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RESEARCH ARTICLE

Perverse sheaves on Riemann surfaces as Milnor sheaves

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Abstract
Constructible sheaves of abelian groups on a stratified space can be equivalently described in terms of representations of the exit-path category. In this work, we provide a similar presentation of the abelian category of perverse sheaves on a stratified surface in terms of representations of the so-called paracyclic category of the surface. The category models a hybrid exit–entrance behaviour with respect to chosen sectors of direction, placing it 'in between' exit and entrance path categories. In particular, this perspective yields an intrinsic definition of perverse sheaves as an abelian category without reference to derived categories and t-structures.

Contents

Introduction

Contents and future applications.

This paper is the first step in a larger project devoted to a systematic development of the theory of *perverse schobers*. The latter are categorical analogs of perverse sheaves, in which vector spaces are replaced by (enhanced) triangulated categories. The idea of perverse schobers was proposed in [31] based on the features of various 'elementary' descriptions of perverse sheaves in terms of quivers. Namely, these descriptions are often of such form that a natural categorical analog (quiver representations formed by categories instead of vector spaces) suggests itself readily. For example, for the classical description [2, 24] of perverse sheaves on the disk in terms of diagrams

$$\begin{tikzcd} \Phi \arrow[r, bend left, "a"] & \Psi, \arrow[l, bend left, "b"] \end{tikzcd} \tag{0.1}$$

with $\mathrm{id} - ab$ and $\mathrm{id} - ba$ invertible, such a categorical analog is found in the concept of a *spherical adjunction*; see [31].

However, the quiver descriptions do not give satisfying *definitions* of the category of perverse sheaves since they depend on auxiliary choices. For example, in the above case, a choice of a direction at the origin is needed to define vanishing and nearby cycles. On the other hand, from the customary point of view, a perverse sheaf is an object of an abelian category that arises as the heart of a certain t-structure on the derived category of constructible sheaves on a stratified topological space. It is not clear whether such an approach can be categorified directly.

In this paper, we identify perverse sheaves (not yet schobers) on a stratified surface X with so-called *Milnor sheaves* (Theorem 3.1.13). Similarly to the description of constructible sheaves as representations of the exit path category (see [45]), our result follows from an alternative parametrization in terms of a hybrid of the exit and entrance path categories, called the *Milnor category of the surface*. Its objects, *Milnor disks*, are given by disks in X together with a choice of a finite number of *boundary intervals*. These intervals determine the interaction with the stratification: A disk may move on the surface via isotopy such that the points in the zero-dimensional stratum exit the disk through the chosen boundary intervals and enter the disk through their complement. In addition, the boundary intervals themselves can interact in a way familiar from Connes' cyclic category (see below for more details). A Milnor sheaf is then defined as a representation of the Milnor category subject to certain natural gluing conditions that arise from cutting Milnor disks into pieces.

As a result, we obtain an intrinsic definition of perverse sheaves on Riemann surfaces that is internal to the framework of abelian categories, without reference to derived categories, and which can therefore serve as an alternative to the definition given in [3]. Our main incentive is that the definition has a comparatively straightforward categorification offering a good framework for perverse schobers. This approach will be elaborated in sequels to this paper.

Even in the uncategorified context of perverse sheaves, Milnor sheaves provide a novel perspective on classical aspects of the theory. For example, one motivation for the introduction of perverse sheaves is the fact that, in contrast to constructible sheaves, they are preserved under Verdier duality. This phenomenon becomes almost self-evident in the Milnor sheaf model. Namely, it is a direct consequence of a canonical self-duality of the Milnor category obtained by swapping the boundary intervals with their complements (generalizing the well-known self-dualities of the cyclic and paracyclic categories).

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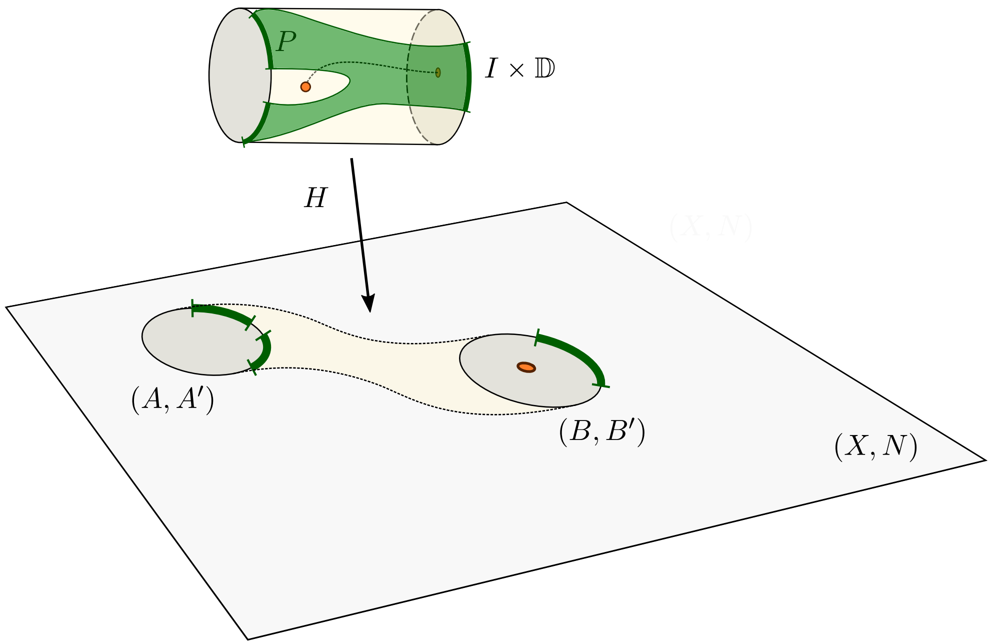
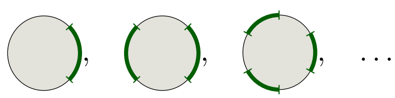


Figure 1. *A morphism in $M(X, N)$ from (A, A') to (B, B') represented given by the isotopy H.*

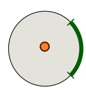
In higher complex dimensions, a possible generalization could involve mimicking more closely the topology related to forming perverse sheaves of vanishing cycles associated to holomorphic functions. When such a perverse sheaf is supported at a single point (the 'isolated microlocal singularity' case), it reduces to a single vector space so we have purity just like for Riemann surfaces. We hope to explore this approach in future work.

Details of the main result

Fundamental for us is the concept of a *Milnor disk*, a pair (A, A') where $A \subset X$ is a closed disk, containing at most one point from the zero-dimensional stratum N, and $A' \subset \partial A$ is a finite nonempty disjoint union of closed intervals. These Milnor disks will be depicted by the symbols



We call the points in the zero-dimensional stratum N *special* and signify them via the symbol . For example, a Milnor disk (A, A') with one boundary interval containing a special point will be referred to as

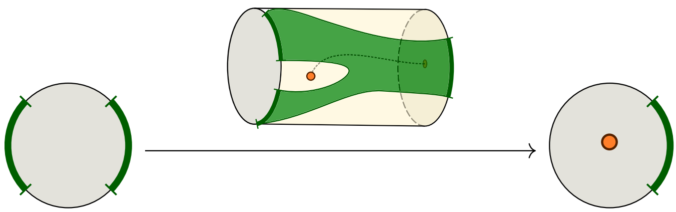
$(A, A') =$ 

leaving the embedding of A into the surface X implicit. Milnor disks form the objects of the *Milnor category* $M(X, N)$ where a morphism from (A, A') to (B, B') is given by an equivalence class of isotopies $H : I \times \mathbb{D} \to X$ with $H_0 : \mathbb{D} \cong A$ and $H_1 : \mathbb{D} \cong B$, together with a choice of bordism $P \subset I \times S^1$ from $H_0^{-1}(A')$ to $H_1^{-1}(B')$ such that the inclusion $H_1^{-1}(B') \subset P$ is a homotopy equivalence (see Figure 1). Here, roughly speaking, the trajectories $H^{-1}(N)$ of the special points are required to enter the cylinder through $(I \times S^1) \setminus P$ and exit through P. This hybrid exit–entry behaviour puts the Milnor category 'in between' the exit and entrance path categories of (X, N). As will be explained in the main body of this work, this phenomenon can be regarded as a geometric manifestation of the fact that the perverse t-structure lies 'in between' the standard t-structure and its Verdier dual.

In particular, while the exit and entrance path categories are dual to one another, the Milnor category is self-dual: On objects, the duality is given by

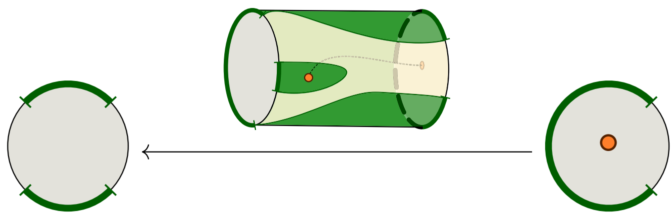
$$(A, A') \mapsto (A, \overline{\partial(A) \setminus A'})$$

on morphisms, it is obtained by replacing the bordism P by the closure of $(I \times S^1) \setminus P$ and reversing the direction of the isotopy H. For example, the action of the self-duality associates to the morphism



(0.2)

depicted in Figure 1, the morphism



(0.3)

Given an object $\mathcal{F}$ of the derived constructible category $D(X, N; \mathcal{A})$ and a morphism (H, P) : $(A, A') \to (B, B')$ of Milnor disks, we obtain a correspondence on relative (hyper) cohomology

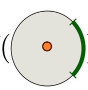
$$\mathrm{R}\Gamma(A, A'; \mathcal{F}) \xleftarrow[\mathrm{R}\Gamma]{} (I \times \mathbb{D}, P; H^*\mathcal{F}) \xrightarrow{\simeq} \mathrm{R}\Gamma(B, B'; \mathcal{F}) \tag{0.4}$$

and hence a functor

$$\mathrm{R}\Gamma(-; \mathcal{F}) : M(X, N)^{\mathrm{op}} \longrightarrow D(\mathcal{A}). \tag{0.5}$$

We note that $\mathrm{R}\Gamma(A, A'; \mathcal{F})$ can be identified with $\Phi_f(\mathcal{F})$, the sheaf of vanishing cycles for $\mathcal{F}$ with respect to an appropriate holomorphic function f (possibly with a zero of arbitrary order), hence the name 'Milnor disk', modelled after 'Milnor fibers' in singularity theory. In particular, we may now express the local classification data (0.1) at a special point $\bullet \in N$ in terms of our terminology:

1. The space of *vanishing cycles*:

$\Phi \cong \mathcal{F}($  $)$

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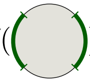
2. The space of *nearby cycles*:

$$\Psi \cong \mathcal{F}($$  $$)$$

3. The *variation* map

$$a = \mathrm{var} : \Phi \to \Psi$$

   is the value of $\mathrm{R}\Gamma(-;\mathcal{F})$ on the morphism (0.2).
4. The *canonical* map

$$b = \mathrm{can} : \Psi \to \Phi$$

   is the value of $\mathrm{R}\Gamma(-;\mathcal{F})$ on the morphism (0.3).

See §4.2 for a discussion of how to recover the relations $T_\Psi = \mathrm{id} - ab$ and $T_\Phi = \mathrm{id} - ba$, expressing the monodromy in terms of these data.

Our main result is based on the observations that

(1) Perverse sheaves can be characterized by the fact that their relative (hyper) cohomology on Milnor disks is concentrated in degree 0,
(2) A perverse sheaf F is completely described by its values $\mathcal{F}(A, A') = H^0(A, A'; F)$ on Milnor disks.

Observation (1) immediately implies that, for a perverse sheaf $\mathcal{F}$, the functor $\mathrm{R}\Gamma(-,\mathcal{F})$ from equation (0.5) takes values in the *abelian* category $\mathcal{A} \subset D(\mathcal{A})$ given by the heart of the standard t-structure. Observation (2) then leads to the main result of this work: Theorem 3.1.13 establishes that the association $\mathcal{F} \mapsto \mathrm{R}\Gamma(-;\mathcal{F})$ provides an equivalence between the abelian category of perverse sheaves on the stratified Riemann surface (X, N) and the category of Milnor sheaves: $\mathcal{A}$-valued presheaves on the Milnor category $M(X, N)$ that satisfy descent conditions with respect to the cutting and pasting Milnor disks.

Method of proof: ∞-categorical Kan extension

Although the statement of Theorem 3.1.13 is 'purely abelian', the proof utilizes the ambient derived category and relies on ∞-categorical techniques. That is, we establish a result (Corollary 3.1.12) identifiying constructible sheaves with values in a stable ∞-category $\mathcal{D}$, and appropriately defined Milnor sheaves valued in $\mathcal{D}$. When $\mathcal{D} = \mathcal{D}(\mathcal{A})$ is the ∞-categorical enhancement of the derived category of a Grothendieck abelian category $\mathcal{A}$, then perverse sheaves are recovered among all constructible complexes via the observation (1) above.

The method of proof of Corollary 3.1.12 is as follows. In general, identifying two given ∞-categories is hard to achieve by hand due to the infinite amount of coherence data involved. The technique of Kan extensions allows for an efficient means of handling such data and 'mediating' it across parametrizing diagram categories (see Proposition A.3). Using this technique, we produce equivalences between representations of various subcategories of the larger paracyclic category $\Lambda(X, N)$ to mediate the subcategories of *standard disks*, *Milnor disks*, and *bounded disks*. In this framework, we provide an alternative construction of the presheaf $\mathrm{R}\Gamma(-,\mathcal{F})$ on the Milnor category $M(X, N)$ as a Kan extension from the category of standard disks (cf. §3).

Corollary 3.1.12 and various technical tools developed for its proof provide not only a stepping stone for the more classical-looking Theorem 3.1.13 but also present a possible framework for the generalization to perverse schobers. In that generalization, a stable ∞-categorical enhancement of triangulated categories is important from the very beginning.

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The role of paracyclic Segal objects

Our approach to perverse sheaves via Milnor sheaves naturally involves structures familiar in the theory of cyclic homology [11, 21, 36]. One of them is the *paracyclic category* Λ_∞ which can be regarded as the universal central extension (by $\mathbb{Z}$) of the cyclic category Λ of Connes [11].

Namely, in the most classical case, when (X, N) is the disk $(\mathbb{D}, \{0\})$ with the origin as special point, a Milnor sheaf can be uniquely recovered from its values on Milnor disks containing 0. These disks form a subcategory of $M(\mathbb{D}, \{0\})$ equivalent to the paracyclic category Λ_∞, and our approach identifies $\mathcal{A}$-valued perverse sheaves, with the following structures: *paracyclic objects* $Y : \Lambda_\infty^{\mathrm{op}} \to \mathcal{A}$ *whose restriction to* $\Delta^{\mathrm{op}} \subset \Lambda_\infty^{\mathrm{op}}$ *is a Segal [5, 20] simplicial objects* (see Corollary 4.3.2). Further, the equivalence of such structures with the more customary classification data (0.1) can be understood as a special instance the duplicial Dold–Kan correspondence (see §4.4).

This point of view turns out to be important for the generalization to perverse schobers. The corresponding analog of a perverse sheaf on the disk is, as mentioned above, a spherical adjunction. It turns out that any such adjunction gives, via a variant of the relative Waldhausen $S_\bullet$-construction [48], rise to a *paracyclic object whose restriction to* Δ^{op} *is 2-Segal*, that is, satisfies a two-dimensional generalization of the Segal condition introduced in [20]. Such data then form the local data comprising the structure of a perverse schober, as will be explained in subsequent work.

Relation to previous work

The dream of defining perverse sheaves in a way that would be at the same time topological (avoiding analysis and D-modules) and abelian-categorical (avoiding derived categories) is of course as old as the theory of perverse sheaves itself. We should particularly mention the 1990 preprint of MacPherson [40] that introduced (in arbitrary dimension) the concept of *Fary sheaves* which are certain 'cohomology theory' data on an appropriate class of pairs (U_+, U_-) of opens in a stratified manifold. Our concept of a Milnor sheaf can be seen as an adaptation and a simplification of that of a Fary sheaf to the case of two real dimensions, when instead of a functor associating a *graded* vector space (i.e., *several* cohomology groups) to a pair of opens, we have a functor associating a single vector space, more in line with the idea of a 'sheaf'.

1. Perverse sheaves on stratified surfaces

1.1. Perverse sheaves with values in abelian categories

Sheaves with values in abelian categories.
Let $\mathcal{A}$ be an Grothendieck abelian category. In particular, $\mathcal{A}$ has arbitrary products and projective limits.

For any topological space X, we denote by $\mathrm{Sh}(X, \mathcal{A})$ the category of $\mathcal{A}$-valued sheaves over X. By definition, such a sheaf $\mathcal{F}$ is a contravariant functor from the poset of opens in X into $\mathcal{A}$, satisfying descent. That is, for any open covering $\{U_i\}$ of an open set U, the map

$$\mathcal{F}(U) \longrightarrow \mathrm{Ker}\left\{\prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j)\right\}$$

is an isomorphism.

By $D(X, \mathcal{A})$, we denote the (unbounded) derived category of $\mathrm{Sh}(X, \mathcal{A})$. We consider it as a triangulated category.

For any continuous map $f : X \to Y$ of topological spaces, we have the standard adjoint functors

$$f^* : D(Y, \mathcal{A}) \to D(X, \mathcal{A}), \quad Rf_* : D(X, \mathcal{A}) \to D(Y, \mathcal{A}).$$

If X, Y are locally compact, we also have the functors

$$Rf_! : D(X, \mathcal{A}) \to D(Y, \mathcal{A}), \quad f^! : D(Y, \mathcal{A}) \to D(X, \mathcal{A}),$$

with their standard adjunctions; cf. [33].

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Decompositions, stratifications and exit paths

Concerning stratified spaces, we follow the terminology of [27] part.II §1.1-2.

Thus, a *decomposition* of a topological space X is a collection $\mathcal{S}$ of locally closed subsets $S \in \mathcal{S}$ called *strata* such that $X = \bigsqcup_{S\in\mathcal{S}} S$ is a disjoint decomposition and the closure of a stratum is a union of strata. The set $\mathcal{S}$ acquired then a partial order $\leq$ by inclusion of the closures, that is, $S \leq S'$ if $S \subset \overline{S'}$. For each $x \in X$, we denote by $S_x \in \mathcal{S}$ the stratum containing x. A *decomposed space* $(X, \mathcal{S})$ is a space equipped with a decomposition.

The concept of decomposition is identical to that of an $(\mathcal{S}, \leq)$-stratification in the sense of [38] Definition A.5.1. Recall that the latter defined as a continuous map $f : X \longrightarrow \mathcal{S}$, where the poset $\mathcal{S}$ is given the topology consisting of *upwardly closed sets*, that is, of $\mathcal{I} \subset \mathcal{S}$ such that $S \in \mathcal{I}$ implies $S' \in \mathcal{I}$ whenever $S \leq S'$. Explicitly, the map f is given by $f(x) = S_x$.

Let $(X, \mathcal{S})$ be a decomposed space. We denote the inclusions of the strata by $i_S : S \to X$. By $\mathrm{Sh}(X, \mathcal{S}, \mathcal{A}) \subset \mathrm{Sh}(X, \mathcal{A})$, we denote the category of sheaves $\mathcal{F}$ which are constructible with respect to $\mathcal{S}$, that is, such that each $i_S^*\mathcal{F}$ is locally constant on S. By $D(X, \mathcal{S}; \mathcal{A}) \subset D(X, \mathcal{A})$, we denote the subcategory of complexes of sheaves $\mathcal{F}$ whose cohomology sheaves $\underline{H}^i(\mathcal{F})$ are constructible with respect to $\mathcal{S}$.

Let us recall the concept of exit paths for $(X, \mathcal{S})$, originally introduced by MacPherson; see [45] for a more detailed treatment. For $x \in X$, we denote by $S_x \in \mathcal{S}$ the stratum containing x. This gives a partial order $\leq$ on X (as a set) given by $x \leq y$, if $S_x \leq S_y$, that is, $S_x \subset \overline{S_y}$. An *exit path* for $(X, \mathcal{S})$ is a continuous parametrized path $\gamma : [0, 1] \longrightarrow X$ which is monotone with respect to $\prec$, that is, such that for $t_1 \leq t_2$ we have $\gamma(t_1) \leq \gamma(t_2)$. The *category of exit paths* $\mathrm{Exit}(X, \mathcal{S})$ has, as objects, all points $x \in X$, with $\mathrm{Hom}_{\mathrm{Exit}(X,\mathcal{S})}(x, y)$ being the set of isotopy classes of exit paths γ with $\gamma(0) = x$ and $\gamma(1) = y$. Thus, $\mathrm{Exit}(X, \mathcal{S})$ can be considered as a stratified version of the fundamental groupoid of X (to which it reduces in the particular case when $\mathcal{S}$ consists of just one stratum X). By reversing the direction of the paths (or passing to the opposite category), we get the *category of entrance paths* $\mathrm{Entr}(X, \mathcal{S}) = \mathrm{Exit}(X, \mathcal{S})^{\mathrm{op}}$.

We will use some particular types of decompositions in which one imposes various 'conicity' conditions describing the neighborhood of a stratum in the closure of a larger stratum:

(1) Whitney stratifications, see [27] part II §1.2. In this case, the strata are C^∞-manifolds.
(2) Topological stratifications, see [26] and [45] §3.1. In this case, the strata are topological manifolds.
(3) Conical stratifications, see [38] Definition A.5.5. In this case, strata are not required to be manifolds, but near a stratum S, the space X is locally identified with the product of S and the cone over another decomposed space with strata labelled by $S' \in \mathcal{S}$ with $S \prec S'$.

It is known that these three conditions are of increasing generality, that is, (1)⇒(2)⇒(3).

Proposition 1.1.1. *Let $(X, \mathcal{S})$ be a space with a conical stratification. The category* $\mathrm{Sh}(X, \mathcal{S}, \mathcal{A})$ *is equivalent to* $\mathrm{Fun}(\mathrm{Exit}(X, \mathcal{S}), \mathcal{A})$ *(the category of covariant functors).*

Proof. For topological stratifications, this is the original result of MacPherson; see [45] Theorem 1.2. For conical stratifications, this follows from [38] Theorem A.9.3 which gives an ∞-categorical upgrade of $\mathrm{Exit}(X, \mathcal{S})$. □

Suppose now that X is a complex manifold and $\mathcal{S}$ is a complex analytic Whitney stratification of X. By $\mathrm{PS}(X, \mathcal{S}, \mathcal{A}) \subset D(X, \mathcal{S}, \mathcal{A})$, we denote the subcategory of *perverse sheaves* (with respect to the middle perversity). Recall [3][33] that $\mathcal{F} \in \mathcal{P}(X, \mathcal{S}, \mathcal{A})$ iff two conditions are satisfied:

(P^+) For every $S \in \mathcal{S}$, we have $\underline{H}^n(i_S^*\mathcal{F}) = 0$ for $n > -\dim_{\mathbb{C}}(S)$,
(P^-) For every $S \in \mathcal{S}$, we have $\underline{H}^n(i_S^!\mathcal{F}) = 0$ for $n < -\dim_{\mathbb{C}}(S))$.

It is well known [3] that the category $\mathrm{PS}(X, \mathcal{S}; \mathcal{A})$ is the heart of a t-structure and so is an abelian category.

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The case of stratified surfaces

We specialize to the case of $\dim_{\mathbb{C}}(X) = 1$, so X is a Riemann surface, possibly noncompact and with nonempty boundary. We fix a finite subset $N \subset X$ of interior points which we refer to as *special points* and denote the corresponding stratification $X = N \cup (X \setminus N)$ by $\mathcal{S} = \mathcal{S}_N$. This gives a topological stratification, and we adopt the following definition.

Definition 1.1.2. By a *stratified surface*, we mean a pair (X, N) consisting of:

(1) A topological manifold X of real dimension 2, possibly noncompact and with boundary.
(2) A finite subset $N \subset X$ of interior points which we refer to as *special points*.

We denote by $j : X \setminus N \to X$ and $i : N \to X$ the embeddings of the strata.

Let us fix a Grothendieck abelian category $\mathcal{A}$. We denote by $D(X, N; \mathcal{A}) \subset D(X, \mathcal{A})$ the full subcategory of complexes whose cohomology sheaves are constructible with respect to the stratification $\mathcal{S}_N$, that is, in our case, locally constant on $X \setminus N$.

Further, the concept of a perverse sheaf makes sense in this context and is given explicitly as follows.

Definition 1.1.3. Let (X, N) be a stratified surface and $\mathcal{A}$ a Grothendieck abelian category. An object $\mathcal{F}$ of $D(X, N; \mathcal{A})$ is called *perverse* if

(1) $j^*\mathcal{F}$ is isomorphic to $L[1]$, where L is a local system on $X \setminus N$ with values in $\mathcal{A}$,
(2) $H^n(i^*\mathcal{F}) = 0$ for $n > 0$,
(3) $H^n(i^!\mathcal{F}) = 0$ for $n < 0$.

The category of perverse sheaves with respect to N will be denoted $\mathrm{PS}(X, N; \mathcal{A})$. As explained above, it is an abelian category.

1.2. Milnor disks, Milnor pairs and the purity property

We denote by $\mathbb{D} \subset \mathbb{C}$ the closed unit disk. Let (X, N) be a surface X with a set of special points $N \subset X$ as in §1.1. By a *closed disk*, we mean a subspace $A \subset X$ homeomorphic to $\mathbb{D}$.

