t\right]$$
$$= \int_0^\infty e^{-rt}\mathbb{E}\left[{}_tp_x^{1-\epsilon}\,\mathbb{E}\left[\frac{nd(t)}{N_\varphi(t)}\,\middle|\,\zeta_\varphi > t,\ \epsilon = \varphi\right]\right]\mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\mathbb{E}\left[{}_tp_x^{1-\epsilon}\cdot\sum_{k=0}^{n-1}\frac{nd(t)}{k+1}\binom{n-1}{k}({}_tp_x^{1-\epsilon})^k(1-{}_tp_x^{1-\epsilon})^{n-1-k}\right]\mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\mathbb{E}\left[\sum_{k=1}^{n}\binom{n}{k}({}_tp_x^{1-\epsilon})^k(1-{}_tp_x^{1-\epsilon})^{n-k}\right]d(t)\,\mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\mathbb{E}\left[1-(1-{}_tp_x^{1-\epsilon})^n\right]d(t)\,\mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\int_{-\infty}^1 \left(1-(1-{}_tp_x^{1-\varphi})^n\right) f_\epsilon(\varphi)\,\mathrm{d}\varphi\ d(t)\,\mathrm{d}t, \tag{2.5}$$

where in the third line we used that, conditional on $\zeta_\varphi > t$, the number of pool members $N_\varphi(t)$ is at least 1. Then, from the perspective of one surviving pool member and conditional on a mortality shock φ, the total pool size is distributed according to $N_\varphi(t) - 1 \sim \text{Bin}\,(n-1, {}_tp_x^{1-\varphi})$.

Remark 1 (Relation to Milevsky and Salisbury, 2015). *Our net premium* (2.5) *slightly differs from Milevsky and Salisbury* (2015)*, who instead obtain the (higher) premium*

$$P_0 = \int_0^\infty e^{-rt} d(t) \mathrm{d}t. \tag{2.6}$$

The difference results from the fact that in case of (2.6)*, the net premium is set such that $N_\epsilon(t) = 0$ is excluded, that is, there is always some participants that live forever. Hence, the insurer does not bear any longevity risk and always retains a systematic profit. In contrast, our premium* (2.5) *is actuarially fair and thus leaves some longevity risk with the insurer, that is, the risk related to the time of death of the last survivor. The approximation of* (2.5) *by* (2.6) *is plausible for large portfolios.*

2.3. Optimality of $c(t)$ and $d(t)$

What still needs to be determined are the optimal payout functions $c(t)$ and $d(t)$ of the two retirement products. Therefore, we assume that policyholders are risk-averse with regard to longevity risk and evaluate their payoff streams using CRRA utility (see Assumption 1 for a formal introduction).

Assumption 1 (Utility function). *The policyholder evaluates consumption by using a CRRA utility function, that is*

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{2.7}$$

where $x \geq 0$ is the payment the policyholder receives and $\gamma \in (0, \infty) \setminus \{1\}$ is the policyholder's relative risk aversion.

Following, for example, Yaari (1965), we make the following assumption about the policyholder's preferences:

Assumption 2 (Policyholder's preferences). *Optimally, a rational retiree without bequest motives would choose the product payoff $b(t)$ that maximizes the expected discounted lifetime utility*

$$\mathbb{E}\left[\int_0^{\zeta_\epsilon} e^{-\eta t}\, u\big(b(t)\big)\, \mathrm{d}t \right] \tag{2.8}$$

subject to the constraint that the initially provided net premium of the annuity and tontine is given by (2.4) *and* (2.5)*, respectively. The utility function u is as introduced in Assumption 1 and the policyholder's subjective discount rate is η.*

A policyholder whose preferences are described by (2.7) and (2.8) can choose optimal payout functions $c^*(t)$ and $d^*(t)$ of annuity and tontine, respectively. The result is given by Theorem 2. For deterministic survival functions, it has been obtained by Milevsky and Salisbury (2015).

Theorem 2 (Optimal payout function: annuity and tontine). *Assume that the policyholder's preferences for consumption can be described by Assumption 2.*

(a) For an annuity product, maximizing the expected discounted lifetime utility (2.8) subject to the constraint (2.4) leads to

$$c^*(t) = e^{\frac{1}{\gamma}(r-\eta)t} \cdot P_0 \cdot \left(\int_0^\infty e^{\left(\frac{r-\eta}{\gamma}-r\right)t} {}_tp_x \cdot m_\epsilon(-\log {}_tp_x)\,\mathrm{d}t \right)^{-1}. \tag{2.9}$$

The policyholder's expected discounted lifetime utility is then given by

$$U^{[0]} := \int_0^\infty e^{-\eta t} {}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \cdot u\big(c^*(t)\big)\,\mathrm{d}t. \tag{2.10}$$

(b) For a tontine product, maximizing the expected discounted lifetime utility (2.8) subject to the constraint (2.5) leads to

$$d^*(t) = \frac{e^{\frac{1}{\gamma}(r-\eta)t}}{(\lambda^*)^{\frac{1}{\gamma}}} \cdot \frac{\big(\kappa_{n,\gamma,\epsilon}({}_tp_x)\big)^{\frac{1}{\gamma}}}{\left(\int_{-\infty}^{1} \big(1-\big(1-{}_tp_x^{1-\varphi}\big)^n\big) f_\epsilon(\varphi)\,\mathrm{d}\varphi\right)^{\frac{1}{\gamma}}}, \tag{2.11}$$

where

$$\kappa_{n,\gamma,\epsilon}({}_tp_x) := \sum_{k=1}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^{\gamma} \int_{-\infty}^{1} \big({}_tp_x^{1-\varphi}\big)^k \big(1-{}_tp_x^{1-\varphi}\big)^{n-k} f_\epsilon(\varphi)\,\mathrm{d}\varphi, \tag{2.12}$$

$$\lambda^* := \left(\frac{1}{P_0} \int_0^\infty e^{\left(\frac{r-\eta}{\gamma}-r\right)t} \frac{\big(\kappa_{n,\gamma,\epsilon}({}_tp_x)\big)^{\frac{1}{\gamma}}}{\left(\int_{-\infty}^{1} \left(1-\left(1-{}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\,\mathrm{d}\varphi\right)^{\frac{1}{\gamma}-1}}\,\mathrm{d}t \right)^{\gamma}.$$

For $\gamma \in \mathbb{N}\backslash\{1\}$, we can exploit that $n^\gamma \cdot \kappa_{n,\gamma,\epsilon}$ is the γth moment of a binomial distribution, that is

$$\kappa_{n,\gamma,\epsilon}({}_tp_x) = \frac{1}{n^\gamma} \sum_{l=1}^{\gamma} a_l \cdot {}_tp_x^l \cdot m_\epsilon(-l \log {}_tp_x), \tag{2.13}$$

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with real coefficients a_l defined in Lemma 4 in Appendix A.1. The policyholder's expected discounted lifetime utility is then given by

$$U^{[\infty]} := \int_0^\infty e^{-\eta t} \kappa_{n,\gamma,\epsilon}({}_t p_x) \cdot u\big(d^*(t)\big)\, \mathrm{d}t. \tag{2.14}$$

Proof. See Appendix A.1. Note that the implementation of (2.12) gets challenging for portfolio sizes $n \geq 100$ as then the binomial coefficients $\binom{n}{k}$ are difficult to compute. Equation (2.13), in contrast, can easily be implemented also for large portfolio sizes. If one is interested in non-integer valued risk-aversion coefficients γ, it might make sense to approximate (2.12) by (2.13) using suitable interpolation techniques.

