Bull. Austral. Math. Soc. 20E34, 20F22
Vol. 40 (1989) [243–254]

PERMUTABLE WORD PRODUCTS IN GROUPS

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Dedicated to Professor B.H. Neumann on his eightieth birthday

Let $u(x_1, \ldots, x_n) = x_{11} \ldots x_{1m}$ be a word in the alphabet $x_1, \ldots, x_n$ such that $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. If $(H_1, \ldots, H_n)$ is an n-tuple of subgroups of a group G then denote by $u(H_1, \ldots, H_n)$ the set $\{u(h_1, \ldots, h_n) \mid h_i \in H_i\}$. If $\sigma \in S_n$ then denote by $u_\sigma(H_1, \ldots, H_n)$ the set $u\big(H_{\sigma(1)}, \ldots, H_{\sigma(n)}\big)$. We study groups G with the property that for each n-tuple $(H_1, \ldots, H_n)$ of subgroups of G, there is some $\sigma \in S_n$, $\sigma \neq 1$ such that $u(H_1, \ldots, H_n) = u_\sigma(H_1, \ldots, H_n)$. If G is a finitely generated soluble group then G has this property for some word u if and only if G is nilpotent-by-finite. In the paper we also look at some specific words u and study the properties of the associated groups.

1. Introduction

Let n be a fixed positive integer, $X = \{x_1, \ldots, x_n\}$ a set of n symbols and $F = F(X)$ the free group on X. Let $U = \{u_1, u_2, \ldots\}$ and $V = \{v_1, v_2, \ldots\}$ be non-empty sets of elements in F. Define the class $P(U, V)$ to consist of groups G such that given an n-tuple $(g_1, \ldots, g_n)$ of elements in G, $u(g_1, \ldots, g_n) = v(g_1, \ldots, g_n)$ for some $u \in U$ and some $v \in V$, $v \neq u$. Some examples:

1.1. Let $U = \{u\}$ where $u(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n$ and $V = \{u_\sigma \mid \sigma \in S_n \backslash 1\}$ where S_n is the symmetric group of degree n and

$$u_\sigma(x_1, \ldots, x_n) = u\big(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big) = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}.$$

Then every group in $P(U, V)$ is finite-by-abelian-by-finite. Conversely every finite-by-abelian-by-finite group is in $P(U, V)$ for some suitable n. This was shown by Curzio, Longobardi, Maj and Robinson in [**2**]. These groups are more commonly referred to as P_n-groups.

1.2. Let $u = u(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n$, $U = \{u_\sigma \mid \sigma \in S_n\}$ and $V = U$. Then $P(U, V)$-groups, more commonly referred to as Q_n-groups or rewritable groups, are again finite-by-abelian-by-finite groups as shown by Blyth in [**1**]. That an abelian-by finite group is in Q_n for some n is implicit in Theorem 1 of Kaplansky [**4**].

Received 28 March 1989
Partial support by NSERC is gratefully acknowledged by the second author.

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https://doi.org/10.1017/S0004972700004354 Published online by Cambridge University Press

1.3. If we take $n = 2$, $u = u(x_1, x_2) = (x_1x_2)^r$, $v = v(x_1, x_2) = (x_2x_1)^r$ where $r > 0$ is fixed and $U = \{u\}$, $V = \{v\}$ then $G \in P(U,V)$ if and only if $G/Z(G)$ is of exponent r. This is not difficult to verify.

In general the classes $P(U,V)$ may be viewed as generalising varieties and, except for some specific sets U and V, it is very difficult to describe them. We now turn to related classes of groups.