Definition 1.2.1. A *Milnor disk* in (X, N) is a pair (A, A'), where:

(1) $A \subset X$ is a closed disk containing at most one special point.
(2) $A' \subset \partial A \simeq S^1$ is a disjoint union of finitely many closed arcs, different from $\emptyset$ and the whole ∂A.

See the left of Figure 2. The concept of a Milnor disk can be compared with the following possibly more intuitive concept.

Definition 1.2.2. A *Milnor pair* for (X, N) is a pair (U, U'), $U' \subset U$, of closed subsets of X such that

(1) U is a closed disk containing at most one special point.
(2) U' is a finite, nonempty, disjoint union of closed disks $\{U_i\}_{i \in I}$ such that $K = U \setminus U'$ is contractible.

Thus, a Milnor disk can be seen as a Milnor pair (U, U') with U' being very thin, reducing to a union of boundary arcs; see Figure 2. Up to homotopy equivalence, there is no difference between the two concepts.

Example 1.2.3. Let X be a Riemann surface (one-dimensional complex manifold), z be a holomorphic coordinate near an interior point $x \in X$ and f be a holomorphic function defined near x such that $f(x) = 0$. Then for sufficiently small $\varepsilon > \delta > 0$ the pair formed by

$$U = \{|z| \leq \varepsilon\}, \quad U' = \{|z| \leq \varepsilon, \ \Re(f(z)) \leq \delta\}$$

is a Milnor pair. This explains our terminology, motivated by the concept of Milnor fibers in singularity theory. Note that the cardinality $|\pi_0(U')|$ is equal to $\mathrm{ord}_x(f)$, the order of vanishing of f at x.

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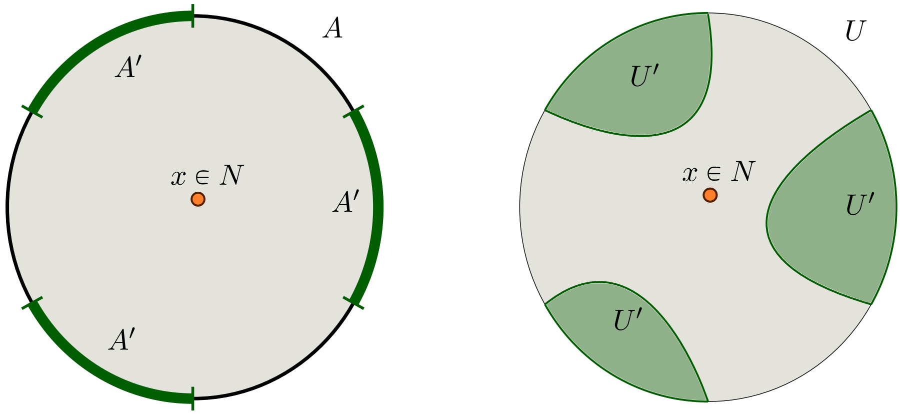

Figure 2. *A Milnor disk (A, A') and a Milnor pair (U, U').*

The role of Milnor disks for our purposes stems from the following:

Proposition 1.2.4 (Purity property). *Let (X, N) be a stratified surface, let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{F}$ be an object of the derived constructible category $D(X, N; \mathcal{A})$. Then the following are equivalent:*

(i) *$\mathcal{F}$ is a perverse sheaf.*
(ii) *For every Milnor disk (A, A'), the relative hypercohomology $H^i(A, A'; \mathcal{F})$ vanishes for $i \neq 0$.*

We will refer to the condition (ii) as *purity*.

Proof of Proposition 1.2.4. (i) $\Rightarrow$ (ii): Assume that $\mathcal{F}$ is perverse.

Assume first that A either contains no special point or contains exactly one special point x in its interior. Note that the first possibility is really a particular case of the second, as we can always introduce a 'dummy' special point, where a singularity is allowed but not present. So we assume that the second possibility holds. Denote by by $i_x : \{x\} \to X$ the inclusion of the point. Note that $R\Gamma_{\{x\}}(A, \mathcal{F}) \simeq i_x^! \mathcal{F}$, and so its cohomology, by Definition 1.1.3(3), is concentrated in degrees ≥ 0. Further, $R\Gamma(A, F) \simeq i_x^* \mathcal{F}$, and so its cohomology, by Definition 1.1.3(2), is concentrated in degrees ≤ 0. Consider now the following diagram with rows and columns being exact triangles:

$$\begin{tikzcd}
R\Gamma_{\{x\}}(A, A'; \mathcal{F}) \arrow[r] \arrow[d] & R\Gamma(A, A'; \mathcal{F}) \arrow[r] \arrow[d] & R\Gamma(A \backslash \{x\}, A' \backslash \{x\}; \mathcal{F}) \arrow[d] \\
R\Gamma_{\{x\}}(A; \mathcal{F}) \arrow[r] \arrow[d] & R\Gamma(A, \mathcal{F}) \arrow[r, "a"] \arrow[d, "c"] & R\Gamma(A \backslash \{x\}; \mathcal{F}) \arrow[d, "b"] \\
R\Gamma_{\{x\}}(A', \mathcal{F}) \arrow[r] & R\Gamma(A'; \mathcal{F}) \arrow[r, "d"] & R\Gamma(A' \backslash \{x\}; \mathcal{F}).
\end{tikzcd} \tag{1.2.5}$$

Note that $\mathcal{F}|_{A \backslash \{x\}} \simeq L[1]$, a local system in degree (-1) and $A \backslash \{x\}$ is homotopy equivalent to S^1. So $R\Gamma(A \backslash \{x\}; \mathcal{F})$ has cohomology only in degrees $\{-1, 0\}$. The long exact sequence (LES) of cohomology of the middle row of the diagram gives, using the information above, the following:

$$\begin{aligned} H^n(A, \mathcal{F}) &= 0 \quad \text{for } n \notin \{-1, 0\}, \\ H^n_{\{x\}}(A, \mathcal{F}) &= 0 \quad \text{for } n \notin \{0, 1\}. \end{aligned} \tag{1.2.6}$$

Look now at the middle column of the diagram. Since $R\Gamma(A'; \mathcal{F})$ is concentrated in degree (-1), in order to show that $R\Gamma(A, A'; \mathcal{F})$ has cohomology only in degree 0, it suffices to show that

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$c : H^{-1}(A; F) \to H^{-1}(A'; \mathcal{F})$ is injective. For this, it suffices to prove that the maps induced by a and b on H^{-1} are injective. For a, it follows from the fact (1.2.6) that $R\Gamma_{\{x\}}(A; \mathcal{F})$ has no cohomology in degree (-1). For b, we use the identification $\mathcal{F}|_{A\backslash\{x\}} \simeq L[1]$ as above. Then the statement becomes that $H^0(A\backslash\{x\}; L) \to H^0(A'; L)$ is injective which is clear.

Suppose now that the special point x lies in ∂A. If $x \in A'$, then by excision we reduce to the case when $A \cap N = \emptyset$ treated above. So let $x \in \partial A \backslash A'$. In this case, the argument is similar to the above, as $A\backslash\{x\}$ is contractible, and so $\mathcal{F}|_{A\backslash\{x\}} = L[1]$ has cohomology only in degree (-1).

(ii) $\Rightarrow$ (i): Vice versa, suppose that $\mathcal{F}$ is an object of $D(X, N; \mathcal{A})$ satisfying the purity condition. Let $A \subset X$ be a closed disk not containing any special points. Let $A' \subset \partial A$ be a disjoint union of two closed arcs so that (A, A') is a Milnor disk. Since by our assumptions, $\mathcal{F}|_A$ has locally constant, hence constant cohomology, it is straightforward to conclude that

$$\mathrm{R}\Gamma(A, \mathcal{F}) \simeq \mathrm{R}\Gamma(A, A'; \mathcal{F})[1].$$

By purity, this implies that $j^*\mathcal{F}[-1] \simeq L$ is quasi-isomorphic to a single local system with values in $\mathcal{A}$. This shows Condition (1) of Definition 1.1.3.

Now, let A be an closed disk that contains exactly one special point x in its interior. Let $A' \subset \partial A$ be the disjoint union of two arcs. We consider again the diagram (1.2.5), arguing now 'in the other direction'.

That is, look at the middle column. By purity, $R\Gamma(A, A'; \mathcal{F})$ has cohomology only in degree 0. But since $j^*\mathcal{F}[-1] = L$ is a single local system in degree 0, the complex $R\Gamma(A'; F)$ has cohomology only in degree (-1). Therefore, $R\Gamma(A, F) \simeq i_x^*\mathcal{F}$ has cohomology only in degrees $\{-1, 0\}$, thus establishing Condition (2) of Definition 1.1.3.

Next, look at the left column. Clearly, $R\Gamma_{\{x\}}(A'; \mathcal{F}) = 0$, as $x \notin A'$, and so $i_x^! F \simeq R\Gamma_{\{x\}}(A, \mathcal{F})$ is identified with $R\Gamma_{\{x\}}(A, A'; \mathcal{F})$. Now, the latter can be analyzed via the top row of the diagram, which contains $R\Gamma(A, A'; \mathcal{F})$, with cohomology in degree 0 and $R\Gamma(A\backslash\{x\}, A'\backslash\{x\}; \mathcal{F})$ which, we claim, has cohomology only in degree 0. This follows from looking at the right column, where the statement reduces to the claim that $H^0(A\backslash\{x\}; L) \to H^0(A'; L)$ is injective. Therefore, $i_x^!\mathcal{F}$ has cohomology only in degrees $\{0, 1\}$, thus establishing Condition (3) of Definition 1.1.3. □

Remark 1.2.7. Assume that we are in the situation of Example 1.2.3. Then $R\Gamma(U, U'; \mathcal{F})$ is identified with $\Phi_f(\mathcal{F})_x$, the stalk at x of the complex of vanishing cycles for $\mathcal{F}$ with respect to f; see [33]. It is well known (*loc. cit.*) that $\Phi_f(\mathcal{F})$ is itself a perverse sheaf which, in our case, amounts to saying that $\Phi_f(\mathcal{F})_x$ is quasi-isomorphic to a single vector space in degree 0. This provides an alternative proof of purity for such Milnor pairs, at least in the classical case when $\mathcal{A}$ is the category of vector spaces over a field.

2. The paracyclic category and constructible sheaves

In this section, we will introduce the paracyclic category $\Lambda(X, N)$ of a stratified surface and explain how the formalism of Kan extensions, applied to a directed version of $\Lambda(X, N)$, can be used to describe the Verdier duality of the derived constructible category. The ideas and constructions introduced in this section serve as a preparation for the main part of this work, §3, where we will apply similar techniques to parametrize perverse sheaves in terms of the subcategory $M(X, N) \subset \Lambda(X, N)$ of Milnor disks.

2.1. The standard paracyclic category and the Ran space of the circle

Recall that the standard *simplex category* Δ has, as objects, the standard finite nonempty ordinals $[n] = \{0, 1, \cdots, n\}$, $n \geq 0$, with morphisms being monotone maps. The morphisms of Δ are generated by the coface and codegeneracy maps

$$\delta_i : [n-1] \longrightarrow [n], \quad i = 0, \cdots, n \quad \text{(omitting } i\text{);}$$
$$\sigma_j : [n+1] \longrightarrow [n] \quad j = 0, \cdots, n \quad \text{(repeating } j\text{),}$$

subject to well-known relations; see, for example, [11], Chapter III, Appendix A, Proposition 2. We denote by $\Delta^{\mathrm{surj}} \subset \Delta$ the subcategory with the same objects and only surjective maps as morphisms. In other words, morphisms of Δ^{surj} are generated by the σ_j only. As usual, we call a *simplicial object* in a category $\mathcal{A}$ a contravariant functor $Z : \Delta \to \mathcal{A}$. Thus, Z consists of objects $Z_n = Z([n]) \in \mathcal{A}, n \geq 0$ and morphisms (face and degenaracy maps)

$$\partial_i : Z_n \longrightarrow Z_{n-1},\ i = 0, \cdots, n; \quad s_j : Z_n \longrightarrow Z_{n+1},\ j = 0, \cdots, n+1,$$

satisfying the relations dual to those among the δ_i and σ_j. We will also use the term *half-simplicial object* for a contravariant functor $\Delta^{\mathrm{surj}} \to \mathcal{A}$. Thus, a half-simplicial object has only degeneracy maps but no face maps.

Definition 2.1.1 ([11] Chapter III Appendix A, [36] Definition 6.1.1). (a) The standard *paracyclic category* Λ_∞ has the objects $\langle n \rangle$, $n \geq 0$ which are in bijection with those of Δ. Its morphisms are generated by those of Δ (i.e., the $\delta_i : \langle n-1 \rangle \to \langle n \rangle$ and $\sigma_j : \langle n+1 \rangle \to \langle n \rangle$ as above satisfying the same relations) together with additional *automorphisms* $\tau_n : \langle n \rangle \to \langle n \rangle$ which are subject to the following relations:

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \text{ for } 1 \leq i \leq n, \quad \tau_n \delta_0 = \delta_n;$$
$$\tau_n \sigma_i = \sigma_{i+1} \tau_{n+1} \text{ for } 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2;$$

(b) The *cyclic category* Λ is obtained from Λ_∞ by imposing the additional relations $\tau_n^{n+1} = \mathrm{Id}$.

The following proposition is well known; see [13]. It can be expressed by saying that Λ_∞ is a *central extension of* Λ *by* $\mathbb{Z}$.

Proposition 2.1.2.
(a) *The automorphisms* $\tau_n^{n+1} \in \mathrm{Hom}_{\Lambda_\infty}(\langle n \rangle, \langle n \rangle)$ *form a central system (i.e., define a natural transformation from the identity functor to itself).*
(b) *Let* $p : \Lambda_\infty \to \Lambda$ *be natural functor (identical on objects, surjective on morphisms). The fibers of each induced map*

$$\mathrm{Hom}_{\Lambda^\infty}(\langle m \rangle, \langle n \rangle) \longrightarrow \mathrm{Hom}_{\Lambda}(\langle m \rangle, \langle n \rangle)$$

are principal homogeneous spaces with respect to the action of $\mathbb{Z}$ *given by composition with powers of* τ_m^{m+1} *or, what by (a) is the same, by composition with powers* τ_n^{n+1}.

We also denote $\Lambda_\infty^{\mathrm{surj}} \subset \Lambda_\infty$ the subcategory on the same objects with the morphisms generated by the σ_j and τ_n only. By a *paracyclic object* in a category $\mathcal{A}$, we will mean a contravariant functor $Z : \Lambda_\infty \to \mathcal{A}$. As for simplicial objects, we write Z_n for the value of Z on $\langle n \rangle$ and ∂_i, s_j, t_n for the values on $\delta_i, \sigma_j, \tau_n$. By a *half-paracyclic object* we will mean a contravariant functor $\Lambda_\infty^{\mathrm{surj}} \to \mathcal{A}$.

Remark 2.1.3. The categories Λ and Λ_∞ are self-dual, that is, isomorphic to their opposite categories [11] [21]. In fact, by introducing the additional codegeneracies $\sigma_{n+1} = \tau_n \sigma_n \tau_{n+1}^{-1} : \langle n+1 \rangle \to \langle n \rangle$, one can write their presentations in a manifestly self-dual way, so that cofaces and codegeneracies will be dual to each other.

A partial interpretation via the Ran space

We recall the topological version of the Ran space construction [4]. As pointed out in [4], this version goes back to Borsuk and Ulam [7].

Let M be a C^∞-manifold. The *Ran space* of M is the set $\mathrm{Ran}(M)$ of all finite nonempty subsets $I \subset M$ equipped with a natural (Vietoris) topology. If we choose a metric on M inducing the topology, then $\mathrm{Ran}(M)$ can be metrized using the corresponding Hausdorff distance. The space $\mathrm{Ran}(M)$ has a filtration by closed subspaces $\mathrm{Ran}^{\leq d}(M) = \{I \subset M : |I| \leq d\}$, and the complement

$$\mathrm{Ran}^{\leq d}(M) \setminus \mathrm{Ran}^{\leq d-1}(M) \simeq \mathrm{Sym}^d_{\neq}(M)$$

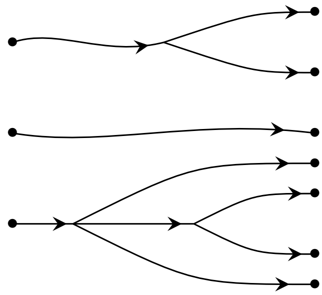

Figure 3. *An exit path in* Ran(M).

is the configuration space of unordered d-tuples of distinct points in M. In this way, each $\mathrm{Ran}^{\leq d}(M)$ becomes a Whitney stratified space, and $\mathrm{Ran}(M)$ can be considered as a (infinite-dimensional) space with a conical stratification; see §1.1. In particular, we can speak about the category of exit paths $\mathrm{Exit}(\mathrm{Ran}(M))$ and, for a Grothendieck abelian category $\mathcal{A}$, about $\mathcal{A}$-valued constructible sheaves on $\mathrm{Ran}(M)$ (with respect to the stratification by the $\mathrm{Sym}^d_{\neq}(M)$).

Remarks 2.1.4.

(a) An exit path in $\mathrm{Ran}(M)$ can be seen as a history of a colony of bacteria living in M which can move and multiply (by splitting) but not merge together, and cannot die; see Figure 3.

(b) A constructible sheaf $\mathcal{F}$ on $\mathrm{Ran}(M)$ assigns to any finite nonempty $I \subset M$ an object $\mathcal{F}_I \in \mathcal{A}$ (the stalk). When I 'evolves' into J by moving and splitting, we have a morphism $\mathcal{F}_I \to \mathcal{F}_J$ (the generalization map).

Let us focus, in particular, on the Ran spaces of the real line $\mathbb{R}$ and the circle S^1.

Example 2.1.5. It goes back to Bott [8] that $\mathrm{Ran}^{\leq 3}(S^1)$ is homeomorphic to the 3-sphere S^3. Further, inside this sphere $\mathrm{Ran}^{\leq 1}(S^1) = S^1$ is embedded as a trefoil knot, and $\mathrm{Ran}^{\leq 2}(S^1)$ is a Moebius band bounding this knot. See [41] for a beautiful treatment using elliptic functions. The topology and homotopy type of $\mathrm{Ran}^{\leq d}(S^1)$ for higher d was studied in [46].

The following result was proven in [9]:

Proposition 2.1.6.

(a) *The category* $\mathrm{Exit}(\mathrm{Ran}(\mathbb{R}))$ *is equivalent to* $(\Delta^{\mathrm{surj}})^{\mathrm{op}}$. *In particular, $\mathcal{A}$-valued constructible sheaves on* $\mathrm{Ran}(\mathbb{R})$ *can be identified with half-simplicial objects in $\mathcal{A}$.*

(b) *The category* $\mathrm{Exit}(\mathrm{Ran}(S^1))$ *is equivalent to* $(\Lambda_\infty^{\mathrm{surj}})^{\mathrm{op}}$. *In particular, $\mathcal{A}$-valued constructible sheaves on* $\mathrm{Ran}(S^1)$ *can be identified with half-paracyclic objects in $\mathcal{A}$.*

Proof. (a) An exit path γ in any $\mathrm{Ran}(M)$ going from I to J gives, for any $x \in I$, a tree of descendents of x which terminates in a subset of J. This gives a surjection $a_\gamma : J \to I$ (the 'ancestry map'). Isotopic exit paths lead to the same surjection. If $M = \mathbb{R}$, then the order of $\mathbb{R}$ makes both I and J into nonempty finite ordinals and the surjection a_γ is monotone.

(b) Recall from [11] Chapter III, Appendix A the geometric definition of the cyclic category Λ. For this, we identify $\langle n\rangle$ with the set of $(n+1)$st roots of 1 in the standard circle S^1. Then $\mathrm{Hom}_\Lambda(\langle m\rangle, \langle n\rangle)$ is the set of connected components of the space of degree 1 monotone maps $f : (S^1, \langle m\rangle) \to (S^1, \langle n\rangle)$. Each such connected component has the homotopy type of S^1, and $\mathrm{Hom}_{\Lambda_\infty}(\langle m\rangle, \langle n\rangle)$ is obtained by passing to the universal coverings of these components. That is, $\mathrm{Hom}_{\Lambda_\infty}(\langle m\rangle, \langle n\rangle)$ is the set of isotopy classes of data (f, s) consisting of f as above together with a homotopy s between f and the identity (as maps $S^1 \to S^1$). Note now that for $M = S^1$, an exit path γ as in (a) gives not only a surjection a_γ but a well-defined isotopy class of pairs (f, s), where $f : (S^1, J) \to (S^1, I)$ is a monotone degree 1 map and s is homotopy of f to the identity. □

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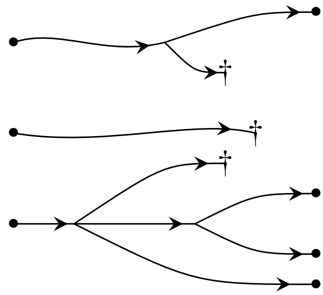

Figure 4. *An exit path in* Ran(M) *with deaths.*

Remark 2.1.7. One would like to extend the approach with the Ran spaces so as to realize the full categories Δ, Λ_∞ or functors out of them in terms of some categories of exit paths or constructible sheaves. For this, in the language of Remark 2.1.4(a), we would need to modify the concept of an exit path as a history of a colony of bacteria so as to allow the bacteria to die; see Figure 4. Then for such a 'history with deaths' evolving from I to J we will still have the ancestry map $J \longrightarrow I$ but it need not be surjective, as some lines may die out.

To account for such 'exit paths with deaths', one needs to consider constructible sheaves $\mathcal{F}$ on Ran(M) equipped with an additional *monotone structure* which is a system of maps $\mathcal{F}_J \longrightarrow \mathcal{F}_I$ given for any nested pair $I \subset J \subset S^1$ of nonempty finite sets and transitive in nested triples.

We do not pursue this approach further but note that our point of view based on Milnor disks (A, A') has A', a finite union of intervals in the circle $\partial A \simeq S^1$, playing the role of a finite subset $I \in \mathrm{Ran}(\partial A)$.

A systematic approach to the matter discussed in Remark 2.1.7 via 'unital' Ran spaces was developed in [9, 10]. The author recovers the paracyclic category and Joyal's categories Θ_n as unital exit path categories associated to the Ran spaces of S^1 and $\mathbb{R}^n$, respectively.

2.2. The paracyclic category of a stratified surface

Let (X, N) be a stratified surface as defined above. Throughout this text, we will assume that, if $X \cong S^2$, then $|N| \geq 2$. In this section, we introduce the paracyclic category $\Lambda(X, N)$ of (X, N) which can be seen as a certain amalgamation of the copies of Λ^∞ associated with the circles of directions at all the points $x \in X$

Pant cobordisms and the paracyclic category

We will use the notation $I = [0, 1]$ for the closed unit interval and, as before, $\mathbb{D}$ for the closed unit disk.

Definition 2.2.1. By a *para-disk* in (X, N), we mean a pair (A, A'), where $A \subset X$ is a closed disk such that $|A \cap N| \leq 1$ and $A' \subset \partial A \cong S^1$ is a compact one-dimensional submanifold, that is one of the following:

(i) the empty set,
(ii) a finite nonempty union of closed intervals,
(iii) the full boundary circle.

Thus, a Milnor disk is a particular case of a para-disk corresponding to the possibility (ii) of Definition 2.2.1. In the other two cases, a para-disk (A, A') will be called:

(a) a *standard disk*, if $A' = \emptyset$,
(b) a *bounded disk*, if $A' = \partial A$.

We now define morphisms between para-disks. Intuitively, such a morphism should be a certain isotopy class of paths $(A_t, A'_t)_{t \in I}$ in the space of para-disks. We want such paths to satisfy the following dynamical requirements as t increases from 0 to 1:

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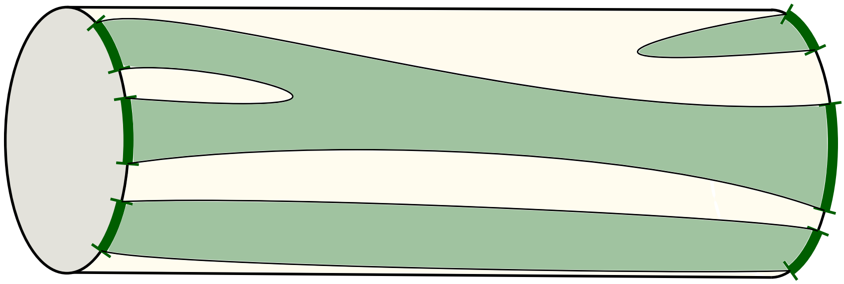


Figure 5. *A pant cobordism.*

(PD1) The components A'_t can merge together and can appear *ex nihilo* (growing out of single points) but cannot split.

(PD2) A special point $x \in N$ can enter the interior of A_t (i.e., A_t can 'run it over') only through the complement $A_t \backslash A'_t$ and exit A_t only through A'_t.

To implement this formally, we represent paths in the space of para-disks via maps $I \times \mathbb{D} \to X$. We start with formalizing the merging behavior of the components A_t as in (PD1).

Definition 2.2.2.

(1) Let $P \subset I \times S^1$ be a subset. For any $t \in I$, we denote by $P_t = P \cap (\{t\} \times S^1)$ the slice of P over t. We can view P_t as a subset in S^1.