By allowing the subjective discount rate η to differ from the risk-free interest rate r, the optimal annuity payout function $c^*(t)$ is not constant over time. For $\eta = r$ and deterministic survival distributions (i.e., $\epsilon \equiv 0$), we are back to the result of Milevsky and Salisbury (2015). Note that the optimal payout functions $c^*(t)$ and $d^*(t)$ are derived by considering the net premium P_0. In the subsequent section, we will add risk capital charges F_0 that depend on the (longevity) risk the insurance provider is taking by issuing a policy. It would of course be desirable to choose the optimal payout functions taking into account also the risk capital charges. This, however, does not lead to analytic solutions and is computationally unfeasible. In other words, the optimal benefit amount computed here is a simplification to avoid the untractability caused by the risk margins. We are implicitly assuming that we assess the optimal benefit amount before computing the charges for longevity risk! ■

2.4. Risk capital charges

So far, the premiums derived in Section 2.2 are actuarially fair on a net basis. We neglect any administration or acquisition charges, but, to fairly compare our retirement products, we add risk charges for longevity risk that compensate the insurance company for the risks taken. The RM ensures that "the value of technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations" (see EIOPA, 2014's technical provisions TP 5.2.). We see this as an alternative to risk premia for longevity risk that are often incorporated in the literature (see e.g., Bauer *et al.*, 2010 for a comparison of different approaches).

We therefore determine the CoC provision for longevity risk following Solvency II regulation in Europe (see also Pitacco *et al.*, 2009; EIOPA, 2014).[4] These costs are then paid as an additional charge F_0 at contract initiation.

Generally, the amount of regulatory capital required by Solvency II standards is consistent with a Value-at-Risk assessment of the basic own funds at a 99.5% confidence level on a 1-year time horizon, see Article 101 of Directive

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2009/138/EC (2009). For the longevity risk sub-module, the standard approach is to consider an instantaneous and permanent 20% decrease in annual death probabilities, see EIOPA (2014). As our payout functions require a continuous survival curve, we need to calibrate our stochastic mortality shock ϵ to the Solvency II annual shock, see Section 3. Following Olivieri and Pitacco (2009), the SCR at any time $t \geq 0$ can be computed as

$$\mathrm{SCR}(t) = \mathrm{BEL}(t \mid \mathrm{shock}) - \mathrm{BEL}(t \mid *), \tag{2.15}$$

that is, as the difference between the value of liabilities subject to a longevity shock $\mathrm{BEL}(t \mid \mathrm{shock})$ and the best-estimate $(\mathrm{BEL}(t \mid *))$ of mortality. Note that, seen from time 0, SCRs at time t are actually random variables, as they depend on the evolution of mortality up to time t. We follow Börger (2010) and take the assumption that mortality, up to time t, evolves according to best-estimate assumptions, that is, the probability to survive time t is given by ${}_tp_x$. Then, at any time $t > 0$, the insurer has to fulfill the capital requirements for $n \cdot {}_tp_x$ contracts. Survival probabilities after time t are then subject to the mortality shock ϵ. Given a realization φ of the mortality shock, the survival curve after time t is then given by $\{{}_{s-t}p_{x+t}^{1-\varphi}\}_{s\geq t}$. Hence, the (per contract) best-estimate of liabilities at time t is given by

$$\mathrm{BEL}^{[0]}(t \mid *) = {}_tp_x \int_t^{\infty} e^{-r(s-t)} \mathbb{E}\left[{}_{s-t}p_{x+t}^{1-\epsilon}\right] \cdot c^*(s)\,\mathrm{d}s. \tag{2.16}$$

For the tontine, the (per contract) best-estimate of liabilities is given by

$$\mathrm{BEL}^{[\infty]}(t \mid *) = {}_tp_x \int_t^{\infty} e^{-r(s-t)} \mathbb{E}\left[\left(1 - \left(1 - {}_{s-t}p_{x+t}^{1-\epsilon}\right)^n\right)\right] \cdot d^*(s)\,\mathrm{d}s. \tag{2.17}$$

Next, we want to compute 99.5% quantiles of the time-t liability value of annuity and tontine. For a realization φ of the shock for future mortality ϵ, we obtain for the annuity a time-t liability value of

$$\mathrm{V}^{[0]}(t, \varphi) = {}_tp_x \int_t^{\infty} e^{-r(s-t)}\, {}_{s-t}p_{x+t}^{1-\varphi} \cdot c^*(s)\,\mathrm{d}s \tag{2.18}$$

and for the tontine

$$\mathrm{V}^{[\infty]}(t, \varphi) = {}_tp_x \int_t^{\infty} e^{-r(s-t)} \left(1 - \left(1 - {}_{s-t}p_{x+t}^{1-\varphi}\right)^n\right) \cdot d^*(s)\,\mathrm{d}s. \tag{2.19}$$

Both (2.18) and (2.19) are strictly increasing functions in φ. For this reason, the 99.5% quantile of the time-t liability value can be expressed by the 99.5% quantile of the mortality shock ϵ, that is, $\mathrm{BEL}^{[0]}(t \mid \mathrm{shock}) = \mathrm{V}^{[0]}(t, z_{0.995})$ and $\mathrm{BEL}^{[\infty]}(t \mid \mathrm{shock}) = \mathrm{V}^{[\infty]}(t, z_{0.995})$, respectively.[5]

According to EIOPA (2014), we now determine the RM for the SCRs:

$$\mathrm{RM} = \mathrm{CoC} \cdot \sum_{t=0}^{\infty} e^{-r(t+1)} \cdot \mathrm{SCR}(t), \tag{2.20}$$

where CoC is the cost of capital rate. The initial single charge for longevity risk is then set to $F_0 = \mathrm{RM}$. For a tontine, we denote this charge by $F_0^{[\infty]}$ and for an annuity by $F_0^{[0]}$.

3. Numerical I: (Dis)advantages of the retirement products

In the following, we give a numerical example to demonstrate the (dis)advantages of tontines and annuities during retirement. The policyholder initially pays the risk capital charge F_0 that is given by the RM (2.20) in addition to the actuarially fair net premium P_0.

In the present paper, we only deal with post-retirement mortality, which is typically at and above age 60 (or 65). Richards (2012) has studied a large number of analytic mortality laws and found that the quality of a given law depends heavily on the age interval it is used for. According to Richards (2012), the Gompertz law, Gompertz (1825), works best for ages 60–90 and is hence used in this paper as it describes the most important age group for retirement products. We use the parameterization for the mortality intensity given by

$$\mu_{x+t} = \frac{1}{b} e^{\frac{x+t-m}{b}}, \tag{3.1}$$

where $b > 0$ is the dispersion coefficient and $m > 0$ the modal age at death, following Gumbel (1958) and, for example, Milevsky and Salisbury (2015). Best-estimate survival probabilities can then be derived to

$$ {}_tp_x := e^{-\int_0^t \mu_{x+s}\,\mathrm{d}s} = e^{e^{\frac{x-m}{b}}\left(1-e^{\frac{t}{b}}\right)}.$$

We follow Milevsky and Salisbury (2015) and set the Gompertz parameters to $m = 88.721$ and $b = 10$. We further assume that the mortality shock ϵ follows a truncated normal distribution on $(-\infty, 1)$, that is, $\mathcal{N}_{(-\infty,1)}(\mu, \sigma^2)$. The parameters μ and σ are determined as follows: (1) The mortality shock ϵ is chosen such that the expected survival probabilities $\mathbb{E}\left[{}_tp_x^{1-\epsilon}\right]$ are close to the best-estimate survival probabilities ${}_tp_x$. (2) Reducing the 1-year death probabilities by 20% (as described in the Solvency II standard approach), we obtain a shocked survival curve ${}_tp_x^{\text{SolvII shock}}$ on a discrete grid $t = 1, 2, \ldots, T$, with $T = 55$. On these annual dates, we want this curve to be close to the 99.5% quantile of our internal mortality shock model ${}_tp_x^{1-z_{0.995}}$, where $z_{0.995}$ is the 99.5% quantile of the shock magnitude ϵ. We equally weigh these two objectives and minimize squared errors of the differences, that is, we use the objective function

$$\min_{\mu,\sigma} \sum_{t=1}^{T} \left| {}_tp_x - \mathbb{E}\left[{}_tp_x^{1-\epsilon}\right] \right|^2 + \left| {}_tp_x^{\text{SolvII shock}} - {}_tp_x^{1-z_{0.995}} \right|^2. \tag{3.2}$$

For our parameter set, we obtain $\mu = -0.0035$ and $\sigma = 0.0814$ with a sum of squared errors of about $6.4 \cdot 10^{-5}$. Then, 0.5% and 99.5% quantiles of the mortality shock are −21.4% and 20.7%, respectively. As Figure 1 shows, a

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TABLE 1

BASE-CASE PARAMETER SET.