(2) By a *pant cobordism*, we will mean a closed two-dimensional (topological) submanifold $P \subset I \times S^1$ with boundary such that:

  (2a) The slices $P_0, P_1 \subset S^1$ are compact one-dimensional submanifolds with boundary, as in Definition 2.2.1.

  (2b) The inclusion $P_1 \subset P$ is a homotopy equivalence.

An example of a pant cobordism is depicted in Figure 5.

Remarks 2.2.3. (a) Strictly speaking, a pant cobordism P is a manifold with corners, not just boundary, the corners being the boundary points of P_0 and P_1, as one can see in Figure 5. Since we consider P as a topological manifold, we ignore this subtlety.

(b) Intuitively, the slices $P_t \subset S^1$ correspond to the one-dimensional submanifolds $A'_t \subset A_t$ in the picture with paths in the space of para-disks. Of course, for some values of t such slices may not be of the form allowed in Definition 2.2.1, in particular, they may have, as components, single points (which can then disappear or grow to become intervals) Nevertheless, the condition (2b) of Definition 2.2.2 corresponds to the requirement (PD1) on the paths. In this way, a pant cobordism can (after time reversal $t \mapsto 1 - t$) be seen as a thickened version of an 'exit path with deaths' from Remark 2.1.7.

Definition 2.2.4. The *paracyclic category* $\Lambda(X, N)$ *of* (X, N) is the category with objects being para-disks (A, A') for (X, N). A morphism

$$f : (A_0, A'_0) \longrightarrow (A_1, A'_1)$$

in $\Lambda(X, N)$ consists of

- a pant cobordism $P \subset I \times S^1$.
- a continuous map $H : I \times \mathbb{D} \to X$, which we also consider as a family of maps $H_t : \mathbb{D} \to X$, $t \in I$ such that
  (1) H is an isotopy, that is, each H_t is an embedding,
  (2) for $i \in \{0, 1\}$, the embedding H_i induces homeomorphisms $\mathbb{D} \cong A_i$ and $P_i \cong A'_i$,
  (3) for every $t \in I$, we have $|H_t(\mathbb{D}) \cap N| \leq 1$,
  (4) for every $t_0 \in I$ and $x \in H_{t_0}(P_{t_0}) \cap N$, there exists $\varepsilon > 0$ such, for every $t_0 \leq t \leq t + \varepsilon$, $x \notin H_t(\mathbb{D} \setminus P_{t_0})$.

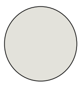
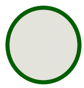
- ◦ two such data $(H, P), (H', P')$ define the same morphism if there exists a homeomorphism $\varphi : I\times\mathbb{D} \to I \times \mathbb{D}$ such that $\varphi|P$ induces a homeomorphism with P', together with a homotopy $\alpha : I^2 \times \mathbb{D} \to X$ with $\alpha_0 = H$ and $\alpha_1 = H'$ such that, for every $s \in I$, α_s satisfies the above conditions.

We denote by $S(X, N) \subset \Lambda(X, N)$ the full subcategory of standard disks, by $B(X, N) \subset \Lambda(X, N)$ the full subcategory of bounded disks and by $M(X, N) \subset \Lambda(X, N)$ the full subcategory of Milnor disks. We refer to $M(X, N)$ as the *Milnor category* of (X, N).

Remarks 2.2.5. (a) Given a morphism f with a representative (P, H), we have, for any $t \in I$, a closed disk $A_t = H_t(\mathbb{D}) \subset X$ and a closed subset $A'_t = H_t(P_t) \subset \partial A_t$. The pair (A_t, A'_t) depends only on f. For generic values of t, the slice P_t belongs to one of the three types described in Definition 2.2.1 and so (A_t, A'_t) is a para-disk by the condition (2) The condition (4) corresponds to the intuitive requirement (PD2) on paths in the space of para-disks while (PD1) corresponds, as mentioned above, to the condition (2b) of Definition 2.2.2 of a pant cobordism.

(b) Our assumption that if $X \cong S^2$, then $|N| \geq 2$ implies that the mapping spaces which appear implicitly in our definition of $\Lambda(X, N)$ have contractible components so that it is justified to consider it as an ordinary category (rather than an ∞-category).

Example 2.2.6. The category $M(\mathbb{C}, \emptyset)$ of Milnor disks in $(\mathbb{C}, 0)$ is equivalent to the paracyclic category Λ_∞. This is shown similarly to the proof of Proposition 2.1.6. Further, the category $\Lambda(\mathbb{C}, \emptyset)$ is equivalent to the category obtained from Λ_∞ by adjoining an initial and a final objects which correspond to the objects

$(A, \varnothing) =$  and $(A, \partial A) =$ ,

respectively.

The Milnor category and perverse sheaves

The role of the category $M(X, N)$ for our purposes is explained by the following.

Proposition 2.2.7. *Let $\mathcal{F} \in \mathrm{PS}(X, N; \mathcal{A})$ be a perverse sheaf on (X, N) with values in a Grothendieck abelian category $\mathcal{A}$. Then the correspondence $(A, A') \mapsto H^0(A, A'; \mathcal{F})$ extends to a functor $h_{\mathcal{F}} : M(X, N)^{\mathrm{op}} \to \mathcal{A}$.*

Proof. Let $f : (A_0, A'_0) \to (A_1, A'_1)$ be a morphism between two Milnor disks represented by a pair (P, H) as in Definition 2.2.4. Let $\widetilde{N} = H^{-1}(N) \subset I \times \mathbb{D}$. Because of condition (1) of that definition, $\widetilde{N}$ is a one-dimensional topological submanifold with boundary, that is, a disjoint union of closed curvilinear intervals in the cylinder $I \times D$, each of them projecting to I in an injective way. We orient these curves following the increase of $t \in I$.

Let $\widetilde{N}^+ \subset \widetilde{N}$ be the union of components that terminate (in the sense of the above orientation) on P. Let $\widetilde{N}^- \subset \widetilde{N}$ be the union of components that terminate on $\{1\} \times \mathbb{D}$. Thus, $\widetilde{N}^+ \cup \widetilde{N}^- = \widetilde{N}$ and $\widetilde{N}^+ \cap \widetilde{N}^-$ is the union of components that terminate on the slice P_1.

Further, let $\widetilde{\mathcal{F}} = H^*(\mathcal{F})$. It is a complex of sheaves on $I \times D$ constructible with respect to the stratification given by $\widetilde{N}$. By Proposition 1.2.4,

$$H^0(A_i, A'_i; \mathcal{F}) \simeq R\Gamma(\{i\} \times \mathbb{D}, P_i; \widetilde{\mathcal{F}}), \quad i \in \{0, 1\} \subset I.$$

Consider the diagram of restrictions

$$R\Gamma(\{1\} \times \mathbb{D}, P_1; \widetilde{\mathcal{F}}) \xleftarrow{\rho_1} R\Gamma(I \times \mathbb{D}, P; \widetilde{\mathcal{F}}) \xrightarrow{\rho_0} R\Gamma(\{0\} \times \mathbb{D}, P_0; \widetilde{\mathcal{F}}).$$

We claim that ρ_1 is a quasi-isomorphism (and therefore, by purity, it reduces to an isomorphism of objects of $\mathcal{A}$). Indeed, denote

$$P^+ = P \cup \widetilde{N}^+ \subset I \times \mathbb{D}, \ \mathbb{D}^- = \{1\} \times \mathbb{D} \cup \widetilde{N}^- \quad \subset \quad I \times \mathbb{D}.$$

Because of the condition (2b) of Definition 2.2.2 and the entry–exit condition (4) of Definition 2.2.4, the inclusion of the slice $P_1 \subset P^+$ is a homotopy equivalence, and the inclusion $\{1\} \times \mathbb{D} \hookrightarrow \mathbb{D}^-$ is a homotopy equivalence as well. This means that each of the two restriction morphisms

$$R\Gamma(I \times \mathbb{D}, P; \mathcal{F}) \longrightarrow R\Gamma(I \times \mathbb{D}, P'; \mathcal{F}) \longrightarrow R\Gamma(\{1\} \times \mathbb{D}, P_1; \mathcal{F})$$

whose composition is ρ_1, is a quasi-isomorphism.

We now define the value of the functor $h_{\mathcal{F}}$ on f, that is, the morphism $h_{\mathcal{F}}(f) : H^0(A_1, A_1'; \mathcal{F}) \to H^0(A_0, A_0'; \mathcal{F})$ to be given by $\rho_2 \rho_1^{-1}$. The necessary verifications are left to the reader. □

Remark 2.2.8. In a similar way, utilizing the ∞-category of spans, one can show that the association $(A, A') \mapsto R\Gamma(A, A'; \mathcal{F})$ extends to an ∞-functor from $\Lambda(X, N)$ to $\mathcal{D}_\infty(\mathcal{A})$, the ∞-categorical enhancement of the derived category of $\mathcal{A}$; see §A.3.

Example 2.2.9. The categories $S(X, N)$ of standard disks and $B(X, N)$ of bounded disks are equivalent to $\mathrm{Entr}(X, N)$ and $\mathrm{Exit}(X, N)$, the categories of entrance and exit paths of the stratified space (X, N) respectively. The first equivalence has the form

$$\mathrm{Entr}(X, N) \to S(X, N), \ x \mapsto (A_x, \emptyset),$$

where $A_x \subset X$ is a disk containing x such that $A_x \cap N = \emptyset$ if $x \notin N$. The second equivalence is defined in the dual way.

The paracyclic duality

Next, we describe an identification of $\Lambda(X, N)$ with its opposite category $\Lambda(X, N)^{\mathrm{op}}$ which will play an important role in interpreting the Verdier duality for perverse sheaves. We start with the following remarks. For a closed subset Z of a topological space Y, we denote by $\mathring{Z}$ the interior of Z. The next two propositions are then clear.

Proposition 2.2.10.

(a) *For a para-disk $(A, A') \subset X$ the pair $(A, A')^* := (A, \partial A \backslash (\mathring{A}'))$ is again a para-disk.*
(b) *Let $\sigma : I \times S^1 \times I \times S^1$ be the involution $(t, \theta) \mapsto (1 - t, \theta)$. For a pant cobordism $P \subset I \times S^1$, the subset $P^* = \sigma(I \times S^1) \backslash \mathring{P}$ is again a pant cobordism.*

Proposition 2.2.11. *Let $i : \Lambda(X, N)' \subset \Lambda(X, N)$ denote the full subcategory consisting of those Milnor disks (A, A') such that $\partial A \cap N = \emptyset$. Then the inclusion i is an equivalence of categories.*

Proposition 2.2.12. *We have a perfect duality (which we call the paracyclic duality)*

$$\xi : \Lambda(X, N) \xrightarrow{\simeq} \Lambda(X, N)^{\mathrm{op}}$$

defined on objects by the association $(A, A') \mapsto (A, A')^*$.

Proof. Using Proposition 2.2.11, it suffices to define a duality on the equivalent subcategories $\xi' : \Lambda(X, N)' \xrightarrow{\simeq} \Lambda(X, N)'^{\mathrm{op}}$, which is given on objects by the desired formula $(A, A') \mapsto (A, A')^*$.

To do this, suppose we have a morphism f represented by (H, P); note that we may assume, replacing (H, P) by an equivalent representative if needed, that special points enter in $I \times S^1 \setminus P$ and exit in $\mathring{P}$. Then we define $\xi(f)$ to be represented by $(H(1 - t, -), P^*)$. It is straightforward to verify that this association yields a well-defined functor squaring to the identity, that is, giving a perfect duality. □

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Note, that the paracyclic duality ξ interchanges the subcategories $S(X,N)$ and $B(X,N)$, identifying them as opposite to one another, and restricts to a self-duality of $M(X,N)$.

2.3. The directed paracyclic category and its localization

Let (X,N) be as before. In this section, we exhibit $\Lambda(X,N)$ as a localization of another category $\overrightarrow{\Lambda}(X,N)$ which we call the directed paracyclic category. This latter category turns out to be more suitable for the use of Kan extensions.

Definition 2.3.1. We define the *directed paracyclic category* $\overrightarrow{\Lambda}(X,N)$ exactly as in Definition 2.3.1 but replacing condition (4) by the following:

(Ent) For every $x \in N$, we have
- (Ent1) if $x \in A_{t_0} = H_{t_0}(\mathbb{D})$ for $t_0 \in I$, then, for all $t \geq t_0$, we have $x \in A_t$,
- (Ent2) if $x \in A'_{t_0} = H_{t_0}(P_{t_0})$ for $t_0 \in I$, then, for all $t \geq t_0$, we have $x \in A'_t$.

A morphism $f : (A,A') \to (B,B')$ in $\overrightarrow{\Lambda}(X,N)$ is called a *weak equivalence* if either

(i) f is an isomorphism, or
(ii) f can be represented by a pair (P,H) such that $H_0^{-1}(A') \subset P$ is a homotopy equivalence and $H^{-1}(N) \subset P$.

We denote $W \subset \mathrm{Mor}(\overrightarrow{\Lambda}(X,N))$ the set of weak equivalences.

Remarks 2.3.2.

(a) The condition (Ent) is a two-step version of the entrance path condition: If a special point x enters A_{t_0}, then it stays in all the A_t for all $t \geq t_0$, and similarly for A'_{t_0}.
(b) The condition (ii) in the definition of a weak equivalence means that a special point x is allowed to enter $A'_{t_0} \subset A_{t_0}$ from the outside of A_{t_0} and stay there for all $t \geq t_0$.

We also denote by $\overrightarrow{S}(X,N), \overrightarrow{B}(X,N), \overrightarrow{M}(X,N) \subset \overrightarrow{\Lambda}(X,N)$ the full subcategories of standard disks, bounded disks and Milnor disks, respectively.

Proposition 2.3.3. *The natural morphism*

$$\pi : \overrightarrow{\Lambda}(X,N) \longrightarrow \Lambda(X,N)$$

exhibits $\Lambda(X,N)$ as a localization of $\overrightarrow{\Lambda}(X,N)$ along W.

Here, by 'localization' we mean $\overrightarrow{\Lambda}(X,N)[W^{-1}]$, the Gabriel–Zisman localization in the sense of ordinary categories [23]. In fact, one can prove stronger statements, identifying $\Lambda(X,N)$ with the ∞-categorical localization or with the Dwyer–Kan simplicial localization [16] of $\overrightarrow{\Lambda}(X,N)$ with respect to W. This can be done by adapting our proof below by using a hammock-type model for the Dwyer–Kan localization. We will not need this generalization for our purposes except for a very particular case in Lemma 2.5.2 below, which is easily proved directly.

Proof. Recall that in $\Lambda(X,N)$ a special point x is allowed to exit A_{t_0} through A'_{t_0}. This process is inverse to entering A_{t_0} through A'_{t_0} from the outside which is, according to Remark 2.3.2(b), a general form of a weak equivalence (apart from an isomorphism). Indeed, the composite process (entering A_{t_0} through A'_{t_0} from the outside and then bouncing back to the original position) is connected to the identity by a homotopy α as in Definition 2.2.4.

Therefore, the functor π inverts weak equivalences and we obtain an induced functor $\overline{\pi}$: $\overrightarrow{\Lambda}(X,N)[W^{-1}] \to \Lambda(X,N)$. We claim that $\overline{\pi}$ is an equivalence. To this end, we study a typical Hom-set

$$\mathrm{Hom}_{\overrightarrow{\Lambda}(X,N)[W^{-1}]}((A,A'),(C,C')). \tag{2.3.4}$$

By definition (cf. [23] §I.1), an element of this set is an equivalence class of zig-zags

$$(A, A') = (A_1, A'_1) \xleftarrow{w_1} (B_1, B'_1) \xrightarrow{f_1} (A_2, A'_2) \xleftarrow{w_2} \cdots \xrightarrow{f_{n-1}} (A_n, A'_n) = (C, C') \tag{2.3.5}$$

of arbitrary length, with $w_i \in W$. The equivalence relation on the set of such zig-zags is generated by two elementary moves:

(M1) For any factorization

$$\begin{tikzcd}
(A_i, A'_i) & (B_i, B'_i) \arrow[l, "w_i"'] \arrow[r, "f_{i+1}"] & (A_{i+1}, A'_{i+1}) \\
(B_{i-1}, B'_{i-1}) \arrow[u, "f_i"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

we can replace the fragment $\xrightarrow{f_i}\xleftarrow{w_i}\xrightarrow{f_{i+1}}$ with $\xrightarrow{f_{i+1}g}$.

(M2) For any factorization

$$\begin{tikzcd}
(B_{i-1}, B'_{i-1}) \arrow[r, "f_{i-1}"] & (A_i, A'_i) \arrow[dr, "h"'] & (B_i, B'_i) \arrow[l, "w_i"'] \arrow[d, "f_i"] \\
& & (A_{i+1}, A'_{i_1})
\end{tikzcd}$$

we can replace the fragment $\xrightarrow{f_{i-1}}\xleftarrow{w_i}\xrightarrow{f_i}$ with $\xrightarrow{h f_{i-1}}$.

These two moves imply the *hammock move*, which is at the basis of Dwyer–Kan localization theory [16] (except that we don't assume that the vertical morphisms are weak equvialences):

(H) Any two zig-zags connected by a *hammock*, that is, by a commutative diagram

$$\begin{tikzcd}
& (B_1, B'_1) \arrow[dl, "w_1"'] \arrow[r, "f_1"] \arrow[dd] & (A_2, A'_2) \arrow[dd] & (B_2, B'_2) \arrow[l, "w_2"'] \arrow[r, "f_2"] \arrow[dd] & \cdots & (B_{n-1}, B'_{n-1}) \arrow[l, "w_{n-1}"'] \arrow[dr, "f_n"] \arrow[dd] & \\
(A, A') & & & & & & (C, C') \\
& (\widetilde{B}_1, \widetilde{B}'_1) \arrow[ul, "\widetilde{w}_1"] \arrow[r, "\widetilde{f}_1"] & (\widetilde{A}_2, \widetilde{A}'_2) & (\widetilde{B}_2, \widetilde{B}'_2) \arrow[l, "\widetilde{w}_2"'] \arrow[r, "\widetilde{f}_2"] & \cdots & (\widetilde{B}_{n-1}, \widetilde{B}'_{n-1}) \arrow[l, "\widetilde{w}_{n-1}"'] \arrow[ur, "\widetilde{f}_n"'] &
\end{tikzcd}$$

are equivalent.

We now compare this with $\mathrm{Hom}_{\Lambda(X,N)}((A, A'), (C, C'))$. An element f of this latter set is an equivalence class of pairs (P, H) as in Definition 2.2.4. As usual, we write $A_t = H_t(\mathbb{D})$, $A'_t = H_t(P_t)$. Without loss of generality, we can assume that:

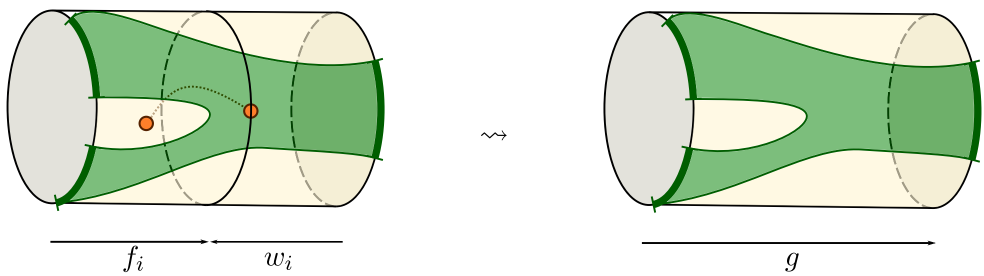
- P is smooth as a manifold with corners, that is, the part of ∂P lying over the open interval $(0, 1) \subset I$ is smooth.
- The projection of this part of ∂P to $(0, 1)$ is a Morse function. This implies that for all but finitely many values of t (which we call *critical values*) the slice P_t has one of the three forms listed in Definition 2.2.1 and therefore (A_t, A'_t) is a para-disk.
- The moments $t_1 < \cdots < t_n$, $t_i \in I$, of exit of special points $x \in N$ out of A_t (happening through A'_t) are noncritical.

Let $t'_i > t_i$, $i = 1, \cdots, n$, be sufficently close. As explained in the beginning of the proof, the restriction of (P, H) to the preimage of each interval $[t_i, t'_i]$ can be seen as an inverse of a weak equivalence in

$\overrightarrow{\Lambda}(X, N)$. while the restriction to each interval in the complement of the union of the $[t_i, t_i']$, is a morphism in $\overrightarrow{\Lambda}(X, N)$. Therefore, we can associate to (H, P) a zig-zag (2.3.5).

We claim that different choices of (H, P) representing the same morphism f, give rise to equivalent zig-zags. Any two such different choices are, by Definition 2.2.4, related by a reparemetrization $\varphi : I \times \mathbb{D} \to I \times \mathbb{D}$ and a homotopy $\alpha : I^2 \times \mathbb{D} \to X$. By choosing α generic enough, we see that any two choices are connected by a sequence of the following moves and their inverses:

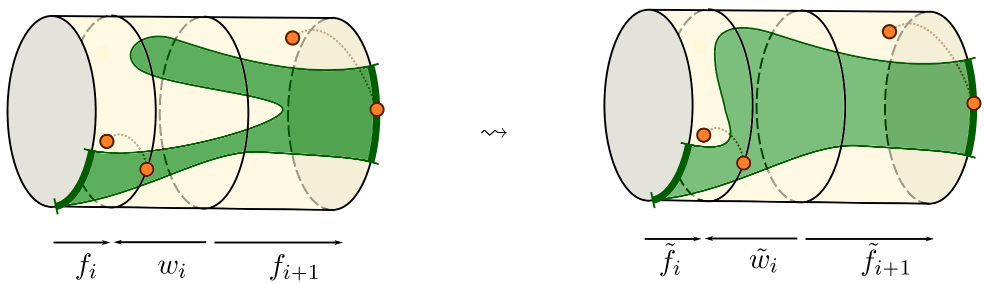
(M'1) replacing a representative (P, H) with a representative $(P, \widetilde{H})$ which, locally around $t \in I$, avoids the special point contained in A_t':



$$(2.3.6)$$

Denote by $\xleftarrow{w_i}$ the slice of (P, H) from the moment of exit of x until shortly afterwards and by $\xrightarrow{f_i}$ the slice from shortly before exit to the moment of exit; see (2.3.6). We see that we have three morphisms g, w_i, f_i $\overrightarrow{\Lambda}(X, N)$ and a factorization $w_i g = f_i$ in $\overrightarrow{\Lambda}(X, N)$ represented by an appropriate homotopy α. Therefore, the move (M′1) yields two zig-zags connected by the move (M1).

(M'2) replacing a representative (P, H) with a representative $(\widetilde{P}, H)$, where $\widetilde{P}$ is obtained by deforming P in a suitable way locally around one of the exit moments t_i so that two intervals in A_{t_i}' are replaced by one:



$$(2.3.7)$$

.

Making four slices of each the two cobordisms as in (2.3.7), we get two zig-zags connected by a hammock:

$$\begin{tikzcd}
 & (A_i, A_i') \arrow[d] & (B_i, B_i') \arrow[l, "w_i"'] \arrow[d] \arrow[dr, "f_{i+1}"] & \\
(B_{i-1}, B_{i-1}') \arrow[ur, "f_i"] \arrow[dr, "\widetilde{f}_i"'] & & & (A_{i+1}, A_{i+1}') \\
 & (\widetilde{A}_i, \widetilde{A}_i') & (\widetilde{B}_i, \widetilde{B}_i') \arrow[l, "\widetilde{w}_i"] \arrow[ur, "\widetilde{f}_{i+1}"'] &
\end{tikzcd}$$

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so they are equivalent by the hammock move. Therefore, the entire zig-zags corresponding to (P, H) and $(\widetilde{P}, H)$ are equivalent as well.

In this way, we define a functor $\overrightarrow{\Lambda}(X, N)[W^{-1}] \to \Lambda(X, N)$ which is easily seen to be quasi-inverse to $\overline{\pi}$. □

Corollary 2.3.8. *The functor π from Proposition 2.3.3 induces an equivalence $\overrightarrow{S}(X, N) \simeq S(X, N)$ and localizations $\overrightarrow{M}(X, N) \to M(X, N)$, $\overrightarrow{B}(X, N) \to B(X, N)$.*

2.4. Constructible sheaves with values in ∞-categories

Let (X, N) be a stratified surface, let $\mathfrak{O}(X)$ denote the poset of open subsets of X and let $\mathcal{D}$ be an ∞-category. The following is an ∞-categorical analog of the discussion for abelian categories in §1.1.

Lemma 2.4.1. *Given a functor $\mathcal{F} : \mathrm{N}(\mathfrak{O}(X))^{\mathrm{op}} \to \mathcal{D}$, an open subset $U \subset X$, and an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U, the following conditions are equivalent:*

(i) *Denote by $\mathfrak{O}(X)/\mathcal{U}$ the poset of open subsets $V \subset X$ such that $V \subset U_i$ for some $i \in I$. Then the canonical map*

$$\mathcal{F}(U) \longrightarrow \lim \mathcal{F}|(\mathfrak{O}(X)/\mathcal{U})^{\mathrm{op}}$$

is an equivalence in $\mathcal{D}$.

(ii) *Denote by $\mathcal{P}(I)$ the poset of nonempty finite subsets of I, and consider the inclusion $\mathcal{P}(I) \subset \mathfrak{O}(X)^{\mathrm{op}}$, $J \mapsto \cap_{j \in J} U_j$. Then the canonical map*

$$\mathcal{F}(U) \longrightarrow \lim \mathcal{F}|\mathcal{P}(I)$$

is an equivalence in $\mathcal{D}$.

Proof. The inclusion $\mathcal{P}(I)^{\mathrm{op}} \subset \mathfrak{O}(X)/\mathcal{U}$ is ∞-cofinal. □

A $\mathcal{D}$-*valued sheaf* on X is a functor

$$\mathcal{F} : \mathrm{N}(\mathfrak{O}(X))^{\mathrm{op}} \to \mathcal{D}$$

such that, for every open $U \subset X$ and every open cover $\mathcal{U}$ of U, the equivalent conditions of Lemma 2.4.1 hold. We denote by

$$\mathrm{Sh}(X; \mathcal{D}) \subset \mathrm{Fun}(\mathrm{N}(\mathfrak{O}(X))^{\mathrm{op}}, \mathcal{D})$$

the full subcategory spanned by the $\mathcal{D}$-valued sheaves on X.

Let $(\mathrm{Disk}^o(X, N, \leq)$ be the poset of standard pairs $(U, \emptyset)$ ordered by inclusion. We will consider it as a category. A morphism in $(\mathrm{Disk}^o(X, N, \leq)$ (i.e., an inclusion $U_1 \subset U_2$ of standard disks) will be called a *weak equivalence*, if $|N \cap U_1| = |N \cap U_2|$. We denote by W the set of weak equivalences. The map

$$i : \mathrm{Disk}^o(X, N) \subset \mathfrak{O}(X), \ (U, \emptyset) \mapsto U$$

identifies $\mathrm{Disk}^o(X, N)$ with a full subposet of $\mathfrak{O}(X)$. A sheaf $\mathcal{F}$ in $\mathrm{Sh}(X; \mathcal{D})$ is called *constructible* if its restriction $\mathcal{F}|\mathrm{Disk}^o(X, N)^{\mathrm{op}}$ maps weak equivalences to equivalences in $\mathcal{D}$. We denote the full subcategory of $\mathrm{Sh}(X; \mathcal{D})$ spanned by the constructible sheaves by $\mathrm{Sh}(X, N; \mathcal{D})$.

Remark 2.4.2. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $\mathcal{D} = \mathcal{D}^+(\mathcal{A})$ denote the corresponding (left-bounded) derived ∞-category as defined in [38, 1.3.2.8]. We equip the stable ∞-category $\mathrm{Sh}(X; \mathcal{D})$ with the t-structure $(\mathrm{Sh}(X; \mathcal{D}_{\geq 0}), \mathrm{Sh}(X; \mathcal{D}_{\leq 0}))$, where the t-structure on $\mathcal{D}$ is the one from [38, 1.3.2.19]. The heart of this t-structure is equivalent to $\mathrm{Sh}(X, \mathcal{A})$. Then, using the recognition principle for derived ∞-categories ([38, 1.3.3.7]), we obtain an equivalence of ∞-categories

$$\mathcal{D}^+(\mathrm{Sh}(X;\mathcal{A})) \xrightarrow{\simeq} \mathrm{Sh}(X;\mathcal{D}^+(\mathcal{A})).$$

In particular, the ∞-category $\mathrm{Sh}(X;\mathcal{D}(\mathcal{A}))$ is really an enhancement of the ordinary derived category of complexes of $\mathcal{A}$-valued sheaves. Further, this equivalence identifies our constructible category $\mathrm{Sh}(X, N;\mathcal{D}^+(\mathcal{A}))$ with the more traditional derived constructible category, defined as the full subcategory of $\mathcal{D}^+(\mathrm{Sh}(X;\mathcal{A}))$ spanned by objects with constructible cohomology sheaves.

We denote by $\mathrm{Disk}^o(X, N)[W^{-1}]_\infty$ the ∞-categorical localization of $\mathrm{Disk}^o(X, N)$ along the weak equivalences W. In particular, we may identify

$$\mathrm{Fun}(\mathrm{Disk}^o(X, N)[W^{-1}]_\infty^{\mathrm{op}}, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Disk}^o(X, N)^{\mathrm{op}}, \mathcal{D})$$

with the full subcategory spanned by those functors that map weak equivalences in $\mathrm{Disk}^o(X, N)$ to equivalences in $\mathcal{D}$.

Proposition 2.4.3. *The functor*

$$i^* : \mathrm{Sh}(X, N;\mathcal{D}) \longrightarrow \mathrm{Fun}(\mathrm{Disk}^o(X, N)[W^{-1}]_\infty^{\mathrm{op}}, \mathcal{D})$$

is an equivalence of ∞-categories.

Proof. Let $\mathcal{F} : \mathfrak{O}(X)^{\mathrm{op}} \to \mathcal{D}$ be a presheaf on X such that $\mathcal{F}|\mathrm{Disk}^o(X, N)^{\mathrm{op}}$ sends weak equivalences to equivalences in $\mathcal{D}$. We claim that the following conditions are equivalent:

(1) $\mathcal{F}$ is a sheaf.
(2) $\mathcal{F}$ is a right Kan extension of $\mathcal{F}|\mathrm{Disk}^o(X, N)^{\mathrm{op}}$.

The claim immediately implies the statement of the proposition. The reason why this statement is not completely formal is that in condition (2), we do not assume that the restriction of $\mathcal{F}$ to $\mathrm{Disk}^o(X, N)^{\mathrm{op}}$ satisfies a descent condition. We rather need to convince ourselves that this is automatic due to the assumption that $\mathcal{F}$ is constructible.

(1) $\Rightarrow$ (2): Suppose that $\mathcal{F}$ is a sheaf. We need to show that, for every open $U \subset X$, $\mathcal{F}(U)$ is the limit of the diagram $\mathcal{F}|(\mathrm{Disk}^o(X, N)/U)^{\mathrm{op}}$. We interpret the set $\mathcal{U} = \mathrm{Disk}^o(X, N)/U$ as an open cover of U so that this statement follows immediately from the hypothesis that $\mathcal{F}$ is a sheaf.

(2) $\Rightarrow$ (1): Suppose that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}|\mathrm{Disk}^o(X, N)^{\mathrm{op}}$. Let $U \subset X$ be an open subset, and let $\mathcal{U} \subset \mathfrak{O}(X)$ be an open cover of U. Let $\mathfrak{O}(X)/\mathcal{U}$ (resp. $\mathrm{Disk}^o(X, N)/\mathcal{U}$) denote the subposet of $\mathcal{U}$ consisting of those opens V (resp. $V \in \mathrm{Disk}^o(X, N)$) such that $V \subset U_i$ for some $U_i \in \mathcal{U}$. We need to show that the map

$$\mathcal{F}(U) \to \lim \mathcal{F}|(\mathfrak{O}(X)/\mathcal{U})^{\mathrm{op}}$$

is an equivalence. Since $\mathcal{F}|(\mathfrak{O}(X)/\mathcal{U})^{\mathrm{op}}$ is a right Kan extension of $\mathcal{F}|(\mathrm{Disk}^o(X, N)/\mathcal{U})^{\mathrm{op}}$, it suffices to show that the composite

$$\mathcal{F}(U) \to \lim \mathcal{F}|(\mathfrak{O}(X)/\mathcal{U})^{\mathrm{op}} \to \lim \mathcal{F}|(\mathrm{Disk}^o(X, N)/\mathcal{U})^{\mathrm{op}}$$

is an equivalence. Via the pointwise formula for $\mathcal{F}(U)$, we deduce that it suffices to show that $\mathcal{F}|(\mathrm{Disk}^o(X, N)/U)^{\mathrm{op}}$ is a right Kan extension along i^{op}, where

$$i : \mathrm{Disk}^o(X, N)/\mathcal{U} \subset \mathrm{Disk}^o(X, N)/U.$$

To this end, let $D \in \mathrm{Disk}^o(X, N)$ with $D \subset U$. We need to show that $\mathcal{F}(D)$ is a limit of $\mathcal{F}|(i/D)^{\mathrm{op}}$. Denote $\mathcal{E} = i/D$, and introduce the category $\mathcal{L}$ with

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- the set of objects of $\mathcal{L}$ is the set of objects of $\mathcal{E}$,
- a morphism between objects V and V' of $\mathcal{L}$ is a homotopy class of paths γ in $\mathrm{Emb}(V, D)$ such that $\gamma(0)$ is the embedding $V \subset D$, $\gamma(1)$ is a homeomorphism $V \cong V'$ and, if $\gamma(t)(V)$ contains the special point for some t, then $\gamma(t')(V)$ contains the special point for all $t' \geq t$.

Denote by $\pi : \mathcal{E} \to \mathcal{L}$ the natural functor. We will show that π is an ∞-cofinal localization at the set of weak equivalences in $\mathcal{E}$.

Step 1. π is ∞-cofinal. To show this claim, we need to show that, for every $V \in \mathcal{L}$, the category V/π is weakly contractible. To this end, we consider the space $E = P' \mathrm{Emb}(V, D)$ of paths γ in $\mathrm{Emb}(V, D)$ that satisfy: If $\alpha(t)(V)$ contains the special point, then $\alpha(t')(V)$ contains the special point for all $t' \geq t$. We then deduce that V/π is weakly contractible, by applying Lemma A.1 to the functor

$$V/\pi \longrightarrow \mathcal{U}(E), (V \xrightarrow{[\gamma]} V') \mapsto U([\gamma]),$$

where $U([\gamma])$ is the open subset of E consisting of paths that end in an embedding $V \hookrightarrow V'$ and whose associated homotopy class, obtained by composing with any path of embeddings from $V \hookrightarrow V'$ to $V \cong V'$, agrees with γ.

Step 2. For $V \in \mathcal{L}$, denote by $j : (V/\pi)^{\cong} \subset V/\pi$ the inclusion of the full subcategory spanned by the isomorphisms in $\mathcal{L}$. By a similar argument as in Step 1, using Lemma A.1, it follows that j is ∞-coinitial. It is then that, for every ∞-category $\mathcal{D}$ with limits, the unit $\mathrm{id} \to \pi_* \pi^*$ is an equivalence, and the counit $\pi^* \pi_* \to \mathrm{id}$ is an equivalence on those functors $\mathcal{E} \to \mathcal{D}$ that map weak equivalences to equivalences. This implies that π^* is fully faithful with essential image consisting precisely of these latter functors $\mathcal{E} \to \mathcal{D}$.

Now, equipped with this statement, we show that $\mathcal{F}(D)$ is a limit of $\mathcal{F}|(i/D)^{\mathrm{op}}$. Namely, by assumption, $\mathcal{F}|(i/D)^{\mathrm{op}}$ maps weak equivalences to equivalences so that it is equivalent to $(\pi^{\mathrm{op}})^* \mathcal{G}$ for some functor $\mathcal{G} : \mathcal{L}^{\mathrm{op}} \to \mathcal{D}$. Since π is ∞-cofinal, we may compute the limit of $\mathcal{F}|(i/D)^{\mathrm{op}}$ as the limit of $\mathcal{G}$. But now the category $\mathcal{L}^{\mathrm{op}}$ has an initial object given by a disk $D' \subset D$ so that, if D contains a special point, then D' also contains the special point. In any case, we have that $D' \subset D$ is a weak equivalence. Therefore, we obtain the desired equivalence $\mathcal{F}(D) \simeq \lim \mathcal{G} \simeq \mathcal{F}|(i/D)^{\mathrm{op}}$. □

In our treatment of Milnor sheaves, it will be important to have a good control on the boundary of disks which is why we now switch from open disks to closed disks. Let $\mathrm{Disk}^{oc}(X, N)$ denote the poset of all open and closed disks in X containing at most one special point. We denote by $\mathrm{Disk}^{o}(X, N) \subset \mathrm{Disk}^{oc}(X, N)$ and $\mathrm{Disk}^{c}(X, N) \subset \mathrm{Disk}^{oc}(X, N)$ the subsets of open and closed disks, respectively. The poset $\mathrm{Disk}^{oc}(X, N)$ comes equipped with a set of weak equivalences W given by those inclusions of disks that preserve the number of special points.

Proposition 2.4.4. *Let (X, N) be a stratified surface, and let $\mathcal{D}$ be an ∞-category. There are equivalences of ∞-categories*

$$\mathrm{Fun}(\mathrm{Disk}^{c}(X, N)[W^{-1}]_{\infty}, \mathcal{D}) \longleftarrow \mathrm{Fun}(\mathrm{Disk}^{oc}(X, N)[W^{-1}]_{\infty}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathrm{Disk}^{o}(X, N)[W^{-1}]_{\infty}, \mathcal{D}).$$

Proof. We claim that the subcategory

$$\mathrm{Fun}(\mathrm{Disk}^{oc}(X, N)[W^{-1}]_{\infty}, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Disk}^{oc}(X, N)\mathcal{D})$$

can be identified with the subcategory of left Kan extensions along $i : \mathrm{Disk}^{c}(X, N) \subset \mathrm{Disk}^{oc}(X, N)$ and the subcategory of right Kan extensions along $j : \mathrm{Disk}^{o}(X, N) \subset \mathrm{Disk}^{oc}(X, N)$. To verify the first claim, suppose that $\mathcal{F} : \mathrm{Disk}^{oc}(X, N) \to \mathcal{D}$ is functor sending weak equivalences in $\mathrm{Disk}^{c}(X, N)$ to equivalences in $\mathcal{D}$. The pointwise Kan extension formula at $U \in \mathrm{Disk}^{o}(X, N)$ exhibits $\mathcal{F}(U)$ as the colimit over i/U. If U contains a special point, then we may replace the category i/U by the cofinal subcategory $(i/U)'$ consisting of those closed disks that contain the special point (otherwise, we set $(i/U)' = i/U$). The category $(i/U)'$ is filtered and hence contractible and the diagram $\mathcal{F}|(i/U)'$ consists

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of equivalences. Hence, by Lemma A.4, $\mathcal{F}$ is a left Kan extension of $\mathcal{F}|\mathrm{Disk}^o(X,N)$ if and only if, for every $A \in (i/U)'$, the map $F(A) \to \mathcal{F}(U)$ is an equivalence. It is now an immediate consequence of the two-out-of-three property of equivalences that $\mathcal{F}$ is a left Kan extension of $\mathcal{F}|\mathrm{Disk}^o(X,N)$ if and only if $\mathcal{F}$ sends all weak equivalences to equivalences in $\mathcal{D}$. The second claim regarding right Kan extensions along j follows from an essentially identical argument. □

Finally, we would like to provide an explicit description of the localization $\mathrm{Disk}^c(X,N)[W^{-1}]_\infty$ which will provide the starting point for our discussion of Milnor disks.

Proposition 2.4.5. *The functor*

$$\pi : \mathrm{Disk}^c(X,N) \longrightarrow S(X,N),\ A \mapsto (A,\emptyset) \tag{2.4.6}$$

exhibits the ordinary category $S(X,N)$ as an ∞-categorical localization along the weak equivalences of $\mathrm{Disk}^c(X,N)$, that is, identifies it with $\mathrm{Disk}^c(X,N)[W^{-1}]_\infty$ as an ∞-category. In particular, for every ∞-category $\mathcal{D}$, the functor

$$\pi^* : \mathrm{Fun}(S(X,N),\mathcal{D}) \longrightarrow \mathrm{Fun}(\mathrm{Disk}^c(X,N),\mathcal{D})$$

is fully faithful with essential image consisting of those functors that send weak equivalences in $\mathrm{Disk}^c(X,N)$ to equivalences in $\mathcal{D}$.

Proof. Let $(A,\emptyset) \in S(X,N)$. Suppose first that A does not contain a special point. Then we have

$$(\pi/(A,\emptyset))^{\simeq} = \pi/(A,\emptyset).$$

Further, we claim that $\pi/(A,\emptyset)$ is contractible. To this end, consider the topological space P of continuous paths $[0,1] \to \mathring{X} \setminus N$ ending in $\mathring{A}$. To an object $(B,\alpha : (B,\emptyset) \to (A,\emptyset))$, we associate the open subset of P consisting of those paths that start in $\mathring{B}$ and lie in the same homotopy class as the class of paths that arises from the isotopy comprising α. This association defines a functor

$$\pi/(A,\emptyset) \longrightarrow \mathcal{U}(P)$$

which satisfies the hypothesis of Lemma A.1 thus proving the contractibility of $\pi/(A,\emptyset)$.

Now, suppose that A does contain a special point $x \in N$. Then we first claim that the inclusion

$$j : (\pi/(A,\emptyset))^{\simeq} \subset \pi/(A,\emptyset)$$

is cofinal. To this end, we need to show that, given an object $b = (B,\alpha : (B,\emptyset) \to (A,\emptyset))$ of $\pi/(A,\emptyset)$, the category b/j is contractible. We consider the space Q of paths in $\mathring{X} \setminus (N \setminus x)$ starting in $\mathring{B}$ and ending in x. To an object $b' = (B',\alpha : (B',\emptyset) \to (A,\emptyset))$ of b/j, we associate the open subset of Q consisting of paths that lie in $\mathring{B}'$. An application of Lemma A.1 proves the claim. Finally, an argument similar to the above shows that $(\pi/(A,\emptyset))^{\simeq}$ is contractible so that the result follows from Proposition A.4. □

As a consequence of the results of this section, we thus obtain the following:

Corollary 2.4.7. *Let (X,N) be a stratified surface, and let $\mathcal{D}$ be an ∞-category. Then there is an equivalence*

$$\mathrm{Sh}(X,N;\mathcal{D}) \simeq \mathrm{Fun}(S(X,N)^{\mathrm{op}},\mathcal{D})$$

of ∞-categories.

Remark 2.4.8. In view of Example 2.2.9, Corollary 2.4.7 recovers the presentation of the constructible derived category in terms of the exit path category. Nevertheless, the description in terms of $S(X,N)$ will be more convenient in what follows.

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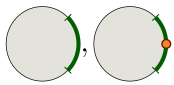
2.5. Verdier duality

In this section, we assume that $\mathcal{D}$ is a stable ∞-category.

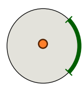
Recall that, in Corollary 2.4.7, we have identified the ∞-category of constructible sheaves on (X, N) with values in any ∞-category $\mathcal{D}$ with the ∞-category $\mathrm{Fun}(S(X,N)^{\mathrm{op}}, \mathcal{D})$ of presheaves on the category $S(X,N) \subset \overrightarrow{\Lambda}(X,N)$. In this section, under the assumption that $\mathcal{D}$ is stable, we illustrate the use of Kan extensions among subcategories of $\overrightarrow{\Lambda}(X,N)$ to provide a proof of Verdier duality. This treatment can be regarded as an introduction to the techniques to be used in §3 to establish our description of $\mathrm{Sh}(X,N;\mathcal{D})$ as Milnor sheaves.

Along with $S = \overrightarrow{S} = \overrightarrow{S}(X,N)$ and $\overrightarrow{B} = \overrightarrow{B}(X,N)$ defined earlier, we consider the following full subcategories of $\overrightarrow{\Lambda}(X,N)$ consisting, respectively, of the following disks:

- $\overrightarrow{D}$: disks (A, A') of the form


$\subset X$

  where $A' \subset \partial A$ is a single closed interval and $A \cap N \subset A'$,
- $\overrightarrow{V}$: disks (A, A') of the form


$\subset X$

  where $A' \subset \partial A$ is a single closed interval and $(A \setminus A') \cap N$ is a singleton.

The category S is equivalent to the entrance path category of (X, N) so that, for any stable ∞-category $\mathcal{D}$, the ∞-category $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{D})$ can be identified with the category of constructible sheaves on (X, N) valued in $\mathcal{D}$ (Corollary 2.4.7). We set $\overrightarrow{Q} := S \cup \overrightarrow{B} \cup \overrightarrow{D} \cup \overrightarrow{V}$.

Theorem 2.5.1. *Let $\mathcal{D}$ be a stable ∞-category. And let*

$$\mathcal{F} : \overrightarrow{Q} \longrightarrow \mathcal{D}$$

be a functor. Then the following are equivalent:

(i) *$\mathcal{F}$ satisfies the following conditions:*
  (1) *$\mathcal{F}|S \cup \overrightarrow{D}$ is a right Kan extension of $\mathcal{F}|S$, and*
  (2) *$\mathcal{F}$ is a left Kan extension of $\mathcal{F}|S \cup \overrightarrow{D}$.*

(ii) (1) *$\mathcal{F}|\overrightarrow{B}$ maps weak equivalences in $\overrightarrow{B}$ to equivalences in $\mathcal{D}$,*
  (2) *$\mathcal{F}|\overrightarrow{B} \cup \overrightarrow{D}$ is a left Kan extension of $\mathcal{F}|\overrightarrow{B}$, and*
  (a) *$\mathcal{F}$ is a right Kan extension of $\mathcal{F}|\overrightarrow{B} \cup \overrightarrow{D}$.*

Before we provide a proof of the theorem, we explain its implications. The following lemma generalizes one of the statements of Corollary 2.3.8.

Lemma 2.5.2. *Let $B \subset \Lambda(X,N)$ denote the full subcategory spanned by the bounded disks. Then the restriction $\overrightarrow{B} \to B$ of the canonical functor $\overrightarrow{\Lambda}(X,N) \to \Lambda(X,N)$ exhibits B as an ∞-categorical localization of $\overrightarrow{B}$ along the weak equivalences.*

Proof. This follows immediately from Proposition A.4. $\square$

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Corollary 2.5.3. *There is a canonical equivalence of stable ∞-categories*

$$\delta : \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}(S, \mathcal{D})$$

identifying constructible sheaves and constructible cosheaves valued in $\mathcal{D}$.

Proof. Let $\mathrm{Fun}(\overrightarrow{Q}, \mathcal{D})', \mathrm{Fun}(\overrightarrow{Q}, \mathcal{D})'' \subset \mathrm{Fun}(\overrightarrow{Q}, \mathcal{D})$ be the full (∞-)subcategories consisting of functors satisfying the conditions (i) and (ii) of Theorem 2.5.1, respectively. By Proposition A.3, the restriction functor

$$p : \mathrm{Fun}(\overrightarrow{Q}, \mathcal{D})' \longrightarrow \mathrm{Fun}(S, \mathcal{D})$$

is an equivalence. For the same reason, the restriction functor

$$q : \mathrm{Fun}(\overrightarrow{Q}, \mathcal{D})'' \longrightarrow \mathrm{Fun}(\overrightarrow{B}[W^{-1}]_\infty, \mathcal{D})$$

is an equivalence. Since (i) and (ii) are equivalent, and $\overrightarrow{B}[W^{-1}]_\infty \simeq B$ by Lemma 2.5.2, we obtain an equivalence

$$\mathrm{Fun}(S, \mathcal{D}) \simeq \mathrm{Fun}(B, \mathcal{D})$$

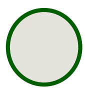
by composing an inverse of p with q. The equivalence $B \simeq S^{\mathrm{op}}$ induced by the duality ξ then yields the desired result. □

Proof of Theorem 2.5.1. Let $\mathcal{F} : \overrightarrow{Q} \longrightarrow \mathcal{D}$ be a functor.

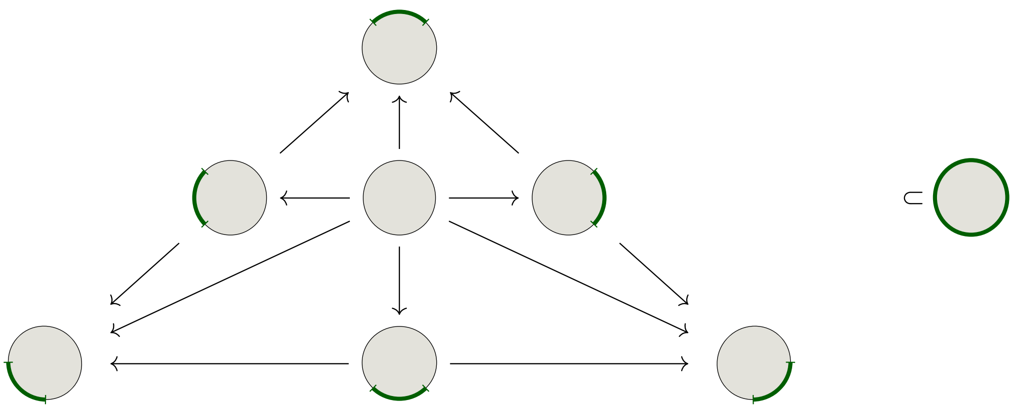
We will provide concrete interpretations of the Kan extension conditions in (i) and (ii) so as the claimed equivalence will become an apparent consequence of the stability of the ∞-category $\mathcal{D}$.

We begin with (i): For every $(A, A') \in \overrightarrow{D}$, the category $(A, A')/S$ is empty so that $\mathcal{F}|S \cup \overrightarrow{D}$ is a right Kan extension of $\mathcal{F}|S$ if and only if, for every $(A, A') \in \overrightarrow{D}$, we have $\mathcal{F}(A, A') \simeq 0$.