Net premium	Pool size	Risk aversion
$P_0 = P_0^{[0]} = P_0^{[\infty]} = 10{,}000$	$n = 100$	$\gamma = 10$
Risk-free rate	Subjective discount rate	CoC rate
$r = 4\%$	$\eta = 4\%$	CoC = 6%[a]
Initial age	Gompertz law	Mortality shock $\epsilon \sim \mathcal{N}_{(-\infty,1)}(\mu, \sigma^2)$
$x = 65$	$m = 88.721$, $b = 10$	$\mu = -0.0035$, $\sigma = 0.0814$

[a]The CoC is typically prescribed by the regulator (see, e.g., EIOPA, 2014). As the level is oriented at the credit spread of a BBB-rated company, the value might change over time or company-specific rates could be implemented. We therefore analyze the impact for various CoC rates.

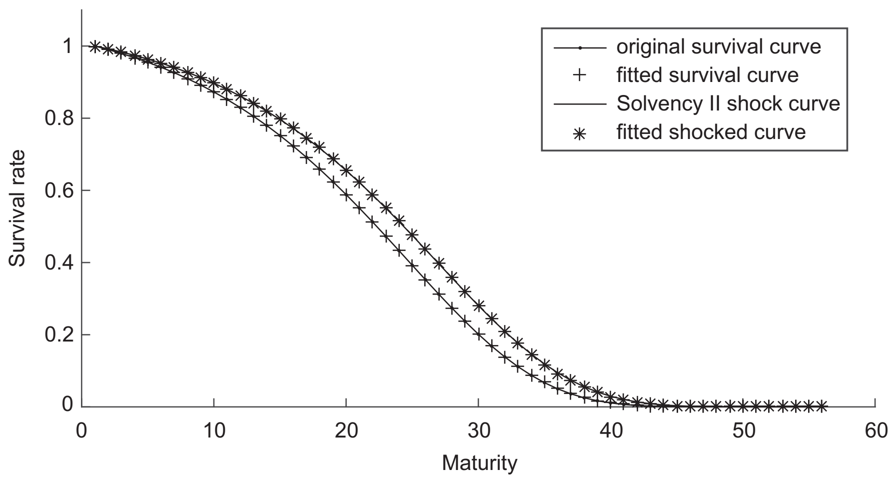

FIGURE 1: Best-estimate survival curve and quantile function of truncated normally distributed shocks compared with original survival curve and Solvency II shock.

truncated normally distributed shock magnitude allows for a very good fit of best-estimate and shocked survival curves in the relevant range of contract durations. In the following, we consider the base-case parameter set as stated in Table 1.

We can now compute the RMs (or risk capital charges F_0) for the products under consideration, see Table 2. We choose the base-case parameter set from Table 1. Even for small pool sizes of $n = 10$, the risk charge for a tontine product is small compared to the net premium $P_0 = 10{,}000$. The capital charge for an annuity contract amounts to a relevant share of 4.8% of the initial contribution. Overall, the risk capital charges for a tontine are much lower than those of an annuity.

TABLE 2

RISK CAPITAL CHARGE F_0 FOR DIFFERENT POOL SIZES n, USING THE BASE-CASE PARAMETER SET FROM TABLE 1.

Product		Risk capital charge F_0
Tontine	$n=10$	101.32
	$n=100$	10.89
	$n=1000$	1.33
Annuity		483.51

The increased (aggregate) mortality risk of a tontine, however, needs to be reflected by a utility loss. We therefore consider annuities and tontines with net premium $P_0 = P_0^{[0]} = P_0^{[\infty]} = 10{,}000$. We then determine the number of tontines $\mathrm{CEQ}_{[\infty]}$ that yield the same expected discounted lifetime utility as one annuity, that is, we solve

$$U^{[0]} \stackrel{!}{=} \int_0^{\infty} e^{-\eta t} \cdot \mathbb{E}\left[u\left(\mathrm{CEQ}_{[\infty]} \cdot b_{[\infty]}(t)\right)\right] \, \mathrm{d}t = \mathrm{CEQ}_{[\infty]}^{1-\gamma} \cdot U^{[\infty]} \tag{3.3}$$

for $\mathrm{CEQ}_{[\infty]}$, using that $\mathrm{CEQ}_{[\infty]}$ tontines have a payoff of $\mathrm{CEQ}_{[\infty]} \cdot b_{[\infty]}(t)$. From (3.3), we obtain

$$\mathrm{CEQ}_{[\infty]} := \left(\frac{U^{[0]}}{U^{[\infty]}}\right)^{\frac{1}{1-\gamma}} \tag{3.4}$$

and denote this term by relative certainty equivalent. The relative certainty equivalent is the number of tontines with net premium $P_0^{[\infty]} = 10{,}000$ that needs to be bought in order to receive the same expected discounted lifetime utility as from an annuity with net premium $P_0^{[0]} = 10{,}000$. Stated differently, the policyholder is indifferent between one annuity and $\mathrm{CEQ}_{[\infty]}$ tontines. Among products with identical expected discounted lifetime utility, the policyholder would choose the one with the lowest gross premium. That is why, in the following analysis, we compare gross premia of $\mathrm{CEQ}_{[\infty]}$ tontines (i.e., $\mathrm{CEQ}_{[\infty]} \cdot (P_0 + F_0^{[\infty]})$) and one annuity (i.e., $P_0 + F_0^{[0]}$). As in Table 2, Table 3 compares tontines with different pool sizes to annuities. Each product presented in this table maintains the same utility level for the policyholder. The lower the gross premium presented, the more attractive is the product for the policyholder. In Table 3, we highlight the lowest gross premium in bold letters. We observe that without risk capital charges (i.e., CoC = 0%), the tontine's gross premium is always higher than the annuity's gross premium $P_0 = 10{,}000$. This result suggests that, neglecting risk capital charges (CoC = 0%), the policyholder clearly prefers the (cheaper) annuity to a tontine product. This is reversed, if CoC rates are positive and tontine pool sizes are $n = 100$ or $n = 1000$: here, tontine products are cheaper (and thus more attractive) than an annuity.

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TABLE 3

GROSS PREMIUM OF $CEQ_{[\infty]}$ TONTINES COMPARED TO THE GROSS PREMIUM OF AN ANNUITY FOR DIFFERENT CoC RATES AND TONTINE POOL SIZES, USING THE BASE-CASE PARAMETER SET FROM TABLE 1. THE PRESENTED PRODUCTS YIELD THE SAME EXPECTED UTILITY FOR THE POLICYHOLDER, WE HIGHLIGHT THE LOWEST GROSS PREMIUM IN BOLD LETTERS. SEE ALSO (3.4).