Suppose now that $\mathcal{F}$ is a left Kan extension of $\mathcal{F}|S \cup \overrightarrow{D}$. We first determine the value of $\mathcal{F}$ at

$(A, A') =$  $\in \overrightarrow{B}$

as determined by the pointwise formula (A.2). The overcategory $S \cup \overrightarrow{D}/(A, A')$ admits an ∞-cofinal subcategory depicted by



(2.5.4)

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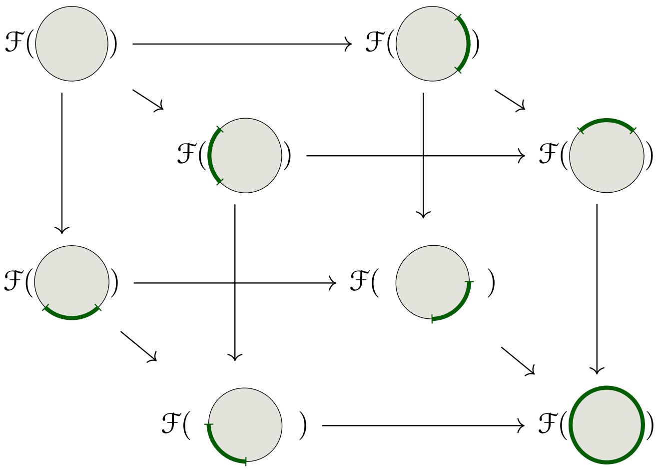
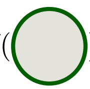
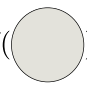
where the morphisms around the boundary of the 2-simplex are given by rotation by the smallest possible angle so that a full turn is obtained by traversing the boundary once. Thus, the value of $\mathcal{F}$ at (A, A') is determined by the colimit cone (i.e., biCartesian cube)

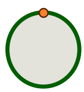


(2.5.5)

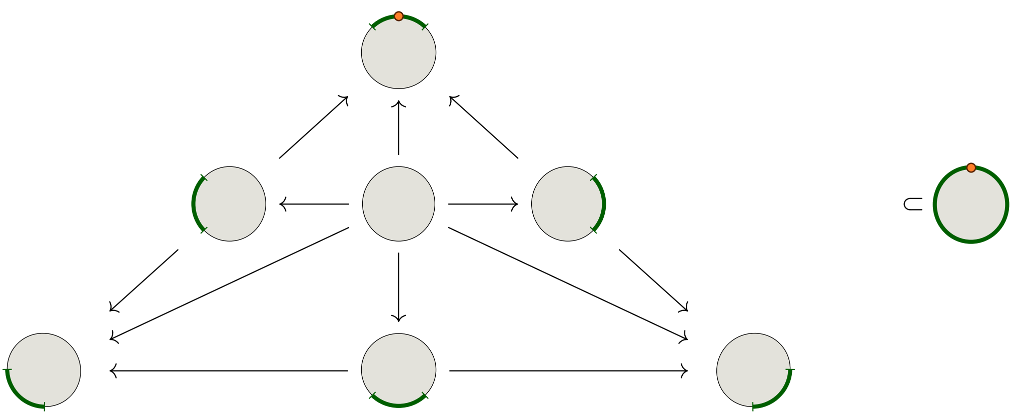
In particular, since $\mathcal{F}|\overrightarrow{D} \simeq 0$, this biCartesian cube induces an equivalence

$\mathcal{F}($$) \simeq \mathcal{F}($$)[2].$

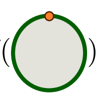
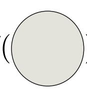
Similarly, for a disk of the form

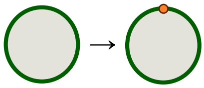
$(A, A') =$  $\in \overrightarrow{B}$

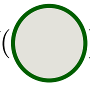
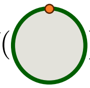
the overcategory $S \cup \overrightarrow{D}/(A, A')$ admits an ∞-cofinal subcategory depicted by



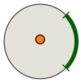
(2.5.6)

again exhibiting an equivalence $\mathcal{F}($$) \simeq \mathcal{F}($$)[2]$. More precisely, we observe that any weak equivalence

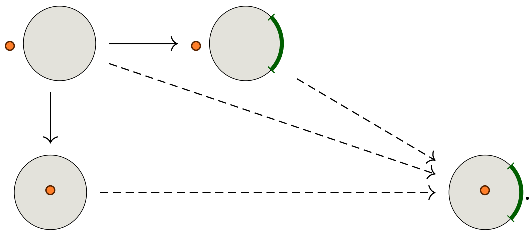


in $\overrightarrow{B}$ induces an equivalence $\mathcal{F}($$) \to \mathcal{F}($$)$ in $\mathcal{D}$ since the induced map relating the above ∞-cofinal subcategories (2.5.6) and (2.5.4) becomes a pointwise equivalence upon applying $\mathcal{F}$.

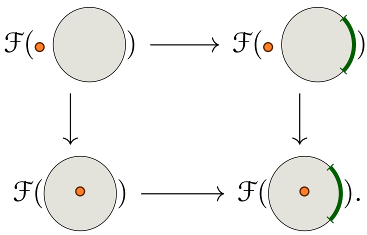
We next describe the value of $\mathcal{F}$ at

$(A, A') =$  $\in \overrightarrow{V}$.

To this end, we argue that the overcategory $(S \cup \overrightarrow{D})/(A, A')$ contains an ∞-cofinal subcategory of the form

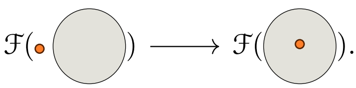


In particular, the value of $\mathcal{F}$ at (A, A') is determined by the colimit cone (i.e., biCartesian square)

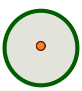


(2.5.7)

Since the top-right object is a zero object, this diagram exhibits $\mathcal{F}(A, A')$ as a cofiber (cone) of the morphism



Finally, it remains to characterize the value of $\mathcal{F}$ at

$(A, A') =$  $\in \overrightarrow{B}$.

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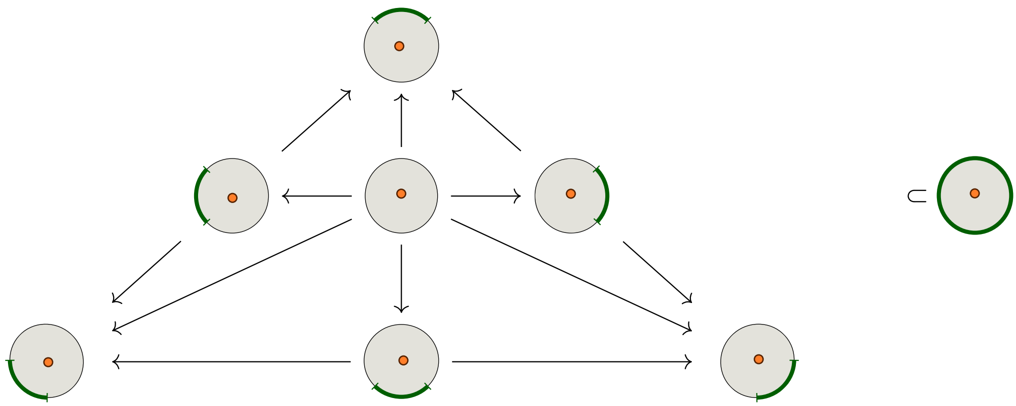
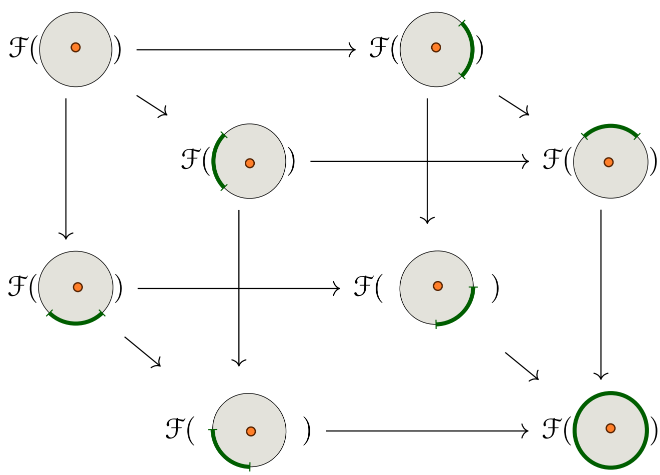
Similarly, as in Step 1, the overcategory $S \cup \overrightarrow{D} \cup \overrightarrow{V}/(A, A')$ admits an ∞-cofinal subcategory depicted by



where the morphisms around the boundary of the 2-simplex are given by rotation by the smallest possible angle. Thus, the value of $\mathcal{F}$ at (A, A') is determined by the colimit cone (i.e., biCartesian cube)



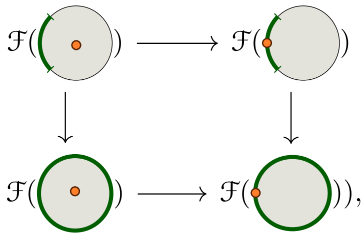
(2.5.8)

In conclusion, we may characterize the functors $\mathcal{F} : \overrightarrow{Q} \longrightarrow \mathcal{D}$ satisfying the Kan extension conditions of (i) as those functors for which $\mathcal{F}|\overrightarrow{D} \simeq 0$ and further the square (2.5.7) as well as the cubes (2.5.5) and (2.5.8) are biCartesian.

We now discuss the Kan extension conditions of (ii). A similar argumentation as the one for (i) show that a functor $\mathcal{F}$ satisfies the conditions of (ii)(2) and (ii)(3) if and only if

(1) $\mathcal{F}|\overrightarrow{D} \simeq 0$,
(2) the cubes (2.5.5) and (2.5.8) are limit cones, and hence biCartesian,

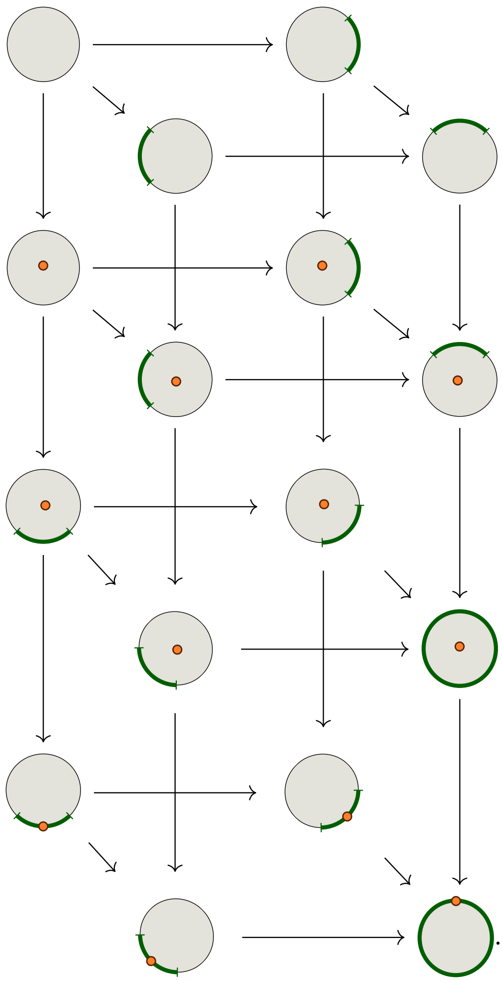
(3) the square



(2.5.9)

is biCartesian.

Thus, to finish the proof, we have to argue why, assuming further (ii)(1), equation (2.5.7) being biCartesian is equivalent to equation (2.5.9) being biCartesian (in the presence of the remaining conditions). To show this, consider the commutative diagram in $\overrightarrow{Q}$ depicted by



(2.5.10)

consisting of three stacked cubes, and further, the diagram in $\mathcal{D}$ obtained by applying $\mathcal{F}$. The middle cube is identical to equation (2.5.8) which is biCartesian. Furthermore, the cube given by the composite

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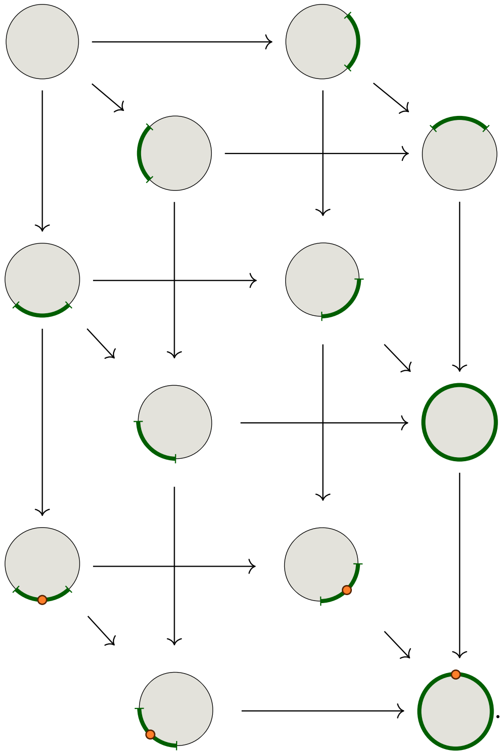
of the three cubes coincides may be decomposed as



(2.5.11)

where the top cube coincides with equation (2.5.5) and the bottom cube is biCartesian since all vertical maps are weak equivalences which, by (ii)(1), are mapped to equivalences by $\mathcal{F}$. Thus, the composite cube is biCartesian as well. The front face of the top cube in equation (2.5.10) is biCartesian since it contains two parallel arrows that are equivalences. Proposition A.4 implies that the top cube is biCartesian if and only if its back face, which coincides with equation (2.5.7), is biCartesian. By the same argument, the bottom cube will be biCartesian if and only if its front face, which coincides with equation (2.5.9), is biCartesian. As a consequence, the two-out-of-three property for the pasting of biCartesian cubes (Proposition A.6) implies that equation (2.5.7) is biCartesian if and only if equation (2.5.9) is biCartesian, concluding our argument. □

3. Milnor sheaves

By Corollary 2.4.7, the ∞-category of constructible sheaves $\mathrm{Sh}(X, N; \mathcal{D})$ with values in a stable ∞-category $\mathcal{D}$ may be parametrized in terms of standard disks: There is an equivalence

$$\mathrm{Sh}(X, N; \mathcal{D}) \simeq \mathrm{Fun}(S(X, N)^{\mathrm{op}}, \mathcal{D}). \tag{3.0.1}$$

If $\mathcal{D}$ is the derived category of an abelian category $\mathcal{A}$, then this equivalence restricts to an equivalence

$$\mathrm{Sh}(X, N; \mathcal{A}) \simeq \mathrm{Fun}(S(X, N)^{\mathrm{op}}, \mathcal{A}).$$

In other words, the equivalence (3.0.1) is compatible with the standard t-structure on $\mathrm{Sh}(X, N; \mathcal{D})$. In this section, we provide yet another parametrization of $\mathrm{Sh}(X, N; \mathcal{D})$, in terms of Milnor disks, which is in the same sense compatible with the perverse t-structure. In particular, it provides an intrinsically abelian description of the category of perverse sheaves.

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3.1. Constructible sheaves as Milnor sheaves

Let (X, N) be a stratified surface, and let $\overrightarrow{\Lambda}(X, N)$ denote its directed paracyclic category. A *collared cut* of an object $(A, A') \in \overrightarrow{\Lambda}(X, N)$ consists of

- a *cut* α, by which we mean an embedding $\alpha : I \to A$ with $\alpha^{-1}(\partial A) = \{0, 1\}$. We denote the two connected components of the complement if $\alpha(I)$ in A by U_1 and U_2.
- a *collar* for α, by which we mean a continuous map $G : [-1, 1] \times I \to A$ such that
  - $G(0, t) = \alpha(t)$,
  - for every $s \in [-1, 1]$, the map $G(s, -)$ is a cut,
  - $G([-1, 1] \times I) \cap \partial A' = \emptyset$,
  - $G(\{-1, 1\} \times I) \cap N = \emptyset$.

  We denote $C = G([-1, 1] \times I)$ and $A_1 = U_1 \cup C$ and $A_2 = U_2 \cup C$.

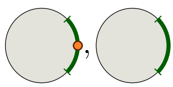
Associated to a collared cut, there is a commutative square

$$\begin{tikzcd} (C, C \cap A') \arrow[r] \arrow[d] & (A_1, A_1 \cap A') \arrow[d] \\ (A_2, A_2 \cap A') \arrow[r] & (A, A') \end{tikzcd} \tag{3.1.1}$$

in $\overrightarrow{\Lambda}(X, N)$.

Definition 3.1.2. Let $\mathcal{D}$ be a pointed ∞-category. A functor $\mathcal{F} : \overrightarrow{M}(X, N) \to \mathcal{D}$ is called a *Milnor cosheaf* if

(1) $\mathcal{F}$ maps weak equivalences in $\overrightarrow{M}(X, N)$ to equivalences in $\mathcal{D}$,
(2) $\mathcal{F}$ maps objects of the form

 $\in \overrightarrow{M}(X, N)$

to a zero object,
(3) for every object (A, A') of $\overrightarrow{M}(X, N)$ and for every collared cut of (A, A'), such that the associated diagram (3.1.1) takes values in $\overrightarrow{M}(X, N)$, $\mathcal{F}$ maps equation (3.1.1) to a coCartesian square in $\mathcal{D}$.

Dually, $\mathcal{F} : \overrightarrow{M}(X, N)^{\mathrm{op}} \to \mathcal{D}$ is called a *Milnor sheaf* if $\mathcal{F}^{\mathrm{op}}$ is a Milnor cosheaf. We denote by $\mathrm{Fun}^\sharp(\overrightarrow{M}(X, N), \mathcal{D})$ the ∞-category of Milnor cosheaves and by $\mathrm{Fun}^\sharp(\overrightarrow{M}(X, N)^{\mathrm{op}}, \mathcal{D})$ the ∞-category of Milnor sheaves defined as full subcategories of the respective functor categories.

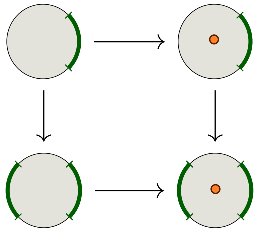
Definition 3.1.3. Let $(A, \emptyset)$ be an object of $S(X, N)$. We denote by Λ_A the subcategory of $\overrightarrow{M}(X, N)$ with objects (A, A') and morphisms, represented by an isotopy $H : I \times \mathbb{D} \to X$ such that, for every $t \in I$, $H_t(\mathbb{D}) = A$. We further denote by $\Lambda_A^+ = \Lambda_A \cup (A, \emptyset)$ obtained by adjoining the initial object $(A, \emptyset)$.

Remark 3.1.4. For every $(A, \emptyset)$, the category Λ_A is equivalent to the paracyclic category Λ_∞.

We say that a functor $\mathcal{F} : \overrightarrow{M}(X, N) \longrightarrow \mathcal{D}$ is *locally Segal* if, for every $(A, \emptyset) \in S(X, N)$, with $\partial A \cap N = \emptyset$, the object $\mathcal{F}|\Lambda_A$ is a Segal object, that is, the restriction along an embedding $\Delta \to \Lambda_A$ is Segal.

Proposition 3.1.5. *Let $\mathcal{D}$ be a stable ∞-category, and let $\mathcal{F} : \overrightarrow{M}(X, N) \to \mathcal{D}$ be a functor. Then $\mathcal{F}$ is a Milnor cosheaf if and only if the following hold:*

(1) *$\mathcal{F}$ maps weak equivalences to equivalences in $\mathcal{D}$.*
(2) *$\mathcal{F}$ is locally Segal.*
(3) *For every $x \in N$, $\mathcal{F}$ maps any square of the form*



to a coCartesian square in $\mathcal{D}$.

Proof. Condition (1) appears directly in the definition of a Milnor cosheaf.

Every Milnor cosheaf satisfies the conditions (2) and (3) since, using the version (3.3.2) of the Segal conditions, the respective coCartesian squares all arise from collared cuts.

Suppose now that $\mathcal{F}$ satisfies (2) and (3). Let $(A, \emptyset) \in \overrightarrow{M}(X, N)$ with $A \cap N = \emptyset$. Then the local Segal conditions imply that equation (3.1.1) is coCartesian for every cut which only intersects one of the boundary intervals. If the cut α intersects two boundary intervals, then it is straightforward to deduce that equation (3.1.1) is coCartesian by considering cuts α_1 of $(A_1, A_1 \cap A')$ and α_2 of $(A_2, A_2 \cap A')$ which are obtained by sliding the endpoint of α out of the boundary interval towards the two possible directions (In the language of [20], this amounts to the statement that every 1-Segal object is 2-Segal).

By the exact same argumentation, we deduce that equation (3.1.1) is coCartesian for $(A, \emptyset) \in \overrightarrow{M}(X, N)$ with $A \cap N = \{x\}$ as long as the cut α runs through the special point x. It remains to verify the coCartesianess of equation (3.1.1) for a cut α which does *not* run through x. But this case can be reduced to equation (3) by induction on the number of boundary intervals: The induction step is obtained by introducing one additional cut which runs either through the special point x or lies completely in the component of $A \setminus \alpha(I)$ which does not contain x. ◻

We denote by

$$\overrightarrow{M}^{+}(X, N) \subset \overrightarrow{\Lambda}(X, N)$$

the full subcategory spanned by the standard and Milnor disks and by

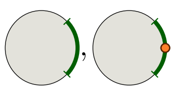
$$\overrightarrow{\Lambda}(X, N)_{\leq n} \subset \overrightarrow{M}^{+}(X, N)$$

the full subcategory consisting of objects (A, A') such that A' has at most n connected components.

Further, we denote by

$$\overrightarrow{D}(X, N) \subset \overrightarrow{\Lambda}(X, N)$$

the full subcategory of objects $(A, A') \in \overrightarrow{\Lambda}(X, N)$ of the form

 $\in \overrightarrow{\Lambda}(X, N).$

Theorem 3.1.6. *Let $\mathcal{F} : \overrightarrow{M}^{+}(X, N) \longrightarrow \mathcal{D}$ be a functor. Then the following are equivalent:*

(1) *$\mathcal{F}|\overrightarrow{D}(X, N) \simeq 0$ and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}|S(X, N) \cup \overrightarrow{D}(X, N)$.*
(2) *$\mathcal{F}|\overrightarrow{M}(X, N)$ is a Milnor cosheaf and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}|\overrightarrow{M}(X, N)$.*

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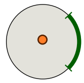
Proof. Suppose that $\mathcal{F}$ is a left Kan extension of $\mathcal{F}|S(X,N) \cup \overrightarrow{D}(X,N)$. To show that $\mathcal{F}|\overrightarrow{M}(X,N)$ is a Milnor cosheaf, we verify conditions (2) and (3) of Proposition 3.1.5. Let $(A, A') \in \overrightarrow{M}(X,N)$ with $A \cap N$ empty. Then the inclusion

$$\Lambda^+_{A,\leq 1}/(A,A') \subset (S(X,N) \cup \overrightarrow{D}(X,N))/(A,A')$$

is an equivalence of categories and hence ∞-cofinal. In particular, by Proposition 3.3.3, $\mathcal{F}|\Lambda_A$ is a Segal object.