Product		CoC = 0%	CoC = 2%	CoC = 4%	CoC = 6%	CoC = 8%
	$n = 10$	11,223	11,261	11,299	11,337	11,375
$CEQ_{[\infty]}$ Tontines	$n = 100$	10,273	10,277	10,281	10,284	10,288
	$n = 1000$	10,103	**10,103**	**10,104**	**10,104**	**10,105**
One annuity		**10,000**	10,161	10,322	10,484	10,645

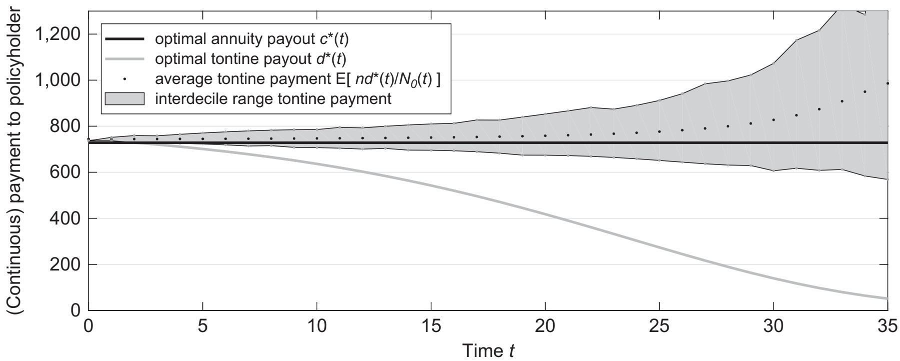

FIGURE 2: Continuous payment stream during the retirement phase using the base-case parameter set from Table 1.

The above analysis indicates that tontines are a useful tool to reduce mortality risk exposure from the insurer's perspective. The policyholder may weigh the tradeoff between the exposure to (aggregate) mortality risk and the risk capital charges. Her preference depends, among others, on longevity risk aversion γ and the CoC rate. Furthermore, he has to be aware of one disadvantage of tontine products: the uncertainty in payoff streams is increasing with age. This is demonstrated in Figure 2 where both the optimal annuity payment $c^*(t)$ (black line) and the optimal tontine payout $d^*(t)$ are given. The tontine payouts $n \cdot d^*(t)$ are shared within the policyholder pool. We thus also give the distribution of the payments to a single surviving policyholder $n \cdot d^*(t)/N_0(t)$ (the gray area presents the interdecile range of these payments, that is, at any time $t \in [0, 35]$ the upper (lower) edge represents a 90% (10%) quantile of the payment to the policyholder). For $t = 25$, that is, at the age of 90, this range is already $[770, 970]$ and thus very unattractive for the risk-averse policyholder.

In the next section, we introduce a product that unites the favorable characteristics of annuities (stable payments over time from the perspective of a policyholder) and tontines (reduced capital requirements from the perspective of an insurer). We refer to this product by the term "tonuity".

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4. Combined product: tonuity

In the previous section, we have shown that both annuities and tontines have desirable characteristics, but also drawbacks. While annuities require a significantly higher portion of risk capital than tontines, their constant and secure payment stream at old ages is preferable to a tontine payout. This suggests combining the two contracts to a tonuity that, up to an initially specified future time point, behaves as a tontine product and afterwards changes to a constant payment of a (deferred) annuity.

- From the policyholder's viewpoint, the additional mortality risk exposure (compared to annuities) reduces capital charges included in the gross premium. This makes the product more attractive. If the policyholder is in good health, he can make up possibly occurring financial losses by, for example, a part-time job.
- From the insurance company's perspective, the longevity risk exposure and required risk capital are lower than a simple annuity contract. Further into the future, when the portfolio has considerably shrunken, insurers may compensate deviations from expected mortality patterns more easily. With a then relatively small portfolio, losses due to a potential deviation from best-estimate survival probabilities might better meet the longevity risk appetite of the insurer.

The differences among the diverse products are summarized in Table 4. More formally, we denote the switching time by $\tau > 0$. This switching time is fixed at contract initiation. In $[0, \tau]$, the contract has a tontine-like payoff. After time τ, the payout is designed as a (deferred) annuity. Let $d_{[\tau]}(t)$ denote the tontine payout function and $c_{[\tau]}(t)$ the continuous (deferred) annuity payment. Hence, a policyholder receives at time t the payment stream of a tonuity:

$$b_{[\tau]}(t) := \mathbb{1}_{\{0 \le t < \min\{\tau, \zeta_\epsilon\}\}} \frac{n d_{[\tau]}(t)}{N_\epsilon(t)} + \mathbb{1}_{\{\tau \le t < \zeta_\epsilon\}} \, c_{[\tau]}(t), \tag{4.1}$$

where ζ_ϵ is again the residual lifetime of the policyholder.

We assume that the payout functions $d_{[\tau]}(t)$ and $c_{[\tau]}(t)$ are fixed at the initiation of the contract. Note that annuity and tontine are just special cases of a tonuity with $\tau = 0$ and $\tau = \infty$, respectively. The (fair) net premium for the tonuity is given by

$$\begin{aligned} P_0 = & \int_0^\tau e^{-rt} \int_{-\infty}^1 \left(1 - (1 - {}_tp_x^{1-\varphi})^n\right) f_\epsilon(\varphi) \, \mathrm{d}\varphi \, d_{[\tau]}(t) \, \mathrm{d}t \\ & + \int_\tau^\infty e^{-rt} {}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \, c_{[\tau]}(t) \, \mathrm{d}t. \end{aligned} \tag{4.2}$$

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TABLE 4

COMPARISON OF PRODUCT TYPES: PERSPECTIVES OF POLICYHOLDERS AND INSURERS.

	Policyholder	Insurer
Annuity	Fixed continuous payments until death	Fully affected by longevity risk and its costs
Tonuity	Longevity risk exposure until fixed date, fixed payments until death afterwards	Risk of longevity is removed for payments until a fixed date, longevity risk and costs only afterwards
Tontine	Full participation in longevity risk, likely fluctuations at high ages	Minimal longevity risk and costs, no fluctuations in payment stream

Again, the payout functions $c^*_{[\tau]}(t)$ and $d^*_{[\tau]}(t)$ can be set optimally by maximizing the expected discounted lifetime utility (2.8) given an initial net premium P_0. Theorem 3 gives the result.

Theorem 3 (Optimal payout function: tonuity). *For a tonuity product with switching time $\tau \in [0, \infty)$ and premium P_0, maximizing the expected discounted lifetime utility (2.8) subject to the constraint (4.2) leads to*

$$d^*_{[\tau]}(t)\Big|_{0\le t\le\tau} = \frac{e^{\frac{1}{\gamma}(r-\eta)t}}{(\lambda^*)^{\frac{1}{\gamma}}} \cdot \frac{\left(\kappa_{n,\gamma,\epsilon}({}_tp_x)\right)^{\frac{1}{\gamma}}}{\left(\int\limits_{-\infty}^{1} \left(1-\left(1-{}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\,\mathrm{d}\varphi\right)^{\frac{1}{\gamma}}},$$
$$c^*_{[\tau]}(t)\Big|_{t>\tau} = \frac{e^{\frac{1}{\gamma}(r-\eta)t}}{(\lambda^*)^{\frac{1}{\gamma}}}, \tag{4.3}$$

where the optimal Lagrangian multiplier is given by

$$\lambda^* = \frac{1}{P_0^\gamma}\left(\int\limits_0^\tau \frac{e^{\left(\frac{r-\eta}{\gamma}-r\right)t}\left(\kappa_{n,\gamma,\epsilon}({}_tp_x)\right)^{\frac{1}{\gamma}}}{\left(\int\limits_{-\infty}^{1}\left(1-\left(1-{}_tp_x^{1-\varphi}\right)^n\right)f_\epsilon(\varphi)\,\mathrm{d}\varphi\right)^{\frac{1}{\gamma}-1}}\,\mathrm{d}t + \int\limits_\tau^\infty e^{\left(\frac{r-\eta}{\gamma}-r\right)t}\,{}_tp_x\cdot m_\epsilon(-\log {}_tp_x)\,\mathrm{d}t\right)^\gamma \tag{4.4}$$

and $k_{n,\gamma,\epsilon}({}_tp_x)$ is as defined in Theorem 2. The policyholder's expected discounted lifetime utility is then given by

$$U^{[\tau]} := \int\limits_0^\tau e^{-\eta t} u\left(d^*_{[\tau]}(t)\right)\cdot \kappa_{n,\gamma,\epsilon}({}_tp_x)\,\mathrm{d}t + \int\limits_\tau^\infty e^{-\eta t}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x)\cdot u\left(c^*_{[\tau]}(t)\right)\mathrm{d}t. \tag{4.5}$$

If $\gamma \in \mathbb{N}\backslash\{1\}$, $\kappa_{n,\gamma,\epsilon}({}_tp_x)$ can again be simplified using (2.13).