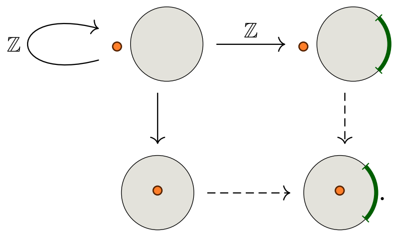
Now, let

$$(A, A') =$$  $$\in \overrightarrow{M}(X,N)$$

such that $A \cap N = \{x\} \subset A \setminus A'$ is a singleton. Then the inclusion

$$(S(A,\{x\}) \cup \overrightarrow{D}(A,\{x\})/(A,A') \subset (S(X,N) \cup \overrightarrow{D}(X,N))/(A,A'),$$

where the first undercategory is taken in $\overrightarrow{M}(U,\{x\})$, is an equivalence, in particular ∞-cofinal. Now, the category $(S(A,\{x\}) \cup D(A,\{x\})/(A,\varphi)$ is equivalent to the category depicted by



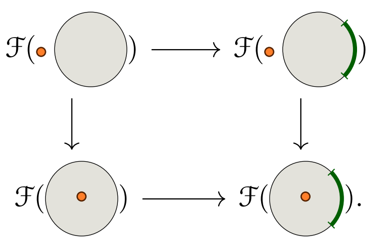
where the automorphisms $\mathbb{Z}$ correspond to the disk moving around the special point x so that it is equivalent to the category depicted by

$$\begin{tikzcd} \bullet \arrow[loop left, "\mathbb{Z}"] \arrow[r, "\mathbb{Z}"] \arrow[d] & \bullet \\ \bullet. & \end{tikzcd}$$

This latter category contains the ∞-cofinal subcategory

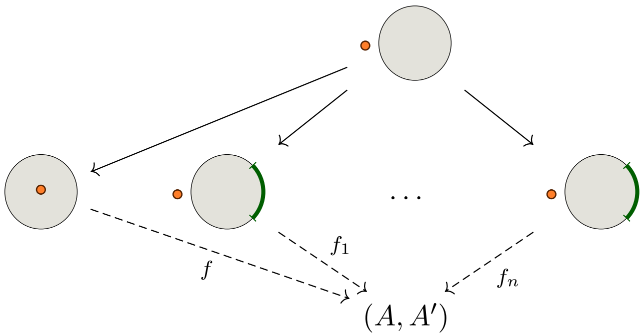
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

so that the pointwise left Kan extension condition for $\mathcal{F}(A, A')$ is thus equivalent to the square



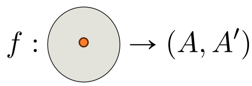
(3.1.7)

being coCartesian. For a more general $(A, A') \in M(X, N)$ with $A \cap N = \{x\} \subset A \setminus A'$, by a similar argument, the category $(S(X, N) \cup D(X, N))/(A, A')$ contains a cofinal subcategory C of the form

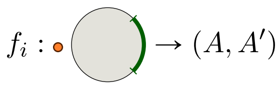


(3.1.8)

where



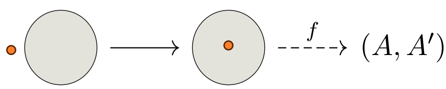
is given by the constant isotopy and the morphisms



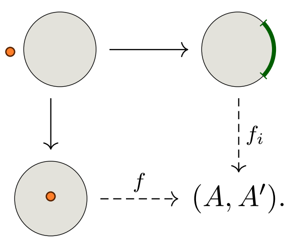
enter the special point and map the unique interval to the ith interval of A (with respect to some chosen order). We have a functor from the category

$$\begin{tikzcd} & & 0 \arrow[dll] \arrow[dl] \arrow[dr] & \\ 1 & 2 & \dots & n \end{tikzcd}$$

to Cat/C by associating to 0 the subcategory



of C and to $i > 0$ the subcategory



An application of [37, 4.2.3.10], using that equation (3.1.7) is a pushout, implies that the pointwise left

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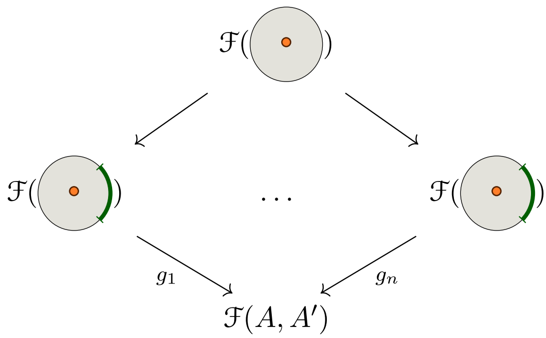
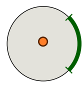
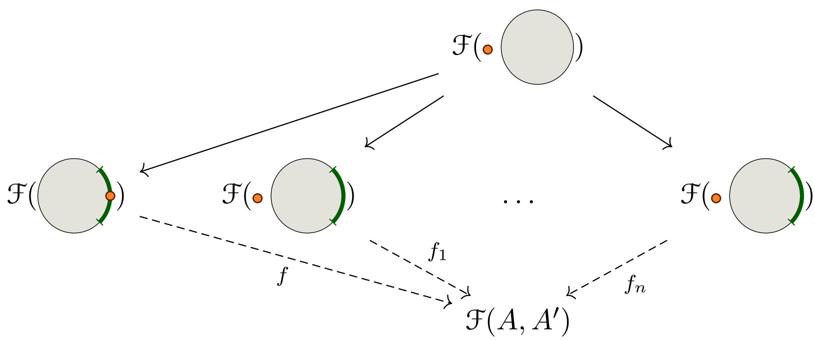
Kan condition for (A, A') is equivalent to the diagram



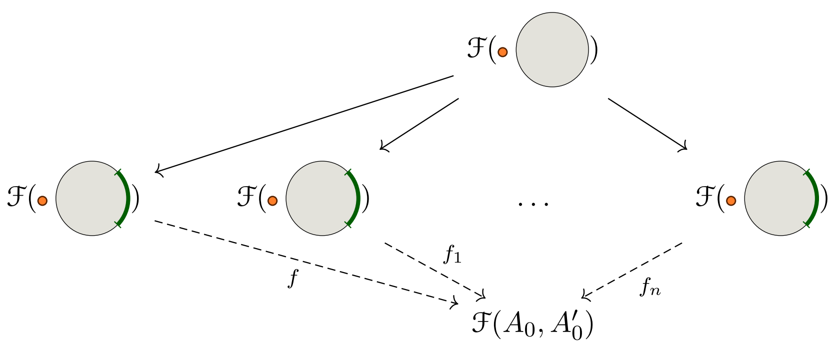
being a colimit cone. Here, the maps g_i :  $\to (A, A')$ are morphisms in Λ_A which move the interval into the various intervals comprising A'. In particular, this implies that the diagram $\mathcal{F}|\Lambda_A^+$ is a left Kan extension of its restriction to $\mathcal{F}|\Lambda_{A,\leq 1}^+$ so that $\mathcal{F}|\Lambda_A$ satisfies the Segal conditions by Proposition 3.3.3. We have thus shown that $\mathcal{F}$ is locally Segal.

A similar argument shows that the value of $\mathcal{F}$ at a disk (A, A') with $A' \cap N = \{x\}$ is determined by the colimit cone



(3.1.9)

Further, the Segal conditions for $\mathcal{F}$ at a disk (A_0, A_0'), obtained by moving (A, A') away from the special point x so that $A_0 \cap N = \emptyset$, imply that the diagram



(3.1.10)

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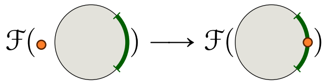
is a colimit cone. Since the map

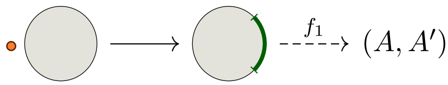


is, as a map between zero objects, an equivalence, we deduce from the induced map on colimit cones that the map $\mathcal{F}(A_0, A_0') \to \mathcal{F}(A, A')$ is an equivalence as well. In particular, $\mathcal{F}$ maps weak equivalences to equivalences in $\mathcal{D}$.

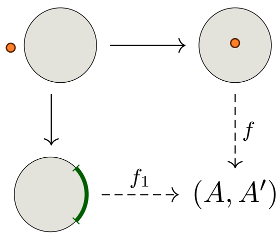
Condition (3) follows by applying [37, 4.2.3.10] to equation (3.1.8) for $n = 2$ with respect to the functor from the category

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

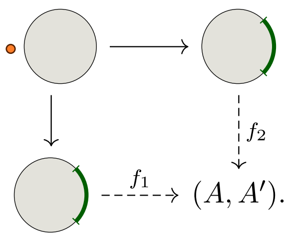
into Cat/C which associates to 0 the subcategory



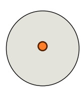
to 1 the subcategory



and to 2 the subcategory



The above statements imply that $\mathcal{F}|\overrightarrow{M}(X, N)$ is a Milnor sheaf. It remains to show that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}|\overrightarrow{M}(X, N)$. To this end, let

$$(A, \varnothing) =$$

$$\in S(X, N)$$

with $A \cap N = \{x\}$ a singleton. Then it is easily seen that the inclusion

$$(A, \emptyset)/\Lambda_A \subset (A, \emptyset)/\overrightarrow{M}(X, N),$$

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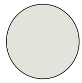
where the left-hand overcategory is taken in the category Λ^{+}_{A}, is ∞-coinitial. Thus, by Proposition 3.3.3 below, the value of $\mathcal{F}$ at $(A, \emptyset)$ is given by right Kan extension of $\mathcal{F}|\overrightarrow{M}(X, N)$.

Finally, consider

$$(A, \varnothing) =$$  $$\in S(X, N)$$

with $A \cap N$ empty. Again, we consider the inclusion

$$j : (A, \emptyset)/\Lambda_A \subset (A, \emptyset)/\overrightarrow{M}(X, N).$$

We claim that j is ∞-coinitial. To this end, we have to verify, for every $f : (A, \emptyset) \to (A_1, A_1') \in (A, \emptyset)/M(X, N)$, that j/f is contractible. This statement is clear if $A_1 \cap N = \emptyset$. Suppose now that $A_1 \cap N = \{x\} \subset A_1 \setminus A_1'$. In this case, we proceed by exhibiting a contractible ∞-cofinal subcategory of j/f: Fix an object a_0 of j/f whose underlying disk (B, B') has $|\pi_0(B')| = |\pi_0(A_1')| + 1$ boundary components and such that the map $(B, B') \to (A_1, A_1')$ includes $|\pi_0(A_1')| - 1$ intervals of B' into respective intervals of A_1' and includes the two intervals adjacent to the entry location of x into the remaining interval of A_1'. There are objects $\{a_i | i \in \mathbb{Z}\}$ of j/f which differ from a_0 in that the entry point of x lies i segments in $S^1 \setminus B'$ away from the entry point of x for a_0. For $i \in \mathbb{Z}$, we denote by a_i^+ and a_i^- the two objects of j/f obtained by omitting one of the intervals of B' adjacent to the entry point of x. The full subcategory of j/f spanned by these objects has the form:

$$\dots \qquad a_{-1} \longleftarrow a_{-1}^+ = a_0^- \longrightarrow a_0 \longleftarrow a_0^+ = a_1^- \longrightarrow a_1 \qquad \dots .$$

It is now straightforward to verify that this subcategory is cofinal in j/f and, since it is further contractible, the claim follows. Finally, the contractibility of j/f in the remaining case where $x \in A_1' \cap N$ is immediate.

Therefore, by Proposition 3.3.3, the value of $\mathcal{F}$ at $(A, \emptyset)$ is also given by right Kan extension of $\mathcal{F}|\overrightarrow{M}(X, N)$ so that, in conclusion, $\mathcal{F}$ is a right Kan extension of $\mathcal{F}|\overrightarrow{M}(X, N)$.

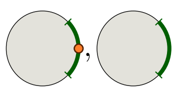
The converse implication (2) $\Rightarrow$ (1) is a consequence of the above argumentation and the converse implication (2) $\Rightarrow$ (1) of Proposition 3.3.3. $\square$

Remark 3.1.11. In the context of Theorem 3.1.6, let

$$\mathrm{Fun}^{\sharp}(\overrightarrow{M}^{+}(X, N), \mathcal{D}) \subset \mathrm{Fun}(\overrightarrow{M}^{+}(X, N), \mathcal{D})$$

denote the full subcategory consisting of those functors that satisfy the equivalent conditions (1) and (2). By arguments analogous to the ones in the proof of Proposition 3.1.5, it can be shown that the objects of $\mathrm{Fun}^{\sharp}(\overrightarrow{M}^{+}(X, N), \mathcal{D})$ are precisely the *cyclic cosheaves*, namely functors $\mathcal{F} : \overrightarrow{M}^{+}(X, N) \to \mathcal{D}$ such that

(1) $\mathcal{F}$ maps objects of the form

 $$\in \overrightarrow{M}^{+}(X, N)$$

to zero objects in $\mathcal{D}$,

(2) for every object (A, A') and for every collared cut of (A, A'), $\mathcal{F}$ maps the associated square (3.1.1) to a coCartesian square in $\mathcal{D}$.

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Corollary 3.1.12. *Let (X, N) be a stratified surface and $\mathcal{D}$ a stable ∞-category. Then there are equivalences of stable ∞-categories*

$$\begin{tikzcd} & \mathrm{Fun}^\sharp(\overrightarrow{M}^+(X,N),\mathcal{D}) \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ \mathrm{Fun}^\sharp(\overrightarrow{M}(X,N),\mathcal{D}) & & \mathrm{Fun}(S(X,N),\mathcal{D}) \end{tikzcd}$$

given by restriction along $\overrightarrow{M}(X,N) \subset \overrightarrow{M}^+(X,N)$ and $S(X,N) \subset \overrightarrow{M}^+(X,N)$, respectively. In particular, via the equivalence $\mathrm{Sh}(X,N;\mathcal{D}) \simeq \mathrm{Fun}(S(X,N)^{\mathrm{op}},\mathcal{D})$ from Corollary 2.4.7, there is a canonical equivalence

$$\mathrm{Sh}(X,N;\mathcal{D}) \simeq \mathrm{Fun}^\sharp(\overrightarrow{M}(X,N)^{\mathrm{op}},\mathcal{D}).$$

Proof. We apply Theorem 4.3.2.15 of [37]. The fact that ρ_1 is an equivalence is then an immediate consequence of the equivalence Theorem 3.1.6. The functor ρ_2 is an equivalence by Theorem 3.1.6 combined with the observation that a functor $S(X,N) \cup \overrightarrow{D}(X,N) \to \mathcal{D}$ is a right Kan extension of its restriction to $S(X,N)$ if and only if $\mathcal{F}|\overrightarrow{D}(X,N) \simeq 0$, that is, two successive applications of loc. cit. □

Theorem 3.1.13. *Let $\mathcal{D} = \mathcal{D}(\mathcal{A})$ be the derived ∞-category of an abelian category $\mathcal{A}$. Then the equivalence*

$$\mathrm{Sh}(X,N;\mathcal{D}) \simeq \mathrm{Fun}^\sharp(\overrightarrow{M}(X,N)^{\mathrm{op}},\mathcal{D})$$

from Corollary 3.1.12 restricts to an equivalence

$$\mathrm{PS}(X,N;\mathcal{A}) \simeq \mathrm{Fun}^\sharp(\overrightarrow{M}(X,N)^{\mathrm{op}},\mathcal{A})$$

identifying perverse sheaves on (X,N) with Milnor sheaves valued in $\mathcal{A}$.

Proof. Under the equivalence

$$\rho : \mathrm{Sh}(X,N;\mathcal{D}) \simeq \mathrm{Fun}^\sharp(\overrightarrow{M}(X,N)^{\mathrm{op}},\mathcal{D}),$$

the value of a Milnor sheaf $\rho(\mathcal{F})$ on a Milnor disk $(A, A') \in \overrightarrow{M}(X,N)$ is equivalent to the value of the corresponding constructible sheaf $\mathcal{F}$ on a Milnor pair (U, U'), where U is a sufficiently small open disk containing the closed disk A and U' is a union of open disks where each disk contains one of the intervals comprising A'. Thus, by Proposition 1.2.4, a constructible sheaf $\mathcal{F} \in \mathrm{Sh}(X,N;\mathcal{D})$ is perverse if and only if $\rho(\mathcal{F})$ takes values in $\mathcal{A}$. Further, since all horizontal morphisms that arise in the Milnor sheaf conditions admit sections, they are Cartesian in $\mathcal{A}$ if and only if they are Cartesian in $\mathcal{D}(\mathcal{A})$. This proves the claim. □

Corollary 3.1.14. *Let $\mathcal{A}$ be an abelian category. Then we have an natural equivalence*

$$\mathrm{Fun}^\sharp(\overrightarrow{M}(X,N)^{\mathrm{op}},\mathcal{A}) \simeq \mathrm{Fun}^\sharp(M(X,N)^{\mathrm{op}},\mathcal{A})$$

where $M(X,N) \subset \Lambda(X,N)$ is the full subcategory of the (undirected) paracyclic category of (X,N) spanned by the Milnor disks.

Proof. This follows from the observation that $\overrightarrow{M}(X,N) \to M(X,N)$ is a localization along the weak equivalences (Corollary 2.3.8). □

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3.2. *Verdier duality for perverse sheaves*

Proposition 3.2.1. *Let $\mathcal{A}$ be an abelian category. Then the self-duality*

$$\xi : \Lambda(X, N) \longrightarrow \Lambda(X, N)^{\mathrm{op}}$$

induces an equivalence

$$\xi^* : \mathrm{Fun}^\sharp(M(X, N)^{\mathrm{op}}, \mathcal{A}) \xrightarrow{\simeq} \mathrm{Fun}^\sharp(M(X, N), \mathcal{A})$$

between Milnor sheaves and cosheaves.

Proof. The Milnor sheaf conditions (in terms of face maps) get swapped with the dual conditions (in terms of degeneracy maps); cf. the proof of Proposition 3.3.4. □

Remark 3.2.2. Suppose $\mathcal{A}$ is an abelian category with exact duality δ. Then the resulting antiequivalence $\delta \circ \xi^*$ of $\mathrm{Fun}^\sharp(M(X, N)^{\mathrm{op}}, \mathcal{A})$ can be identified with the Verdier self-duality of $\mathrm{PS}(X, N; \mathcal{A})$. Note that, even more classically, we may understand the perfect pairing between $R\Gamma(A, A'; \mathcal{F})$ and $R\Gamma(A, \partial A \setminus \mathring{A}'; \mathcal{F}^\vee)$ as an elementary instance of Lefschetz duality for manifolds with boundary.

3.3. *Paracyclic Segal objects*

Let $\mathcal{D}$ be an ∞-category with finite colimits. A cosimplicial object $X : \Delta \to \mathcal{D}$ is called a *Segal object*, if it satisfies the *Segal conditions*: for every $n \geq 1$, the map

$$X_1 \amalg_{X_0} \cdots \amalg_{X_0} X_1 \longrightarrow X_n \tag{3.3.1}$$

induced by the inclusions $[1] \cong \{i, i+1\} \subset [n]$ is an equivalence. Equivalently, X is a Segal object if, for every $1 \leq m < n$, the square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_m \arrow[d] \\ X_{n-m} \arrow[r] & X_n \end{tikzcd} \tag{3.3.2}$$

induced by the diagram

$$\begin{tikzcd} \{m\} \arrow[r] \arrow[d] & \{0, 1, \ldots, m\} \arrow[d] \\ \{m, m+1, \ldots, n\} \arrow[r] & \{0, 1, \ldots, n\} \end{tikzcd}$$

is a pushout square in $\mathcal{D}$.

Proposition 3.3.3. *Let $\mathcal{D}$ be a stable ∞-category, and let Λ^+ be the augmented paracyclic category obtained from Λ by adjoining an initial object $\emptyset$. Let $\mathcal{J} \subset \Lambda^+$ denote the full subcategory spanned by $\emptyset$ and $\langle 0 \rangle$. Then for a functor*

$$\mathcal{F} : \Lambda^+ \longrightarrow \mathcal{D},$$

the following conditions are equivalent:

(1) *$\mathcal{F}$ is a left Kan extension of $\mathcal{F}|\mathcal{J}$.*
(2) *$\mathcal{F}|\Delta$ satisfies the Segal conditions and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}|\Lambda$.*

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Proof. We make two observations:

- The inclusion

$$\Delta \subset \Lambda$$

  is coinitial: The category $\Delta/\langle n\rangle$ is the category of simplices of the simplicial object $\mathrm{Hom}_{\Lambda}(-,\langle n\rangle)|\Delta^{\mathrm{op}}$ whose geometric realization is homeomorphic to $|\Delta^n| \times \mathbb{R}$.
- The inclusion

$$(\Delta^+)_{\leq 1}/[n] \subset (\Lambda^+)_{\leq 1}/\langle n\rangle$$

  is an equivalence and hence cofinal.

Therefore, we have reduced the proof of Proposition 3.3.3 to the statement of Proposition 3.3.4 below. □

Proposition 3.3.4. *Let $\mathcal{D}$ be a stable ∞-category, and let Δ^+ be the augmented simplex category obtained from Δ by adjoining an initial object $\emptyset$. Let $\mathcal{J} \subset \Delta^+$ denote the full subcategory spanned by $\emptyset$ and* $[0]$. *Let*

$$X : \Delta^+ \longrightarrow \mathcal{D}$$

be an augmented cosimplicial object in $\mathcal{D}$. Then the following conditions are equivalent:

(1) *X is a left Kan extension of $X|\mathcal{J}$.*
(2) *$X|\Delta$ satisfies the Segal conditions and X is a right Kan extension of $X|\Delta$.*

Proof. (1) $\Rightarrow$ (2): Suppose that X is a left Kan extension of $X|\mathcal{J}$. The pointwise formula for Kan extensions implies that, for every $n \geq 1$, X_n is a colimit of the restriction of X to $\mathcal{J}/[n]$. We define a functor f from the poset

$$\mathcal{I} = \{0,1\} \leftarrow \{1\} \rightarrow \{1,2\} \leftarrow \{2\} \rightarrow \cdots \leftarrow \{n-1\} \rightarrow \{n-1,n\}$$

to $(\mathrm{Set}_{\Delta})_{/\mathrm{N}(\mathcal{J}/[n])}$ sending a set I to the nerve of the subposet of $\mathcal{J}/[n]$ consisting of those maps with image contained in I. By [37, 4.2.3.10], we may compute the colimit of $X|(\mathcal{J}/[n])$ as the colimit of the diagram

$$\mathcal{I} \to \mathcal{D}, I \mapsto \operatorname{colim} \mathcal{F}|f(I)$$

yielding the nth Segal condition.

To show that X is a right Kan extension of $X|\Delta$, first note that, since $X|\Delta$ is Segal, by Lemma 3.3.5 below, it is a right Kan extension of $X|(\Delta_{\leq 1})$. Therefore, it suffices to show that X is a right Kan extension of $X|(\Delta_{\leq 1})$. By the pointwise criterion, this is equivalent to the statement that X maps the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \{1\} \arrow[r] & \{0,1\} \end{tikzcd}$$

in Δ^+ to a pullback square in $\mathcal{D}$. But, since X is a left Kan extension of $\mathcal{J}$, it maps the square to a pushout square in $\mathcal{D}$ so that the statement follows since $\mathcal{D}$ is stable.

(2) $\Rightarrow$ (1): Suppose that $X|\Delta$ satisfies the Segal conditions. Then, by the above arguments, X is left Kan extension of $X|\mathcal{J}$ if and only if it maps the square

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
\{1\} \arrow[r] & \{0,1\}
\end{tikzcd}$$

to a pushout square. But, by the last part of the argument of (1) $\Rightarrow$ (2), this is equivalent to X being a right Kan extension of $X|\Delta$, concluding the argument. □

Lemma 3.3.5. *Let $\mathcal{D}$ be a stable ∞-category, and let $Y : \Delta \to \mathcal{D}$ be a cosimplicial object in $\mathcal{D}$. Let $\Delta_{\leq 1} \subset \Delta$ denote the full subcategory spanned by the objects $[0]$ and $[1]$. Then Y is a Segal object if and only if Y is a right Kan extension of its restriction $Y|(\Delta_{\leq 1})$.*

Proof. Suppose Y satisfies the Segal conditions. We need to verify that, for every $n \geq 2$, Y_n is a limit of $Y|([n]/\Delta_{\leq 1})$. We prove the statement by induction on n starting with $n = 2$. Consider the commutative diagram in Δ depicted by

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & \{0,1\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\
\{1,2\} \arrow[r] \arrow[d] & \{0,1,2\} \arrow[r] \arrow[d] & \{1,2\} \arrow[d] \\
\{1\} \arrow[r] & \{0,1\} \arrow[r] & \{1\}.
\end{tikzcd}$$

Since all horizontal and vertical composites yields the identity on the respective object, Y maps all $2x1$ and $1x2$ rectangles to biCartesian squares in $\mathcal{D}$. The Segal condition for $n = 2$ is equivalent to Y mapping the top-left square to a pushout, and hence biCartesian, square. The pointwise condition on Y being a right Kan extension is equivalent to Y mapping the bottom-right square to a pullback, hence biCartesian, square. But, by the two-out-of-three property for biCartesian squares ([37, 4.4.2.1]), the top-left square is biCartesian if and only if the bottom-right square is biCartesian. Therefore, for $n = 2$, the Segal condition is equivalent to the corresponding pointwise Kan extension criterion for Y_2.

Assume that the nth Segal condition is equivalent to the pointwise Kan extension formula for Y_n. Consider the diagram

$$\begin{tikzcd}
\{n\} \arrow[r] \arrow[d] & \{0,1,\dots,n\} \arrow[r] \arrow[d] & \{n\} \arrow[d] \\
\{n,n+1\} \arrow[r] \arrow[d] & \{0,1,\dots,n+1\} \arrow[r] \arrow[d] & \{n,n+1\} \arrow[d] \\
\{n\} \arrow[r] & \{0,1,\dots,n\} \arrow[r] & \{n\}
\end{tikzcd}$$

in Δ. A similar argument to the case $n = 2$ implies the equivalence of the $(n+1)$st Segal condition and the pointwise Kan extension formula for Y_{n+1}, concluding the argument. □

4. Perverse sheaves on $(\mathbb{C}, \{0\})$

In this chapter, we consider the classical case when $X = \mathbb{C}$ is the complex plane and $N = \{0\}$. The corresponding category of perverse sheaves is well known, but our approach provides a new point of

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view on it which will be crucial in the further work on categorical generalization to perverse schobers. In what follows, we compare the two approaches and discuss the concepts they lead to.

4.1. The classical (Φ, Ψ)*-description*

Let $\mathcal{A}$ be a Grothendieck abelian category. The following result goes back to the early days of the theory of perverse sheaves [2, 24]. It was originally formulated for perverse sheaves of vector spaces, but the proof given in [24] generalizes easily to the $\mathcal{A}$-valued case.

Proposition 4.1.1. *The category* $\mathrm{PS}(\mathbb{C}, \{0\}; \mathcal{A})$ *is equivalent to the category of data* (Φ, Ψ, a, b)*, where* Φ *and* Ψ *are objects of* $\mathcal{A}$ *and*

$$\begin{tikzcd} \Phi \arrow[r, bend left, "a"] & \Psi \arrow[l, bend left, "b"] \end{tikzcd} \tag{4.1.2}$$

are morphisms such that the monodromy transformations

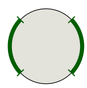
$$T_\Psi := \mathrm{Id}_\Psi - ab \quad \text{and} \quad T_\Phi : \mathrm{Id}_\Phi - ba \tag{4.1.3}$$

are isomorphisms. In fact, T_Ψ *being an isomorphism is equivalent to* T_Φ *being an isomorphism.*

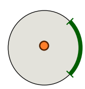
For a given perverse sheaf $F \in \mathrm{PS}(\mathbb{C}, \{0\}; \mathcal{A})$, the corresponding objects $\Phi = \Phi(F)$ and $\Psi = \Psi(F)$ are called the objects of *vanishing* and *nearby cycles* of F. We will now describe the relationship between the classification data in Proposition 4.1.1 and our description of perverse sheaves as Milnor sheaves from Corollary 3.1.14.