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Proof. See Appendix A.2.

For the RM, we need again best-estimate liabilities $\mathrm{BEL}_{[\tau]}(t \mid *)$ and liabilities subject to a longevity shock $\mathrm{BEL}_{[\tau]}(t \mid \text{shock})$ for a tonuity with switching time τ. For $t \leq \tau$, this is given by

$$\mathrm{BEL}^{[\tau]}(t \mid *) = {}_t p_x \int_t^\tau e^{-r(s-t)} \mathbb{E}\left[\left(1-\left(1-{}_{s-t}p_{x+t}^{1-\epsilon}\right)^n\right)\right] \cdot d_{[\tau]}^*(s)\,\mathrm{d}s$$

$$+ {}_t p_x \int_\tau^\infty e^{-r(s-t)} \mathbb{E}\left[{}_{s-t}p_{x+t}^{1-\epsilon}\right] \cdot c_{[\tau]}^*(s)\,\mathrm{d}s, \tag{4.6}$$

$$\mathrm{BEL}^{[\tau]}(t \mid \text{shock}) = {}_t p_x \int_t^\tau e^{-r(s-t)} \left(1-\left(1-{}_{s-t}p_{x+t}^{1-z_{0.995}}\right)^n\right) \cdot d_{[\tau]}^*(s)\,\mathrm{d}s$$

$$+ {}_t p_x \int_\tau^\infty e^{-r(s-t)} {}_{s-t}p_{x+t}^{1-z_{0.995}} \cdot c_{[\tau]}^*(s)\,\mathrm{d}s. \tag{4.7}$$

For $t \geq \tau$, best-estimate liabilities $\mathrm{BEL}^{[\tau]}(t \mid, *)$ and shocked liabilities $\mathrm{BEL}^{[\tau]}(t \mid \text{shock})$ are given by (2.16) and (2.18), respectively, if the payoff stream $c^*(t)$ is replaced by $c_{[\tau]}^*(t)$. Equations (2.15) and (2.20) can then be used to determine the risk capital charges $F_0^{[\tau]} = \mathrm{RM}$. ■

5. Numerical II: Comparison of annuity, tontine and tonuity

We analyze the attractiveness of the tonuity and compare it to the annuity and tontine. Therefore, we use (unless specified differently) the same assumptions and the same parameter set as in Section 3.

Especially from the policyholder's viewpoint, it is necessary to find a way to weigh longevity risk (aversion) and the amount of risk capital charges F_0. A highly longevity risk-averse policyholder might prefer an annuity to a tontine even if the CoC rates (and thus the risk capital charges) are high. As in Section 3, we introduce a relative certainty equivalent $\mathrm{CEQ}_{[\tau]}$ that is chosen such that the policyholder is indifferent between one annuity with net premium $P_0^{[0]} = 10{,}000$ and $\mathrm{CEQ}_{[\tau]}$ tonuities with the same net premium $P_0^{[\tau]} = 10{,}000$, that is, $\mathrm{CEQ}_{[\tau]}$ solves

$$U^{[0]} \stackrel{!}{=} \int_0^\infty e^{-\eta t} \cdot \mathbb{E}\left[u\left(\mathrm{CEQ}_{[\tau]} \cdot b_{[\tau]}(t)\right)\right] \mathrm{d}t = \mathrm{CEQ}_{[\tau]}^{1-\gamma} \cdot U^{[\tau]}. \tag{5.1}$$

Therefore, generalizing (3.4) in Section 3, we obtain relative certainty equivalents of a tonuity with switching time τ as

$$\mathrm{CEQ}_{[\tau]} := \left(\frac{U^{[0]}}{U^{[\tau]}}\right)^{\frac{1}{1-\gamma}}. \tag{5.2}$$

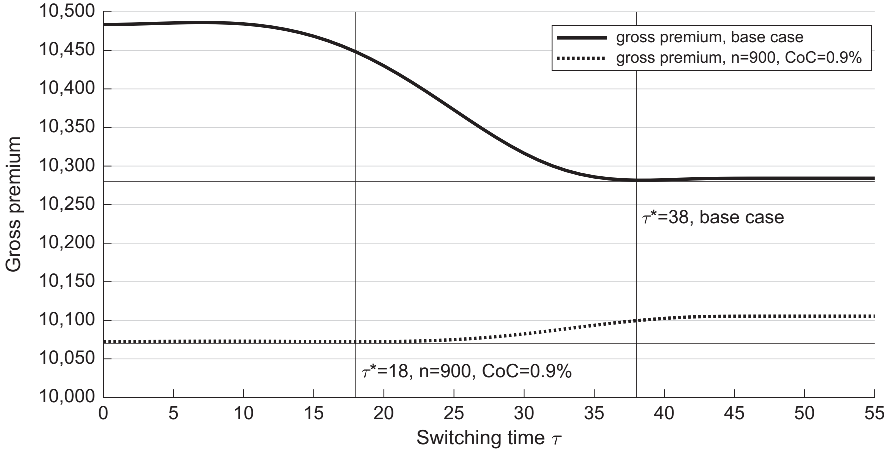

FIGURE 3: Gross premia $\mathrm{CEQ}_{[\tau]} \cdot (P_0 + F_0^{[\tau]})$ using the base-case parameter set from Table 1 and the case $n = 900$ and $CoC = 0.9\%$, varying the switching time τ. For the policyholder, each product has the same expected discounted lifetime utility.

The policyholder is indifferent between the annuity (which is a tonuity with $\tau = 0$) and $\mathrm{CEQ}_{[\tau]}$ tonuities. This is the case as the tonuity payoff $\mathrm{CEQ}_{[\tau]} \cdot b_{[\tau]}(t)$ leads to the same expected discounted lifetime utility as an annuity. Noting that $\mathrm{CEQ}_{[\tau]}$ tonuities require a net premium of $\mathrm{CEQ}_{[\tau]} \cdot P_0$ and a risk capital charge of $\mathrm{CEQ}_{[\tau]} \cdot F_0^{[\tau]}$, we can now compare gross premia $\mathrm{CEQ}_{[\tau]} \cdot (P_0 + F_0^{[\tau]})$ to determine the best retirement product from the policyholder's perspective, that is, the cheapest product that maintains the expected utility level of an annuity. This comparison includes annuities ($\tau = 0$) and tontines ($\tau = \infty$) as special cases.

Figure 3 illustrates the choice of the optimal switching time τ^* and the interplay of switching times and resulting gross premia in a numerical example. For each switching time $\tau \geq 0$, the figure presents risk capital charge $\mathrm{CEQ}_{[\tau]} \cdot F_0^{[\tau]}$ and gross premium $\mathrm{CEQ}_{[\tau]} \cdot (P_0 + F_0^{[\tau]})$ as a function of the switching time. Hence, the left-hand limit ($\tau \downarrow 0$) represents the immediate subscription to an annuity, whereas the right-hand limit ($\tau \uparrow \infty$) corresponds to a lifelong tontine.