4.2. From a Milnor sheaf to vanishing and nearby cycles

Let $\mathcal{F} : M(\mathbb{C}, \{0\})^{\mathrm{op}} \longrightarrow \mathcal{A}$ be a Milnor sheaf. We will explain how to most directly extract from $\mathcal{F}$ the classification data (4.1.2) and verify conditions (4.1.3). First, we define

$$\Psi = \mathcal{F}(A, A') \quad \text{where} \quad (A, A') =$$


is any disk that does not contain the origin 0. Further, we set

$$\Phi = \mathcal{F}(B, B') \quad \text{where} \quad (B, B') =$$


is any disk containing 0 in its interior. The descent conditions force rotation of (A, A') by π to be multiplication by -1: In the local model explained in §3.3, this automorphism corresponds to the paracyclic shift on the Čech nerve of $0 \to \Psi[1]$. The monodromy transformation T_Ψ is obtained by moving (A, A') as a rigid body (parallel to itself) in a circle around the origin $0 \in \mathbb{C}$. The monodromy T_Φ is induced by rotating (B, B') by an angle of 2π around the center of the disk B. The map

$$a : \Phi \longrightarrow \Psi$$

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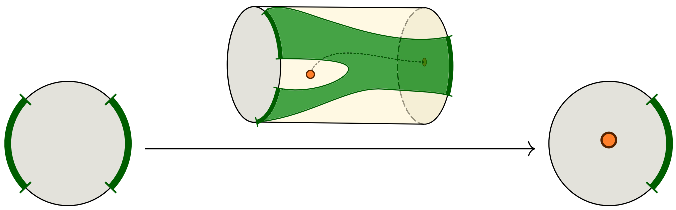
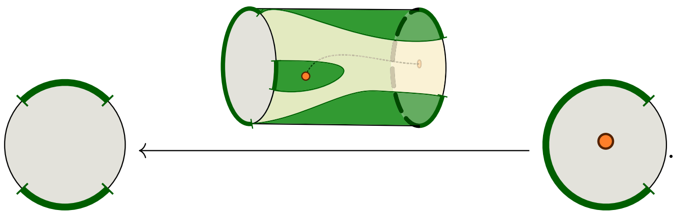
is obtained from the morphism in $M(\mathbb{C}, \{0\})$ that is represented by a bordism of the form



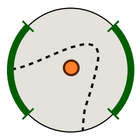
(4.2.1)

while the morphism b corresponds to the dual of equation (4.2.1):



(4.2.2)

To obtain the relations (4.1.3), we investigate the descent condition for

$\mathcal{F}(C, C')$ where $(C, C') =$ 

Namely, introducing the depicted cut, the corresponding descent condition (3.1.1) provides a direct sum decomposition $\mathcal{F}(C, C') \cong \Phi \oplus \Psi$. We then directly observe that, with respect to that decomposition, the transformation induced on $\mathcal{F}(C, C')$ by rotating (C, C') around its center by 2π, is given by the matrix

$$Q = \begin{pmatrix} T_\Phi & 0 \\ 0 & T_\Psi \end{pmatrix}. \tag{4.2.3}$$

On the other hand, this transformation comes equipped with a square root, induced by rotating (C, C') around its center by π. A somewhat more careful analysis shows that, in terms of the above direct sum decomposition, this transformation can be described by the matrix

$$P = \begin{pmatrix} -\mathrm{id} & b \\ -a & \mathrm{id} \end{pmatrix}. \tag{4.2.4}$$

Now, the relation $P^2 = Q$ implies the desired relations (4.1.3). Note that, in order to extract the above data, various choices have to be made – the advantage of the description of $\mathcal{F}$ lies in the intrinsic nature of the parametrizing category $M(\mathbb{C}, \{0\})$ of Milnor disks.

4.3. The equivalence of classical and Milnor sheaf descriptions

In this section, we elaborate on the discussion in §4.2 to provide a direct argument for why these descriptions are equivalent. This can, of course, also be indirectly deduced by combining our Corollary 3.1.14 and [24], but it is nevertheless interesting to provide an explicit dictionary.

The Milnor sheaf description

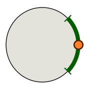
Proposition 4.3.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk. Then the restriction along $\Lambda_{\mathbb{D}} \subset M^+(\mathbb{C}, \{0\})$ induces a fully faithful functor*

$$\mathrm{Fun}^\sharp(M^+(\mathbb{C}, \{0\})^{\mathrm{op}}, \mathcal{A}) \xrightarrow{\simeq} \mathrm{Fun}(\Lambda_{\mathbb{D}}^{\mathrm{op}}, \mathcal{A})$$

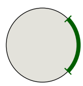
with essential image given by those paracyclic objects whose underlying simplicial object satisfies the Segal conditions.

Proof. For notational convenience, we replace $\mathcal{A}$ by $\mathcal{A}^{\mathrm{op}}$ and prove the cosheaf version of the statement. By Corollary 2.3.8, the category M^+ may be described as the localization of its directed variant $\overrightarrow{M}^+$. In the statement of the proposition, we may therefore replace the category $\mathrm{Fun}^\sharp(M^+(\mathbb{C}, \{0\}), \mathcal{A})$ by the equivalent category $\mathrm{Fun}^\sharp(\overrightarrow{M}^+(\mathbb{C}, \{0\}), \mathcal{A})$, where here, the superscript $\sharp$ also contains the requirement that weak equivalences be sent to isomorphisms in $\mathcal{A}$. We now focus on the following collections of objects of $\overrightarrow{M}^+$ (and the subcategories they span):

- $\overrightarrow{M}_0$: all objects (A, A'), where $0 \in A \setminus A'$,
- $\overrightarrow{M}_1$: $\overrightarrow{M}_0$ together with all objects of the form

$$(A, A') =$$ 

- $\overrightarrow{M}_2$: $\overrightarrow{M}_1$ together with all objects (A, A') such that $0 \in A'$,
- $\overrightarrow{M}_3$: $\overrightarrow{M}_2$ together with all objects of the form

$$(A, A') =$$ 

The fact that the restriction functor of the proposition is an equivalence now follows from the statement that the functors $\mathcal{F} \in \mathrm{Fun}^\sharp(\overrightarrow{M}^+(\mathbb{C}, \{0\}), \mathcal{A})$ can be characterized by the following conditions:

(1) The paracyclic object $\mathcal{F}|\overrightarrow{M}_0 \simeq \Lambda_{\mathbb{D}}$ satisfies the Segal conditions.

(2) $\mathcal{F}$ is obtained from its restriction to $\overrightarrow{M}_0$ via a sequence of left (resp. right) Kan extensions as indicated in

$$\overrightarrow{M}_0 \xrightarrow{\text{right}} \overrightarrow{M}_1 \xrightarrow{\text{left}} \overrightarrow{M}_2 \xrightarrow{\text{left}} \overrightarrow{M}_3 \xrightarrow{\text{right}} \overrightarrow{M}^+(\mathbb{C}, \{0\}).$$

The details are left to the reader. □

Corollary 4.3.2. *The category of Milnor sheaves on $(\mathbb{C}, \{0\})$ with values in $\mathcal{A}$, and therefore the category of perverse sheaves on $(\mathbb{C}, \{0\})$, is equivalent to the category $\mathcal{A}^{\mathrm{Seg}}_{\Lambda_\infty}$ of paracyclic objects in $\mathcal{A}$ whose underlying simplicial object satisfies the Segal conditions.*

In what follows, we provide the relation to the more traditional classification of Proposition 4.1.1 by means of a paracyclic nerve construction which can also be regarded as a special instance of a duplicial variant of the Dold–Kan correspondence established in [17].

Paracyclic structures on the nerve of a Picard groupoid

To compare Propositions 4.1.1 and 4.3.1 in a direct way, we assume for simplicity that $\mathcal{A} = \mathcal{A}b$ is the category of abelian groups. It is classical that a simplicial set is Segal if and only if it is isomorphic to the nerve of a small category. The categories relevant for us are are Picard groupoids of a particular type.

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We recall (cf. [12]) that a *Picard groupoid* is a symmetric monoidal category $(\mathcal{P}, \otimes, \mathbf{1})$ in which each object is invertible with respect to $\otimes$ and each morphism is invertible with respect to the composition.

Example 4.3.3. Let $E^\bullet$ be a two-term complex of abelian groups situated in degrees $[-1, 0]$. It will be suggestive for us to write $E^\bullet$ as $\{\Psi \xrightarrow{b} \Phi\}$ with Φ in degree 0 and Ψ in degree (-1). To such a datum, one associates a Picard groupoid $[E^\bullet] = [\Psi \xrightarrow{b} \Phi]$ with

$$\begin{aligned} \mathrm{Ob}\,[\Psi \xrightarrow{b} \Phi] \ &= \ \Phi; \\ \mathrm{Hom}(\varphi', \varphi) \ &= \ \left\{\psi \in \Psi \,\middle|\, b(\psi) = \varphi - \varphi'\right\}. \end{aligned}$$

Composition of morphisms is given by addition of the ψ. The tensor product of objects is given by addition of the φ. We note that the set of all morphisms in $[\Psi \xrightarrow{b} \Phi]$ (i.e., the disjoint union of all the $\mathrm{Hom}(\varphi, \varphi')$) can be described as

$$\mathrm{Mor}\,[\Psi \xrightarrow{b} \Phi] \ = \ \Psi \oplus \Phi,$$

with the source and target maps $s, t : \mathrm{Mor} \to \mathrm{Ob}$ given by

$$s(\psi, \varphi) = \varphi - b(\psi), \quad t(\psi, \varphi) = \varphi. \tag{4.3.4}$$

See [12] for more details.

The nerve $N[\Psi \xrightarrow{b} \Phi]$ is a simplicial abelian group with n-simplices

$$N_n[\Psi \xrightarrow{b} \Phi] \ = \ \Psi^{\oplus n} \oplus \Phi. \tag{4.3.5}$$

Passing from a two-term complex $\{\Psi \xrightarrow{b} \Phi\}$ to the simplicial object $N[\Psi \xrightarrow{b} \Phi]$ is a particular case of the Dold–Kan correspondence between nonpositively graded cochain complexes of abelian groups and simplicial abelian groups; see §4.4 below.

Proposition 4.3.6. *Let $b : \Psi \to \Phi$ be a morphism of abelian groups. Then the following are in bijection:*

(i) *Morphisms $a : \Phi \to \Psi$ such that the data (Φ, Ψ, a, b) satisfy the conditions of Proposition 4.1.1, that is, define a perverse sheaf $F \in \mathrm{PS}(D, 0; \mathcal{A}b)$.*

(ii) *Extensions of the structure of a simplicial abelian on $N[\Psi \xrightarrow{b} \Phi]$ to that of a paracyclic abelian group, that is, systems of automorphisms $t_n \in \mathrm{Aut}\big(N_n[\Psi \xrightarrow{b} \Phi]\big)$ (actions of the $\tau_n \in \mathrm{Aut}_{\Lambda_\infty}\langle n\rangle$) satisfying the relations dual to those imposed in Definition 2.1.1(a).*

Under this bijection, the automorphism t_n^{n+1} corresponds, via the identification (4.3.5), to the direct sum $T_\Psi^{\oplus n} \oplus T_\Phi$ of the monodromies.

Proof. Explicitly, the convention (4.3.4) on labelling the source and target of a morphism implies that the simplicial face and degeneracy operators on $N_n[\Psi \xrightarrow{b} \Phi]$ are given by

$$\partial_i : \Psi^{\oplus n} \oplus \Phi \to \Psi^{\oplus(n-1)} \oplus \Phi, \quad (\psi_1, \cdots, \psi_n; \varphi) \mapsto \begin{cases} (\psi_2, \cdots, \psi_n; \varphi), & i = 0, \\ (\psi_1, \cdots, \psi_i + \psi_{i+1}, \cdots \psi_n; \varphi), & 1 \le i < n; \\ (\psi_1, \cdots, \psi_{n-1}; \varphi - b(\psi_n)), & i = n; \end{cases}$$

$$s_j : \Psi^{\oplus n} \oplus \Phi \to \Psi^{\oplus(n+1)} \oplus \Phi, \quad (\psi_1, \cdots, \psi_n; \varphi) \mapsto (\psi_1, \cdots, \psi_{j-1}, 0, \psi_{j+1}, \cdots \psi_n; \varphi), \ j = 0, \cdots, n.$$

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Now, let $a : \Phi \to \Psi$ be as in (a). For each $n \geq 0$, define an endomorphism t_n of $\Psi^{\oplus n} \oplus \Phi$ by

$$t_n(\psi_1, \cdots, \psi_n, \varphi) = \big(-\psi_1 - \cdots - \psi_n + a(\varphi), \psi_2, \cdots, \psi_{n-1}; \varphi - b(\psi_n)\big). \tag{4.3.7}$$

We then check directly that the relations dual to those of Definition 2.1.1(a) are satisfied. We also check that $t_n^{n+1} = T_\Psi^{\oplus n} \oplus T_\Phi$ which implies that t_n is invertible.

Conversely, suppose we have automorphisms t_n as in (b). The relation $\partial_0 t_n = \partial_n$ implies that t_n has the form

$$t_n(\psi_n, \cdots, \psi_n; \varphi) = \Big(-\sum_{i=1}^{n} x_i^{(n)}(\psi_i) + a_n(\varphi);\ \psi_2, \cdots, \psi_{n-1}, \varphi - b(\psi_n)\Big)$$

for some linear maps $x_i^{(n)} : \Psi \to \Psi$ and $a_n : \Phi \to \Psi$. We denote $a_1 = a$ and will prove that

$$x_i^{(n)} = \mathrm{Id}, \quad a_n = a, \quad \forall\, n,\ i = 1, \cdots, n, \tag{4.3.8}$$

that is, that all the t_n are given by the formula (4.3.7). This will imply the invertibility of $T_\Psi = \mathrm{Id} - ab$ and $T_\Phi = \mathrm{Id} - ba$ by identifying t_n^{n+1} as above.

The equalities (4.3.8) are proved recursively, using the relations of Λ^∞. To start, the relation $\partial_1 t_2 = t_1 \partial_0$ implies that

$$\partial_1 t_2(\psi_1, \psi_2; \varphi) = (-x_1^{(2)}\psi_1 - x_2^{(2)}\psi_2 + a_2\varphi + \psi_1;\ \varphi - b\psi_2)$$

is equal to

$$t_1\partial_0(\psi_1, \psi_2; \varphi) = (-x_1^{(1)}\psi_2 + a\varphi;\ \varphi - b\psi_2),$$

which entails

$$x_2^{(2)} = x_1^{(1)}, \quad x_1^{(2)} = \mathrm{Id}\,.$$

The relation $\partial_2 t_2 = t_1 \partial_1$ then implies that

$$\partial_1 t_2(\psi_1, \psi_2; \psi) = (-x_1^{(2)}\psi_1 - x_2^{(2)}\psi_2 + a_2\varphi;\ \varphi - b\psi_2 - b\psi_1)$$

is equal to

$$t_1\partial_1(\psi_1, \psi_2; \psi) = (-x_1^{(1)}\psi_1 - x_1^{(1)}\psi_2 + a(\varphi);\ \varphi - b\psi_2 - b\psi_1),$$

which entails

$$x_1^{(2)} = x_2^{(2)} = x_1^{(1)}, \quad a_2 = a.$$

Since we already know that $x_1^{(2)} = \mathrm{Id}$, we see that $x_2^{(2)} = x_1^{(1)} = \mathrm{Id}$. Continuing like this, we prove equation (4.3.8). □

Remark 4.3.9. One can consider paracyclic structures on the nerves of more general Picard groupoids, not necessarily those corresponding to two-term complexes. It would be interesting to understand the relation of such structures to perverse sheaflike objects. We recall [30] that Picard groupoids correspond to spectra (stable homotopy types in the sense of homotopy topology) which have only two nontrivial homotopy groups in adjacent degrees, say only π_0 and π_1 or only π_1 and π_2.

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More generally, unstable homotopy types with only π_1 and π_2 nontrivial, are described by *crossed modules* (see, e.g., [42]), which are two-term complexes of possibly nonabelian groups

$$G^\bullet = \{G^{-1} \xrightarrow{\partial} G^0\}$$

with a compatible action of G^0 on G^{-1}. A crossed module $G^\bullet$ gives rise to a non-abelian Picard groupoid (also known as a 2-*group*) $[G^\bullet]$, defined similarly to Example 4.3.3. One can ask about the meaning of paracyclic structures on the nerve of $[G^\bullet]$ and the possibility of defining perverse sheaves of nonabelian groups in one complex dimension.

4.4. *Relation to the duplicial Dold–Kan correspondence*

The classical Dold–Kan
Let $\mathcal{A}$ be an abelian category and $\mathrm{C}^{\leq 0}(\mathcal{A})$ be the (abelian) category of cochain complexes over $\mathcal{A}$ situated in degrees ≤ 0. As usual, by $\mathcal{A}_\Delta$ we denote the category of simplicial objects of $\mathcal{A}$. The *Dold–Kan correspondence* (see, e.g., [25]) is the pair of mutually quasi-inverse (in particular, adjoint) equivalences of categories

$$C_{\mathrm{DK}} : \mathcal{A}_\Delta \overset{\sim}{\longleftrightarrow} \mathcal{C}^{\leq 0}(\mathcal{A}) : \mathrm{N}_{\mathrm{DK}},$$

defined as follows. The functor C_{DK}, called the *normalized chain complex functor*, takes $A_\bullet \in \mathcal{A}_\Delta$ to the complex $C_{\mathrm{DK}}(A_\bullet)$ with

$$C_{\mathrm{DK}}{}^{-n}(A_\bullet) = \bigcap_{i=1}^{n} \mathrm{Ker}\{\partial_i : A_n \longrightarrow A_{n-1}\}, \quad n \geq 0,$$

with the differential given by the remaining face map ∂_0.

The functor N_{DK}, called the *Dold–Kan nerve*, takes a complex $(E^\bullet, d_E) \in \mathrm{C}^{\leq 0}(\mathcal{A})$ into the simplicial object $\mathrm{N}_{\mathrm{DK}}(E^\bullet)$ with

$$\mathrm{N}_{\mathrm{DK}}(E^\bullet)_n = Z^0(\Delta^n, E^\bullet),$$

the object of degree 0 simplicial (hyper)cocycles on Δ^n with values in $E^\bullet$. That is, denoting Δ^n_m the set of m-simplices of Δ^n,

$$Z^0(\Delta^n, E^\bullet) \subset \prod_{m\geq 0} (E^{-m})^{\Delta^n_m}$$

is given by the following 'end' condition: The action of the morphism induced by each $d_E : E^{-m} \to E^{-m+1}$ is equal to the action of the morphism induced by $\sum(-1)^i \partial_i : \Delta^n_{m+1} \to \Delta^n_m$.

Examples 4.4.1.

(a) Let $\mathcal{A} = \mathcal{A}b$. An element of $Z^0(\Delta^n, E^\bullet)$ is in this case a rule γ associating:
   (0) To each vertex e_i, $0 \leq i \leq n$, of Δ^n, an element $\gamma_i \in E^0$.
   (1) To each edge (possibly degenerate) e_{ij}, $0 \leq i \leq j \leq n$, of Δ^n, an element $\gamma_{ij} \in E^{-1}$ so that $d_E(\gamma_{ij}) = e_j - e_i$.
   (2) To each two-face (possibly degenerate) e_{ijk}, $0 \leq i \leq j \leq k \leq n$, of Δ^n, an element $\gamma_{ijk} \in E^{-2}$ so that $d_E(\gamma_{ijk}) = \gamma_{jk} - \gamma_{ik} + \gamma_{ij}$.
   $(\cdots)$ And so on.

(b) In particular, if $E^\bullet = \{E^{-1} \to E^0\}$ is a two-term complex of abelian groups, then $\mathrm{N}_{\mathrm{DK}}(E^\bullet) = \mathrm{N}[E^\bullet]$ is the usual nerve of the Picard groupoid $[E^\bullet]$.

Proposition 4.3.6 extends verbatim to the following.

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Proposition 4.4.2. *Let $\mathcal{A}$ be any abelian category and $b : \Psi \to \Phi$ be a morphism in $\mathcal{A}$. Then the following are in bijection:*

(i) *Morphisms $a : \Phi \to \Psi$ such that the data (Φ, Ψ, a, b) satisfy the conditions of Proposition 4.1.1, that is, define a perverse sheaf $F \in \mathrm{PS}(D, 0; \mathcal{A})$.*

(ii) *Extensions of the simplicial object structure on $\mathrm{N}_{\mathrm{DK}}\{\Psi \xrightarrow{b} \Phi\}$ to a structure of a paracyclic object.*

The duplicial Dwyer–Kan correspodence

Let $\mathcal{A}$ be an abelian category and $E^\bullet \in \mathrm{C}^{\leq 0}(\mathcal{A})$. Proposition 4.4.2 leads to the following question: What is the meaning of a paracyclic structure on $\mathrm{N}_{\mathrm{DK}}(E^\bullet)$ extending the given simplicial structure? An answer to that can be given by the results of Dwyer–Kan [17] which we recall.

We define the *duplex category* Ξ to have objects $\langle n \rangle$, $n \in \mathbb{N} := \mathbb{Z}_{\geq 0}$. A morphism from $\langle m \rangle$ to $\langle n \rangle$ consists of a weakly monotone map $f : \mathbb{N} \to \mathbb{N}$ satisfying the following periodicity condition: For all $i \in \mathbb{N}$, we have $f(i + m + 1) = f(i) + n + 1$. The simplex category Δ is naturally a subcategory of Ξ obtained by restricting to those morphisms between $\langle m \rangle$ and $\langle n \rangle$ that map the interval $[0, m]$ to $[0, n]$. A *duplicial object* in a category $\mathcal{C}$ is a functor $\Xi^{\mathrm{op}} \to \mathcal{C}$.

We recall [21] that the paracyclic category Λ_∞ can be defined in a very similar way, except we consider weakly mononote maps $f : \mathbb{Z} \to \mathbb{Z}$ (instead of $\mathbb{N} \to \mathbb{N}$) satisfying the same periodicity condition. In particular, the shift map

$$\tau_n : \langle n \rangle \longrightarrow \langle n \rangle, \quad i \mapsto i + 1$$

is invertible as en element of $\mathrm{Hom}_{\Lambda_\infty}(\langle n \rangle, \langle n \rangle)$ (with i running in $\mathbb{Z}$) but is not invertible as an element of $\mathrm{Hom}_{\Xi}(\langle n \rangle, \langle n \rangle)$ (with i running in $\mathbb{N}$). In fact, comparing [17] and [21] leads to the following.

Proposition 4.4.3. $\Lambda_\infty \simeq \Xi[\tau_n^{-1} | n \geq 0]$ *is identified with the localization of Ξ with respect to the morphisms τ_n, $n \geq 0$.*

In fact, the powers τ_n^{n+1} forming a central system (a natural transformation from Id_Ξ to itself), it is easy to see that the ∞-categorical localization $\Xi[\tau_n^{-1} | n \geq 0]_\infty$ is also identified with Λ_∞. In particular, Ξ, like Λ_∞, is generated by the coface and codegeneracy morphisms

$$\begin{aligned} \delta_i^{(n)} &: \langle n-1 \rangle \to \langle n \rangle, \quad (n \geq 1,\ 0 \leq i \leq n), \\ \sigma_i^{(n)} &: \langle n \rangle \to \langle n-1 \rangle, \quad (n \geq 1,\ 0 \leq i \leq n), \end{aligned}$$

satisfying the same quadratic relations as in Λ_∞. The morphisms

$$\delta_i^{(n)},\ 0 \leq i \leq n, \quad \sigma_i^{(n)},\ 0 \leq i \leq n-1$$

generate the simplex category $\Delta \subset \Xi$. The shift map is expressed as $\tau_n = \delta_0^{(n-1} \sigma_{n-1}^{(n)}$. Accordingly, a duplicial object $Y_\bullet$ in a category $\mathcal{C}$ can be identified with a sequence of objects $X_0, X_1, \ldots$ equipped with face and degeneracy maps

$$\begin{aligned} \partial_i &: X_n \longrightarrow X_{n-1} \quad (n \geq 1, 0 \leq i \leq n), \\ s_i &: X_{n-1} \longrightarrow X_n \quad (n \geq 1, 0 \leq i \leq n), \end{aligned}$$

subject to relations dual to those among the $\delta_i^{(n)}, \sigma_i^{(n)}$. The action of τ_n is then $t_n = s_{n+1}\partial_0 : Y_n \to Y_n$. A paracyclic object is a duplicial object such that all the t_n are isomorphisms.

Following Dwyer–Kan, we call a *connective ducomplex* in $\mathcal{A}$ a diagram

$$\cdots \underset{d}{\overset{\delta}{\rightleftarrows}} B^{-2} \underset{d}{\overset{\delta}{\rightleftarrows}} B^{-1} \underset{d}{\overset{\delta}{\rightleftarrows}} B^0$$

satisfying $d^2 = 0$, $\delta^2 = 0$ and no further relations. We denote $\mathrm{DC}^{\leq 0}(\mathcal{A})$ the category of connective ducoplexes in $\mathcal{A}$.

Theorem 4.4.4 (Dwyer–Kan). (a) *There is an equivalence of categories*

$$\mathcal{A}_{\Xi} \xrightarrow{\simeq} \mathrm{DC}^{\leq 0}(\mathcal{A})$$

given by associating to a duplicial abelian group $A_\bullet$ the ducomplex $B^\bullet$ with

$$\begin{aligned} B^{-n} &= \bigcap_{i=1}^{n} \mathrm{Ker}\{\partial_i : A_n \to A_{n-1}\}, \quad n \geq 0, \\ d &= \partial_0 : B^{-n} \to B^{-n+1}, \\ \delta &= \sum_{i=0}^{n} (-1)^i s_i : B^{-n+1} \to B^{-n}. \end{aligned}$$

(b) *Under this equivalence, paracyclic objects correspond to ducomplexes satisfying*

$$\mathrm{Id}_{B^{-n}} + (-1)^n (d\delta - \delta d) : B^{-n} \longrightarrow B^{-n} \quad \textit{is invertible for any } n \geq 0.$$

Proof. Part (a) is Theorem 3.5 of [17]. Part (b) follows from the interpretation of t_n^{n+1} in terms of ducomplexes given in Proposition 6.5 of [17]. □

The equivalence between the descriptions of the category of perverse sheaves on $(\mathbb{C}, \{0\})$ from Proposition 4.3.1 and Proposition 4.1.1, respectively, is then a consequence of restricting the equivalence from Theorem 4.4.4 to paracyclic Segal objects.

Appendix A. ∞-categorical preliminaries

Appendix A.1. Generalities on ∞-categories

In the rest of the paper, we will use freely the language of ∞-categories [37]. The following is intended to fix the terminology and notation and to recall the main tools that will be used.

We denote by Set_Δ the category of simplicial sets. For a simplicial set $S = (S_n)_{n\geq 0}$, we denote

$$\partial_i : S_n \longrightarrow S_{n-1}, \quad 0 \leq i \leq n$$

the simplicial face maps. By $\Delta^n \in \mathrm{Set}_\Delta$, we denote the standard n-simplex.

Following [37], we will use the term ∞-*category* for a weak Kan complex. Thus, an ∞-category $\mathcal{C}$ is a simplicial set $(\mathcal{C}_n)_{n\geq 0}$ satisfying the lifting condition for intermediate horns $\Lambda_i^n \subset \Delta^n$, $0 < i < n$. Any ordinary category can be considered as an ∞-category by passing to the nerve.

Each ∞-category $\mathcal{C}$ contains the maximal Kan subcomplex $\mathcal{C}^{\mathrm{Kan}} \subset \mathcal{C}$, which is can be interpreted as 'the ∞-groupoid of equivalences in $\mathcal{C}$'.

We follow the usual notation and terminology: 0-simplices of $\mathcal{C}$ are called *objects*, and we denote $\mathrm{Ob}(\mathcal{C}) = \mathcal{C}_0$, while 1-simplices are called *morphisms* and we denote $\mathrm{Mor}(\mathcal{C}) = \mathcal{C}_1$. For any two $x, y \in \mathrm{Ob}(\mathcal{C})$, we denote $\mathrm{Hom}_{\mathcal{C}}(x, y) \subset \mathrm{Mor}(\mathcal{C})$ the set of 1-simplices f such that $\partial_1(f) = x$ and $\partial_0(f) = y$.

To any ∞-category $\mathcal{C}$, one associates its *homotopy category* $\mathrm{Ho}(\mathcal{C})$ which is an ordinary category with the set of objects $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x, y)$ defined as the quotient of $\mathrm{Hom}_{\mathcal{C}}(x, y)$ by the homotopy relation: $f \sim g$ if there is $\sigma \in \mathcal{C}_2$ with $\partial_1(\sigma) = f$ and $\partial_2(\sigma) = g$. An *equivalence* in $\mathcal{C}$ is a morphism which becomes an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

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For an ∞-category $\mathcal{C}$ and any $x, y \in \mathrm{Ob}(\mathcal{C})$, the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ can be upgraded to a simplicial set $\mathrm{Map}_{\mathcal{C}}(x, y)$ (the *mapping space*), in such a way as to make out of $\mathcal{C}$ a category enriched in simplicial sets. See [37] §1.2.2 for details.

This leads to another point of view on ∞-categories: as categories enriched in topological spaces (or simplicial sets). Several ∞-categorical concepts can be formulated in this language. For example, an *initial object* of an ∞-category $\mathcal{C}$ is an object 0 such that, for each $x \in \mathrm{Ob}(\mathcal{C})$, the space $\mathrm{Map}_{\mathcal{C}}(0, x)$ is contractible.

Example A.1 (Kan simplicial sets as an ∞-category). Any Kan simplicial set is an ∞-category. The ∞-category Sp of *spaces* is defined as the simplicial nerve of the category of Kan simplicial sets [37, §1.2.16].

Appendix A.2. Dg-categories

We denote by $\mathcal{A}b$ the category of abelian groups and by $C(\mathcal{A}b)$ the category of cochain complexes of abelian groups, with its standard symmetric monoidal structure. By a *dg-category*, we mean a category $\mathcal{A}$ enriched in $C(\mathcal{A}b)$. For such $\mathcal{A}$, we have the ordinary categories $Z^0(\mathcal{A})$, $H^0(\mathcal{A})$ with the same objects as $\mathcal{A}$ and

$$\mathrm{Hom}_{Z^0(\mathcal{A})}(x, y) = Z^0 \,\mathrm{Hom}^\bullet_{\mathcal{A}}(x, y), \quad \mathrm{Hom}_{H^0(\mathcal{A})}(x, y) = H^0 \,\mathrm{Hom}^\bullet_{\mathcal{A}}(x, y).$$

Here, Z^0 is the subgroup of 0-cocycles in the Hom-complex.

A dg-category $\mathcal{A}$ gives an ∞-category $N_{\mathrm{dg}}(\mathcal{A})$ known as the *dg-nerve* of $\mathcal{A}$. As a simplicial set, $N_{\mathrm{dg}}(\mathcal{A})$ was introduced in [28]. For a given $n \geq 0$, the set $N_{\mathrm{dg}}(\mathcal{A})_n$ consists of weakly commutative n-simplices in $\mathcal{A}$ (called *Sugawara simplices* in [28]), which are data of:

$$\begin{gathered} x_0, \cdots, x_n \in \mathrm{Ob}(\mathcal{A}); \\ u_{ij} \in \mathrm{Hom}^0_{\mathcal{A}}(x_i, x_i), \quad d(u_{ij}) = 0, \quad i < j; \\ u_{ijk} \in \mathrm{Hom}^{-1}_{\mathcal{A}}(x_i, x_k), \quad d(u_{ijk}) = u_{jk}u_{ij} - u_{ik}, \quad i < j < k; \\ \text{and so on.} \end{gathered}$$

It was shown in [38] that $N_{\mathrm{dg}}(\mathcal{A})$ is in fact a ∞-category. By construction, we have

$$\mathrm{Ho}(N_{\mathrm{dg}}(\mathcal{A})) \simeq H^0(\mathcal{A}).$$

Appendix A.3. The derived ∞-category of an abelian category

Let $\mathcal{A}$ be a Grothendieck abelian category. In particular, $\mathcal{A}$ has enough injectives. Denote by $\mathrm{C}(\mathcal{A})$ the dg-category of all cochain complexes over $\mathcal{A}$. Thus, $Z^0(\mathrm{C}(\mathcal{A}))$ is the 'usual' category of complexes (morphsims = morphisms of complexes) and $H^0(\mathrm{C}(\mathcal{A}))$ is the homotopy category. The classical (unbounded) derived category of $\mathcal{A}$, denoted $\mathcal{D}(\mathcal{A})$, is defined as the categorical localization of $H^0(\mathrm{C}(\mathcal{A}))$ by the class of quasi-isomorphisms. It is a triangulated category.

The *(unbounded) derived ∞-category* of $\mathcal{A}$, denoted $\mathcal{D}(\mathcal{A})$, can be defined in one of two equivalent ways; see [38] §1.3.5, especially Proposition 1.3.5.16 and before.

(i) As the ∞-categorical localization of the usual (abelian) category $Z^0(C(\mathcal{A}))$ by the class of quasi-isomorphisms.
(ii) As the full ∞-subcategory in $N_{\mathrm{dg}}(\mathrm{C}(\mathcal{A}))$ spanned by *fibrant complexes*. A fibrant complex is a possibly unbounded complex of injective objects with some additional properties; see [38] §1.3.5 and [44].

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We have

$$\mathrm{Ho}(\mathcal{D}_\infty(\mathcal{A})) \simeq \mathcal{D}(\mathcal{A}).$$

Appendix A.4. Stable ∞-categories

The derived ∞-categories from §A.3 are examples of stable ∞-categories. Here, we recall the definition of a stable ∞-category from [38] and discuss some basic results that we will use. Let $\mathcal{D}$ be a pointed ∞-category, and consider a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{A.1}$$

in $\mathcal{D}$ where 0 is a zero object. The square is called a *fiber sequence* if it is a pullback square. In this case, the morphism f is called a fiber of g. Dually, the square is called a *cofiber sequence* if it is a pushout square. In this case, we say that g is a cofiber of f. The category $\mathcal{D}$ is called *stable* if

1. every morphism admits a fiber and a cofiber,
2. a square of the form (A.1) is a fiber sequence if and only if it is a cofiber sequence.

We collect some basic results about stable ∞-categories (cf. [38]):

Proposition A.2. *Let $\mathcal{D}$ be a stable ∞-category. Then:*

(1) *$\mathcal{D}$ admits finite limits and colimits.*
(2) *The homotopy category of $\mathcal{D}$ admits a triangulated structure.*
(3) *A square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

in $\mathcal{D}$ is Cartesian if and only if it is coCartesian.

Cartesian squares in a stable ∞-category (which are hence also coCartesian) will be called biCartesian squares. The statement of Proposition A.2 (3) has a useful generalization to higher-dimensional cubes: Let $\mathcal{D}$ be a stable ∞-category, $n \geq 1$, and let $\mathcal{P}(\{1, \dots, n\})$ be the poset of all subsets of the set $\{1, \dots, n\}$. Consider a diagram

$$q : \mathrm{N}(\mathcal{P}(\{1, \dots, n\})) \longrightarrow \mathcal{D}$$

which, due to the apparent isomorphism $\mathrm{N}(\mathcal{P}(\{1, \dots, n\})) \cong (\Delta^1)^n$, has the shape of an n-dimensional cube. Note that, we may either interpret q as

1. a cone over the diagram $q|\,\mathrm{N}(\mathcal{P}(\{1, \dots, n\}) \setminus \{\emptyset\})$, or
2. a cone under the diagram $q|\,\mathrm{N}(\mathcal{P}(\{1, \dots, n\}) \setminus \{\{1, \dots, n\}\})$.

If the first cone is a limit cone, then we call q *Cartesian*; if the second cone is a colimit cone, then we call q *coCartesian*. We recall some results from [38]:

Proposition A.3. *A cube q is Cartesian if and only if it is coCartesian.*

Proof [38, 1.2.4.13]. □

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We will refer to cubes in a stable ∞-category which are Cartesian (and hence coCartesian) as *biCartesian*, generalizing the above terminology in the case $n = 2$. We further recall:

Proposition A.4. *An n-cube is biCartesian if and only if the $(n-1)$-cube, obtained by passing to cofibers along all morphisms parallel to one coordinate axis, is biCartesian.*

Proof [38, 1.2.4.15]. □

We also note the following immediate consequences of Proposition A.4.

Proposition A.5. *Suppose we are given an n-cube q with one face f biCartesian. Then q is biCartesian if and only if the face parallel to f is also biCartesian.*

Proposition A.6. *The property for cubes being biCartesian satisfies the two-out-of-three property with respect to pasting of cubes.*

Appendix A.5. Limits and Kan extensions

Let $\mathcal{C}$ be an ∞-category. As usual, a (∞-) functor $F : I \to \mathcal{C}$, where I is a (small) ∞-category, will be called a *diagram* in $\mathcal{C}$. We will also use the notation $(F_i)_{i\in I}$ for such a diagram, with $F_i = F(i)$, $i \in \mathrm{Ob}(I)$. The ∞-categorical limit and colimit of $(F_i)_{i\in I}$ (when they exist) will be denoted by

$$\lim_{i\in I}{}^{\mathcal{C}} F_i, \qquad \operatorname{colim}_{i\in I}{}^{\mathcal{C}} F_i,$$

or, in the functor notation, simply $\lim F$, $\operatorname{colim} F$.

Let $\alpha : I \to J$ be a functor of small ∞-categories and $F : I \to \mathcal{C}$ be another functor. In this case, we can speak about the *left* and *right Kan extensions* which are functors

$$\alpha_! F, \alpha_* F : J \longrightarrow \mathcal{C}$$

characterized by universal properties. More precisely, the functors (when they exist)

$$\alpha_! : \mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(J, \mathcal{C}),\ \ F \mapsto \alpha_! F, \quad \alpha_* : \mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(J, \mathcal{C}),\ \ F \mapsto \alpha_* F$$

are, respectively, left and right adjoints to the pullback functor

$$\alpha^* : \mathrm{Fun}(J, \mathcal{C}) \longrightarrow \mathrm{Fun}(I, \mathcal{C}), \quad G \mapsto \alpha^* G := G \circ \alpha.$$

See [37] §4.3. While the general concept of adjunction in the ∞-categorical context is somewhat subtle (see [37] §5.2), it implies identifications (weak equivalences) of mapping spaces having the familiar shape, which in our case read

$$\mathrm{Map}_{\mathrm{Fun}(J,\mathcal{C})}(G, \alpha_* F) \simeq \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(\alpha^* G, F), \quad \mathrm{Map}(\alpha_! F, G) \simeq \mathrm{Map}(F, \alpha^* G) \tag{A.1}$$

for any $F \in \mathrm{Fun}(I, \mathcal{C})$, $G \in \mathrm{Fun}(J, \mathcal{C})$.

We recall the *pointwise formulas* for Kan extensions which describe their values on an object $j \in J$. More precisely, assuming the existence of all the relevant (co)limits, we have

$$(\alpha_! F)(j) = \operatorname{colim}{}^{\mathcal{C}}_{i\in\alpha/j}\ F(j), \qquad (\alpha_*(F)_j = \lim{}^{\mathcal{C}}_{i\in j/\alpha}\ F(i), \tag{A.2}$$

where α/j resp. j/α are the *overcategory* and *undercategory*, whose objects are pairs

$$\big(i \in \mathrm{Ob}(I), a : \alpha(i) \to j\big), \text{ resp. } \big(i \in \mathrm{Ob}(I), b : j \to \alpha(i)\big);$$

see [37] §1.2.9.

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Recall, further, that α is called ∞-*cofinal*, resp. ∞-*coinitial*, if any j/α, resp. α/j is contractible, (i.e., its nerve is a contractible simplicial set). If this is the case, then for any $G; J \to \mathcal{C}$ we have equivalences

$$\operatorname{colim} \alpha^* G \simeq \operatorname{colim} G, \quad \text{resp.} \quad \lim \alpha^* G \simeq \lim G$$

in $\mathcal{C}$.

The following result [37, 4.3.2.15] will be a fundamental tool for us to establish equivalences of ∞-categories.

Proposition A.3. *Let J be an ∞-category and $I \subset J$ a full subcategory. Let $\mathcal{C}$ be an ∞-category with colimits let $\mathcal{E} \subset \mathrm{Fun}(I, \mathcal{C})$ be a full subcategory and let $\mathcal{E}^!$ be the full subcategory in $\mathrm{Fun}(J, \mathcal{C})$ spanned by functors of the form $\alpha_! F$ for F in $\mathcal{E}$. Then the restriction functor $\mathcal{E}^! \to \mathcal{E}$ is an equivalence. The analogous statement holds for right Kan extensions if $\mathcal{C}$ has limits.*

We also note, for future use, the following fact [37, 4.4.4.10].

Lemma A.4. *Let K be a weakly contractible simplicial set, and let $\mathcal{C}$ be an ∞-category. Let*

$$F : K \to \mathcal{C}$$

be a diagram sending every edge of K to an equivalence in $\mathcal{C}$. Then a cone

$$F^+ : K^{\triangleright} \longrightarrow \mathcal{C}$$

is a colimit cone if and only if every edge from a vertex in K to the cone vertex $*$ is mapped to an equivalence in $\mathcal{C}$. In particular, for every vertex k of K, the value $F(k)$ is a colimit of F.

Appendix A.6. ∞-categorical localization

Let $\mathcal{C}$ be an ∞-category and $W \subset \mathrm{Mor}(\mathcal{C})$ be a set of 1-morphisms. For any ∞-category $\mathcal{E}$, we denote

$$\mathrm{Fun}(\mathcal{C}, \mathcal{E})_{[W \to \mathrm{Eq}]} \subset \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

the full ∞-subcategory spanned by (∞-)functors that take elements of W to equivalences in $\mathcal{E}$.

Definition A.1. Let $\pi : \mathcal{C} \to \mathcal{D}$ be an ∞-functor. We say that π *exhibits $\mathcal{D}$ an an ∞-categorical localization of $\mathcal{C}$ by W*, or, simply, that π is an ∞-*localization of $\mathcal{C}$ by W*, if:

(1) $\pi \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})_{[W \to \mathrm{Eq}]}$.
(2) For any ∞-category $\mathcal{E}$, composition with π gives an equivalence

$$\pi^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})_{[W \to \mathrm{Eq}]}.$$

Given $\mathcal{C}$ and W, the datum $(\mathcal{D}, \pi)$ as above is known to exist and be unique up to a contractible space of choices; see [38], §5.2.7. We will therefore denote such $\mathcal{D}$ by $\mathcal{C}[W^{-1}]_\infty$.

We will be particularly interested in the case when $\mathcal{C}$ is a usual category. In this case, $\mathcal{C}[W^{-1}]_\infty$ is the ∞-categorical analog of the Dwyer–Kan simplicial localization [16, 15]. In particular,

$$\mathrm{Ho}\ \mathcal{C}[W^{-1}]_\infty = \mathcal{C}[W^{-1}]$$

is the usual categorical localization of $\mathcal{C}$ by W.

We will be further interested in the cases when $\mathcal{C}[W^{-1}]_\infty$ is equivalent to a usual category, that is, reduces to $\mathcal{C}[W^{-1}]$.

Given a functor $\pi : \mathcal{C} \to \mathcal{D}$ of usual categories and an object $d \in \mathcal{D}$, we denote by

$$j_d : (d/\pi)^{\mathrm{iso}} \hookrightarrow d/\pi$$

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the embedding of the full subcategory spanned by pairs $(c \in \mathcal{C}, a : d \to \pi(c))$ for which a is an isomorphism in $\mathcal{D}$. Then we have (cf. [47]):

Proposition A.2. *Let $\pi : \mathcal{C} \to \mathcal{D}$ be a functor of usual categories and $W \subset \mathrm{Mor}(\mathcal{C})$. Suppose that:*

(1) *Elements of W are precisely the morphisms of $\mathcal{C}$ sent by π into isomorphisms.*
(2) *For any $d \in \mathcal{D}$, the category $(d/\pi)^{\mathrm{iso}}$ is contractible.*
(3) *For any $d \in \mathcal{D}$, the functor j_d is ∞-coinitial.*

Then π exhibits $\mathcal{D}$ as the ∞-categorical localization of $\mathcal{C}$ by W.

Proof. **Step 1.** Let $\mathcal{E}$ be any ∞-category with limits and $G : \mathcal{D} \to \mathcal{E}$ be an ∞-functor. Then the natural transformation $G \longrightarrow \pi_* \pi^* G$ is an equivalence. Indeed, by the pointwise formula for Kan extensions and ∞-coinitiality of j_d, we have

$$(\pi_* \pi^* G)(d) = \lim_{c \in d/\pi} G(\pi(c)) = \lim_{c \in (d/pi)^{\mathrm{iso}}} G(\pi(c)).$$

But $(d/\pi)^{\mathrm{iso}}$ consists of isomorphisms $d \longrightarrow \pi(c)$, so by inverting them, we can say that it consists of isomorphisms $\pi(c) \xrightarrow{\simeq} d$. So we have

$$(\pi_* \pi^* G)(d) = \lim_{\{\pi(c) \xrightarrow{\simeq} d\}} G(\pi(c)) = G(d)$$

since the last limit is taken over a cone-shaped diagram (one with an initial object).

Step 2. Further, let $F : \mathcal{C} \longrightarrow \mathcal{E}$ be any ∞-functor which takes elements of W into equivalences. Then the natural transformation $\pi^* \pi_* F \longrightarrow F$ is an equivalence. Indeed, as above, for any $c \in \mathcal{C}$,

$$(\pi^* \pi_* F)(c) = \lim_{c' \in \pi(c)/\pi} F(c') = \lim_{c' \in (\pi(c)/\pi)^{\mathrm{iso}}} F(c'). \tag{A.3}$$

But $(\pi(c)/\pi)^{\mathrm{iso}}$ has, as objects, isomorphisms $\pi(c) \xrightarrow{b} \pi(c')$, while a morphism

$$[\pi(c) \xrightarrow{b} \pi(c'_1)] \longrightarrow [\pi(c) \xrightarrow{b} \pi(c'_2)]$$

between two such objects is a morphism $u : c'_1 \longrightarrow c'_2$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} & \pi(c'_1) \arrow[d, "\pi(u)"] \\ \pi(c) \arrow[ur, "b_1"] \arrow[r, "b_2"'] & \pi(c'_2) \end{tikzcd}$$

commutes. This means that $\pi(u)$ is an isomorphism and so $u \in W$ by the assumption (1). Thus, $F(u)$ is an equivalence. So the limit in (A.3) is a limit of a diagram of equivalences parametrized by a category that is contractible by assumption (3). So (e.g., by Lemma A.4 for limits instead of colimits) it is identified with any term of the diagram, in particular, the natural map from this limit to $F(c)$ is an equivalence.

Step 3. Now, consider the pullback functor

$$\pi^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E}).$$

Step 1 implies that π^* is fully faithful (the embedding of a full ∞-subcategory). Step 2 means that the essential image of F is $\mathrm{Fun}(\mathcal{C}, \mathcal{E})_{[W \to \mathrm{Eq}]}$. This means that π satisfies the condition (2) of Definition A.1 for any $\mathcal{E}$ with limits.

Finally, we note that in the above reasoning it is not necessary to require that $\mathcal{E}$ has all limits as all the limits we need, automatically exist and are explicitly identified. This proves Proposition A.2. $\square$

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For future use, we note a dual version of Proposition A.2. For a functor $\pi : \mathcal{C} \to \mathcal{D}$ of usual categories and an object $d \in \mathcal{D}$, we consider the embedding

$$j^d : (\pi/d)^{\mathrm{iso}} \hookrightarrow \pi/d,$$

where $(\pi/d)^{\mathrm{iso}}$ is the full subcategory of π/d formed by pairs $(c \in \mathcal{C}, b : \pi(c) \to d)$ for which b is an isomorphism.

Proposition A.4. *Let $\pi : \mathcal{C} \to \mathcal{D}$ be a functor of usual categories. Suppose that:*

(1) *Elements of W are precisely the morphisms of $\mathcal{C}$ sent by π into isomorphisms.*
(2) *For any $d \in \mathcal{D}$ the category $(\pi/di)^{\mathrm{iso}}$ is contractible.*
(3) *For any $d \in \mathcal{D}$ the functor j^d is ∞-cofinal.*

Then:

(a) *π exhibits $\mathcal{D}$ as the ∞-categorical localization of $\mathcal{C}$ by W.*
(b) *For any ∞-category $\mathcal{E}$ with colimits and any functor $G : \mathcal{D} \to \mathcal{E}$, the natural transformation $\pi_! \pi^* G \to \mathcal{G}$ is an equivalence.*
(c) *For any functor $F : \mathcal{C} \to \mathcal{E}$ sending elements of W to equivalences, the natural transformation $F \to \pi^* \pi_! F$ is an equivalence.*

Proof. Obtained from that of Proposition A.2 by dualization. □

Appendix A.7. A covering lemma

We recall the following lemma which generalizes various classical statements of the kind that a space is homotopy equivalent to the nerve of its sufficiently fine open covering.

Lemma A.1. *Let T be a small category. Let E be a topological space, and let $\mathfrak{O}(E)$ denote the poset of open subsets of E. Let*

$$\chi : T \longrightarrow \mathfrak{O}(E))$$

be a functor. For any $e \in E$, let $\chi^{-1}(e) \subset T$ be the full subcategory spanned by t such that $e \in \chi(t)$. Suppose that:

(1) *for every $t \in T$, the open set $\chi(t)$ is contractible,*
(2) *for every $e \in E$, the category $\chi^{-1}(e)$ is contractible.*

Then there is a weak homotopy equivalence $|\,\mathrm{N}(T)| \simeq E$.

Note that the assumption (2) implies, in particular, that the $\chi(t)$ form an open covering of E, as a contractible category is nonempty.

Proof. Let $\pi : K \to \mathrm{N}(T)$ be the relative nerve ([37, 3.2.5]) associated to the functor

$$\mathrm{N}(T) \to \mathrm{Sp},\ t \mapsto \mathrm{Sing}(\chi(t)).$$

Then π is a left fibration whose fibers are, by assumption, contractible Kan complexes. By [37, 2.1.3.4], it is a trivial Kan fibration so that $|K| \simeq |N(T)|$. On the other hand, $\mathrm{Sing}(|K|)$ is a model for the colimit of π ([37, 3.3.4.6]) which by Lurie's Seifert-van Kampen theorem [38, A.3.1] is weakly equivalent to $\mathrm{Sing}(E)$. □

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