Since all tonuity products with payout function $\mathrm{CEQ}_{[\tau]} b_{[\tau]}(t)$ yield the same lifetime utility $U^{[0]}$ for the policyholder, the decision on which tonuity to choose is a tradeoff between the risk capital charge $F_0^{[\tau]}$ (which is highest for the annuity and lowest for the tontine) and the net premium $\mathrm{CEQ}_{[\tau]} \cdot P_0$ (which is lowest for the annuity and highest for the tontine). Overall, for a given parameter set, we can determine the optimal switching time τ^* that minimizes gross premia $\mathrm{CEQ}_{[\tau]} \cdot (P_0 + F_0^{[\tau]})$.[6]

Here, the optimal switching time is $\tau^* = 18$ and $\tau^* = 38$, in case of a large portfolio ($n = 900$) and low CoC rate ($CoC = 0.9\%$) and the base case (see

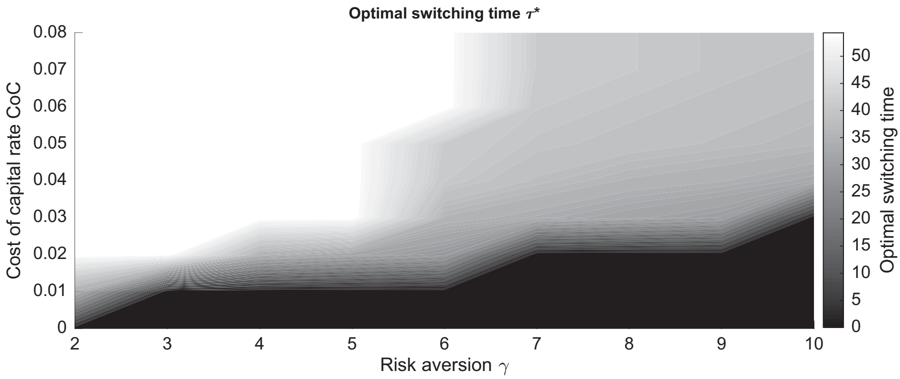

FIGURE 4: Optimal switching time τ^* using the base-case parameter set from Table 1 varying the CoC rate and risk aversion γ.

Table 1), respectively. Note that, in the base case, an expected utility maximizing individual would only be willing to switch to an annuity well above age 90, if he is charged the full cost of longevity risk transfer within a Solvency II model. For practical purposes, this seems to be a quite high switching time, yet, according to the mortality model, an individual has an expected remaining lifetime of more than 20 years at age 65 and the probability to survive at least until age 90 is above 35%.

We now want to further examine the optimal switching time and determine its sensitivity to the different parameters. Figure 4 first presents the optimal switching time τ^* depending on the CoC rate and the risk aversion coefficient γ. In case that there is no risk capital charge (CoC = 0%), the policyholder prefers the annuity to a tonuity or tontine product (black). This is in line with the results in Section 3. This result changes if longevity risk charges are added: in Figure 4, we observe that for CoC = 6%, as suggested by Solvency II regulation, the retiree prefers either a tontine or a tonuity (gray) to the annuity product. This even holds for very risk-averse retirees, with switching times about 30 years after retirement across different levels of risk aversion.

Figure 5 presents the optimal switching time τ^* depending on the risk aversion γ and the risk-free rate r. For low risk aversion parameters (here $\gamma < 2$), we observe that it is optimal for the policyholder to stay with the tontine and never switch to an annuity product. The increased longevity risk of a tontine seems to be compensated by the reduced risk capital charges. For medium to high risk aversion coefficients ($\gamma \in [6, 10]$), it is always optimal to buy a tonuity. Here, the tonuity seems to be an optimal balance between longevity risk taking and risk capital charges. The optimal switching time τ^* is decreasing in the risk-free interest rate r. This effect, however, is moderate.

Next, Figure 6 presents the optimal switching time τ^* depending on the portfolio size n and the subjective discount rate η. The subjective discount rate η is not relevant in the analysis of optimal switching times—it only has a minor

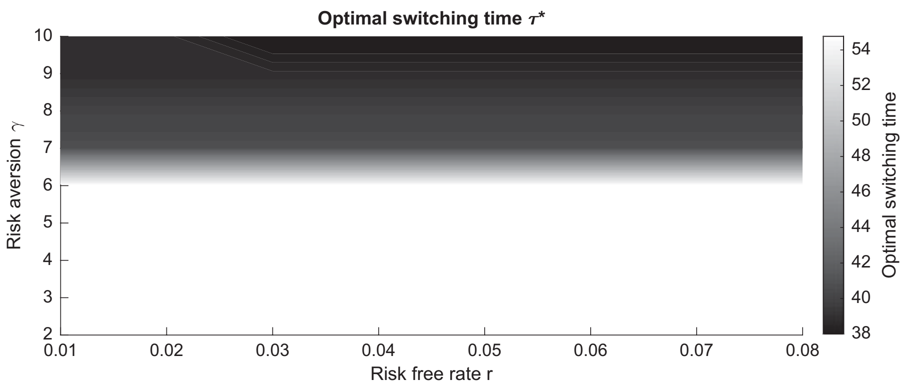

FIGURE 5: Optimal switching time τ^* using the base-case parameter set from Table 1 varying the risk aversion coefficient γ and risk-free rate r.

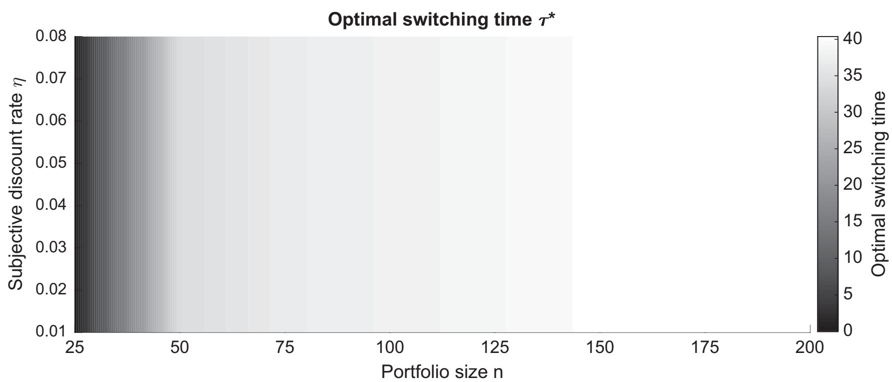

FIGURE 6: Optimal switching time τ^* using the base-case parameter set from Table 1 varying the subjective discount rate η and portfolio size n.

impact on τ^*. In contrast, we observe that the portfolio size is a very important parameter in our analysis. Already a portfolio size of $n = 200$ seems to diversify mortality risk pretty well. Then, it is attractive for the policyholder to invest into a tontine ($\tau^* = \infty$).

6. CONCLUSION

Risk-oriented solvency regulation decreases the attractiveness of retirement products that contain a significant portion of (aggregate) mortality risk. If increased solvency costs are handed over to the policyholder, he/she might prefer to keep part of the product's (aggregate) mortality risk. In this paper, we have suggested a novel retirement product called tonuity that nicely

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combines annuity and tontine products and can more easily adapt to the policyholder's needs. Buying a tonuity, the policyholder benefits from risk capital charges that are lower than those of conventional annuities. Further, a tonuity eliminates income fluctuations at old ages, one of the main drawbacks of a tontine or a pooled annuity fund. In case of an unexpected increase in life expectancy (longevity shock), it is not only the insurance company but also the policyholder that has to fund the additional liquidity need. The policyholder has to accept a decrease in pension income at early ages of retirement where it might be easier to compensate income fluctuations by, for example, a part-time job. In a utility-based framework, we have determined the optimal tonuity payout that maximizes the policyholder's expected discounted lifetime utility. Linking risk capital charges to Solvency II regulation, we have obtained a tailor-made retirement product that better aligns the tradeoff between risk capital charges and longevity risk aversion. While a highly risk-averse policyholder prefers a conventional annuity, a policyholder with medium risk aversion chooses a tonuity. A policyholder whose risk aversion is low would buy a tontine product.

NOTES

1. At the beginning of the 20th century, a total amount of 6 billion dollars has been invested in tontine products, about 2/3 of the life insurance nominal. Shortly afterwards, tontines were prohibited in Great Britain and the US. An increased sensitivity to longevity risk might encourage its re-introduction (see, e.g., Ransom and Sutch, 1987).

2. In the 17th and 18th centuries, the switching between tontine and annuity products is documented. Great Britain's 1693 tontine allowed subscribers to convert the tontine shares into a lifelong annuity (see, e.g., Weir, 1989). In 1770, tontine subscribers in France were forced to convert into life annuities (see, e.g., Weir, 1989).

3. Note that the unsystematic mortality risk borne by the participants of the tontine can initially be diversified by choosing the pool size n large enough. However, at old ages, the portfolio size naturally decreases, exposing the participants in the tontine to both systematic and unsystematic mortality risk.

4. In TP.5.2., the technical provisions for an insurance contract are specified as the best-estimate of liabilities plus the so-called risk margin. The technical provisions are deemed equivalent to the amount another insurer would ask for as compensation for taking over the insurance business. In our approach, we apply this principle to individual contracts, that is, the life insurer sets a premium that is sufficient to cover the expected claims plus the necessary risk capital charge.

5. Here, we use that for a real valued random variable X with cumulative distribution function F_X and quantile function F_X^{-1}, $x \in \mathbb{R}$, confidence level $y \in (0, 1)$ and $g : \mathbb{R} \to \mathbb{R}$ monotonically increasing, and we have that
$F_{g(X)}^{-1}(y) \leq x \Leftrightarrow P(g(X) \leq x) \geq y \Leftrightarrow P(X \leq g^{-1}(x)) \geq y \Leftrightarrow F_X(g^{-1}(x)) \geq y \Leftrightarrow F_X^{-1}(y) \leq g^{-1}(x) \Leftrightarrow g\left(F_X^{-1}(y)\right) \leq x.$

6. We use a rather simple, yet stochastic, model for the calculation of the risks associated with issuing policies for retirement income. More sophisticated models, parameterizations or internal models would affect these results. In particular, the CoC rate seems to be important. The lower the CoC an undertaking uses, the more attractive the annuity will be, and the earlier one would switch from tonuity to annuity.

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APPENDIX A. PROOFS

A.1. Proof of Theorem 2

For a related result, see also Milevsky and Salisbury (2015) (special case $m_\epsilon(s) = 1$). Note first that

$$\int_{-\infty}^{1} {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\,\mathrm{d}\varphi = {}_tp_x \cdot \mathbb{E}\left[e^{-\epsilon \log {}_tp_x}\right] = {}_tp_x \cdot m_\epsilon(-\log {}_tp_x).$$

For part (a) one can use the Lagrangian function

$$\begin{aligned} L(c,\lambda) &:= \int_0^\infty e^{-\eta t} \int_{-\infty}^{1} {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\,\mathrm{d}\varphi \cdot u\big(c(t)\big)\,\mathrm{d}t \\ &\quad + \lambda\left(P_0 - \int_0^\infty e^{-rt} c(t) \int_{-\infty}^{1} {}_tp_x^{1-\varphi} f_\epsilon(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}t\right) \\ &= \int_0^\infty e^{-\eta t}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \cdot u\big(c(t)\big)\,\mathrm{d}t \\ &\quad + \lambda\left(P_0 - \int_0^\infty e^{-rt}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \cdot c(t)\,\mathrm{d}t\right). \end{aligned}$$

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The first-order condition yields $c^*(t) = \left(\lambda \cdot e^{(\eta - r)t}\right)^{-\frac{1}{\gamma}}$. From the budget constraint, we then obtain $\lambda^* = P_0^{-\gamma}\left(\int_0^\infty e^{(\frac{r-\eta}{\gamma}-r)t}\, {}_tp_x \cdot m_\epsilon(-\log {}_tp_x)\, \mathrm{d}t\right)^\gamma$. This directly leads to (2.9). The annuity is constant over time if the subjective discount rate equals the risk-free rate, that is, $\eta = r$.

For part (b), to determine the optimal payout function $d(t)$ to the tontine pool, we again maximize the expected discounted lifetime utility (2.8) subject to (2.5). In case of power utility, $u(x) = x^{1-\gamma}/(1-\gamma)$, $\gamma > 0$ and $\gamma \neq 1$, we find that

$$
\begin{aligned}
L(d, \lambda) &:= \mathbb{E}\left[\int_0^{\zeta_\epsilon} e^{-\eta t}\, u\big(b^{[\infty]}(t)\big)\, \mathrm{d}t\right] \\
&\quad + \lambda\left(P_0 - \int_0^\infty e^{-rt} d(t) \int_{-\infty}^1 \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}t\right) \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{\zeta_\epsilon > t\}} e^{-\eta t} u\left(\frac{nd(t)}{N_\epsilon(t)}\right) \mathrm{d}t\right] \\
&\quad + \lambda\left(P_0 - \int_0^\infty e^{-rt} d(t) \int_{-\infty}^1 \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}t\right) \\
&= \int_0^\infty e^{-\eta t} \mathbb{E}\left[{}_tp_x^{1-\varphi} \cdot \mathbb{E}\left[u\left(\frac{nd(t)}{N_\varphi(t)}\right) \Big| \zeta_\varphi > t, \epsilon = \varphi\right]\right] \mathrm{d}t \\
&\quad + \lambda\left(P_0 - \int_0^\infty e^{-rt} d(t) \int_{-\infty}^1 \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}t\right) \\
&= \int_0^\infty e^{-\eta t} \underbrace{\sum_{k=1}^n \binom{n}{k} \left(\frac{k}{n}\right)^\gamma \int_{-\infty}^1 \left({}_tp_x^{1-\varphi}\right)^k \left(1 - {}_tp_x^{1-\varphi}\right)^{n-k} f_\epsilon(\varphi)\, \mathrm{d}\varphi}_{=:\kappa_{n,\gamma,\epsilon}({}_tp_x)} \cdot u\big(d(t)\big)\, \mathrm{d}t \\
&\quad + \lambda\left(P_0 - \int_0^\infty e^{-rt} d(t) \int_{-\infty}^1 \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\, \mathrm{d}\varphi\, \mathrm{d}t\right)
\end{aligned}
\tag{A.3}
$$

We can then solve the optimization using the Euler–Lagrange theorem, see, for example, Gelfand and Fomin (1963, p. 43). The optimality condition $\partial L(d, \gamma)/\partial d \stackrel{!}{=} 0$ is given by

$$
e^{-\eta t} \kappa_{n,\gamma,\epsilon}({}_tp_x) \cdot \left(d^*(t)\right)^{-\gamma} = \lambda^* e^{-rt} \int_{-\infty}^1 \left(1 - (1 - {}_tp_x^{1-\varphi})^n\right) f_\epsilon(\varphi)\, \mathrm{d}\varphi,
$$

where $u'(x) := \partial u(x)/\partial x = x^{-\gamma}$ and λ^* is the Lagrangian multiplier which makes the budget constraint binding. The optimal tontine payoff can be solved explicitly and owns the following structure:

$$d^*(t) = \frac{e^{\frac{1}{\gamma}(r-\eta)t}}{(\lambda^*)^{\frac{1}{\gamma}}} \cdot \frac{\left(\kappa_{n,\gamma,\epsilon}({}_tp_x)\right)^{\frac{1}{\gamma}}}{\left(\int_{-\infty}^{1} \left(1-(1-{}_tp_x^{1-\varphi})^n\right) f_\epsilon(\varphi)\,\mathrm{d}\varphi\right)^{\frac{1}{\gamma}}},$$

where $\lambda^* > 0$ is chosen such that the budget constraint in (2.5) holds, that is,

$$\lambda^* := \left(\frac{1}{P_0}\int_0^\infty e^{\left(\frac{r-\eta}{\gamma}-r\right)t} \cdot \left(\kappa_{n,\gamma,\epsilon}({}_tp_x)\right)^{\frac{1}{\gamma}} \left(\int_{-\infty}^{1} \left(1-\left(1-{}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\,\mathrm{d}\varphi\right)^{1-\frac{1}{\gamma}} \mathrm{d}t\right)^{\gamma}.$$

The computation of $\kappa_{n,\gamma,\epsilon}({}_tp_x)$ is numerically challenging if the portfolio size n is large. We therefore present a more convenient expression if the risk aversion coefficient is a natural number, that is, $\gamma \in \mathbb{N} \setminus \{1\}$. Consider first Lemma 4.

Lemma 4 (Moments of the binomial distribution). *For $n \in \mathbb{N}$ and $p \in (0, 1)$ consider $Z \sim \mathrm{Bin}\,(n, p)$. For $k, l = 0, 1, 2, \ldots$, we then find that*

$$\mathbb{E}[Z^k] = \sum_{l=1}^{k} a_l \cdot p^l,$$

where $a_l := b_{kl} \cdot n^{\underline{l}}$, $n^{\underline{l}} := n(n-1)\cdots(n-l+1)$, and the b_{kl}'s are determined by the iteration

$$b_{0l} = \delta_{l0}, \qquad b_{kl} = l \cdot b_{k-1,l} + b_{k-1,l-1},$$

where δ_{ij} is the usual Kronecker symbol ($\delta_{ij} = 1$ for $i = j$ and 0 otherwise).

Proof. *See, for example, Theorem 2.2 in Knoblauch (2008). A table for the b_{kl}'s (for the range $0 \le k, l \le 10$) is given by Table 2.1 in Knoblauch (2008).* ■

Introducing $Z_\varphi \sim \mathrm{Bin}\,(n, {}_tp_x^{1-\varphi})$, we can apply Lemma 4 to obtain

$$\begin{aligned}
\kappa_{n,\gamma,\epsilon}({}_tp_x) &:= \mathbb{E}\left[\sum_{k=1}^{n}\binom{n}{k}\left(\frac{k}{n}\right)^{\gamma}({}_tp_x^{1-\varphi})^k(1-{}_tp_x^{1-\varphi})^{n-k}\right] = \frac{1}{n^\gamma}\mathbb{E}\left[\mathbb{E}\left[Z_\varphi^\gamma \,\middle|\, \epsilon=\varphi\right]\right] \\
&\overset{\gamma\in\mathbb{N}\setminus\{1\}}{=} \frac{1}{n^\gamma}\mathbb{E}\left[\sum_{l=1}^{\gamma} a_l\left({}_tp_x^{1-\epsilon}\right)^l\right] \\
&= \frac{1}{n^\gamma}\sum_{l=1}^{\gamma} a_l \cdot {}_tp_x^l \cdot \mathbb{E}\left[\exp\left(-\epsilon l \log {}_tp_x\right)\right] \\
&= \frac{1}{n^\gamma}\sum_{l=1}^{\gamma} a_l \cdot {}_tp_x^l \cdot m_\epsilon(-l\log {}_tp_x),
\end{aligned}$$

where the coefficients a_l are as introduced in Lemma 4.

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A.2. Proof of Theorem 3

To determine the optimal payout, we take the Lagrangian function:

$$L(d,c,\lambda) := \int_0^\tau e^{-\eta t} u\big(d^*_{[\tau]}(t)\big) \sum_{k=1}^n \binom{n}{k} \left(\frac{k}{n}\right)^\gamma \int_{-\infty}^1 \left({}_tp_x^{1-\varphi}\right)^k \left(1 - {}_tp_x^{1-\varphi}\right)^{n-k} f_\epsilon(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}t$$
$$+ \int_\tau^\infty e^{-\eta t}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \cdot u\big(c^*_{[\tau]}(t)\big)\,\mathrm{d}t$$
$$+ \lambda \left(P_0 - \int_0^\tau e^{-rt} \int_{-\infty}^1 \left(1 - (1 - {}_tp_x^{1-\varphi})^n\right) f_\epsilon(\varphi)\,\mathrm{d}\varphi\, d_{[\tau]}(t)\,\mathrm{d}t \right.$$
$$\left. + \int_\tau^\infty e^{-rt}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \cdot c_{[\tau]}(t)\,\mathrm{d}t \right).$$

where we drop the t-dependence of d and c to improve readability. In order to maximize the Lagrangian function, we first take derivatives with respect to d and c, that is,

$$\frac{\partial L(d,c,\lambda)}{\partial d} = \mathbb{1}_{\{0\le t<\tau\}} \Big(e^{-\eta t} \sum_{k=1}^n \binom{n}{k} \int_{-\infty}^1 ({}_tp_x^{1-\varphi})^k (1 - {}_tp_x^{1-\varphi})^{n-k} f_\epsilon(\varphi)\,\mathrm{d}\varphi \cdot u'\left(\frac{nd}{k}\right)$$
$$- \lambda e^{-rt} \int_{-\infty}^1 (1 - (1 - {}_tp_x^{1-\varphi})^n) f_\epsilon(\varphi)\,\mathrm{d}\varphi \Big) \overset{!}{=} 0,$$

$$\frac{\partial L(d,c,\lambda)}{\partial c} = \mathbb{1}_{\{t\ge\tau\}} \Big(e^{-\eta t}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \cdot c^{-\gamma} - \lambda e^{-rt}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x) \Big) \overset{!}{=} 0.$$

Solving for d and c, we can show that the Lagrangian takes global optima if

$$d^*_{[\tau]}(t)\Big|_{0\le t\le\tau} = \frac{e^{\frac{1}{\gamma}(r-\eta)t}}{(\lambda^*)^{\frac{1}{\gamma}}} \cdot \frac{\big(\kappa_{n,\gamma,\epsilon}({}_tp_x)\big)^{\frac{1}{\gamma}}}{\Big(\int_{-\infty}^1 \big(1 - \big(1 - {}_tp_x^{1-\varphi}\big)^n\big) f_\epsilon(\varphi)\,\mathrm{d}\varphi\Big)^{\frac{1}{\gamma}}}, \quad c^*_{[\tau]}(t)\Big|_{t>\tau} = \frac{e^{\frac{1}{\gamma}(r-\eta)t}}{(\lambda^*)^{\frac{1}{\gamma}}}, \tag{A.4}$$

where $\kappa_{n,\gamma,\epsilon}({}_tp_x)$ is as introduced in Theorem 2. The budget constraint (4.2) then leads to the optimal Lagrangian multiplier

$$\lambda^* = \left(\frac{1}{P_0} \int_0^\tau e^{(\frac{r-\eta}{\gamma} - r)t} \frac{\big(\kappa_{n,\gamma,\epsilon}({}_tp_x)\big)^{\frac{1}{\gamma}}}{\Big(\int_{-\infty}^1 \big(1 - \big(1 - {}_tp_x^{1-\varphi}\big)^n\big) f_\epsilon(\varphi)\,\mathrm{d}\varphi\Big)^{\frac{1}{\gamma}-1}}\,\mathrm{d}t \right.$$
$$\left. + \int_\tau^\infty e^{(\frac{r-\eta}{\gamma} - r)t}\,{}_tp_x \cdot m_\epsilon(-\log {}_tp_x)\,\mathrm{d}t \right)^\gamma.$$

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