Let $n > 0$ be fixed, $X = \{x_1, \ldots, x_n\}$ be a set of idempotent variables and $S = S(X)$ the free semigroup generated by X. Thus for any $u \in S$, $u = u(x_1, \ldots, x_n) = x_{11}x_{12}\ldots x_{1m}$ where $x_{1i} \in X$ and $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. If $(H_1, \ldots H_n)$ is an n-tuple of subgroups of a group G then denote by $u(H_1, \ldots, H_n)$ the set $\{u(h_1, \ldots, h_n) \mid h_i \in H_i\}$. Thus if $u(x_1, \ldots, x_n)$ is as above then

$$u(H_1, \ldots, H_n) = H_{11}H_{12}\ldots H_{1m}.$$

If U and V are sets of elements in S then define the class $SP(U,V)$ to consist of groups G such that for any n-tuple $(H_1, \ldots, H_n)$ of subgroups of G, $u(H_1, \ldots, H_n) = v(H_1, \ldots, H_n)$ for some $u \in U$ and some $v \in V$, $v \neq u$. Some examples:

1.4. Let $U = \{u\}$ where $u(x_1, \ldots, x_n) = x_1x_2\ldots x_n$ and $V = \{u_\sigma \mid \sigma \in S_n \backslash 1\}$. As in 1.1, $u_\sigma(x_1, \ldots, x_n) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Finitely generated soluble $SP(U,V)$ groups are finite-by-abelian. Conversely every finite-by-abelian group is an $SP(U,V)$ group for some integer n. These results are contained in [**6**]. From [**5**] we know that periodic $SP(U,V)$-groups are locally finite. The structure of $SP(U,V)$-groups in general seems difficult to determine.

1.5. For each positive integer r let $u_r = u_r(x,y) = (xy)^r$, and $v_r = v_r(x,y) = (yx)^r$. Let $U = \{u_r,\ r = 1, 2, \ldots\}$ and $V = \{v_r,\ r = 1, 2, \ldots\}$. Then the class $SP(U,V)$ is precisely the class of groups in which every subgroup is elliptically embedded. Groups with this property are considered in [**7, 8**]. It is known that a finitely generated soluble group G is in this class if and only if it is finite-by-nilpotent. The same is true if we replace "soluble group" by "residually finite p-group" in the above statement.

In this paper we shall not look at $P(U,V)$-groups, but concentrate our attention on $SP(U,V)$-groups. The two main results are as follows:

THEOREM 1. *Let G be a finitely generated soluble group, $U = \{u\}$ where $u = u(x_1, \ldots, x_n) = (x_1\ldots x_n)^r$ and $V = \{u_\sigma \mid \sigma \in S_n\}$ where $u_\sigma(x_1, \ldots, x_n) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then G is an $SP(U,V)$ group for some $n > 1$, $r > 0$ if and only if G is finite-by-nilpotent.*

THEOREM 2. *Let $U = \{u\}$ where u is a word in idempotent variables $x_1, \ldots, x_n$, $n > 1$; and let $V = \{u_\sigma \mid \sigma \in S_n\}$. If G is a finitely generated soluble group in $SP(U,V)$, then G is nilpotent-by-finite.*

https://doi.org/10.1017/S0004972700004354 Published online by Cambridge University Press

We can not replace "nilpotent-by-finite" in Theorem 2 by the stronger condition "finite-by-nilpotent" of Theorem 1. It is tedious, but we will show that the infinite dihedral group D_∞ lies in $SP(U,V)$ where $U = \{u\}$, $V = \{u_\sigma \mid \sigma \in S_n\}$ and $u = u(x_1,x_2,x_3,x_4) = x_1x_4x_2x_3x_2x_3x_4x_1$ and it is well-known that D_∞ is not finite-by-nilpotent.

At present little is known about the various classes $SP(U,V)$ and $P(U,V)$. The following questions stand a good chance of getting answered, at least partially, in the not too distant future!

QUESTION 1: Let U and V be finite sets of words in $S = S(X)$ where $X = \{x_1,\ldots,x_n\}$ and let G be a finitely generated soluble group in the class $SP(U,V)$. Is G nilpotent-by-finite?

QUESTION 2: For which words u in $S = S(X)$, $X = \{x_1,\ldots,x_n\}$ are the periodic $SP(U,V)$ groups locally finite; where $U = \{u\}$ and $V = \{u_\sigma,\ \sigma \in S_n\}$.

QUESTION 3: Let G be a finitely generated residually finite p-group, p a prime. For which sets U,V of words in $S = S(X)$, $X = \{x_1,\ldots,x_1\}$; would $G \in SP(U,V)$ imply G is nilpotent-by-finite?

Finally we ask if the classes $SP(U,V)$ can contain finitely generated infinite simple groups.

2. Proofs

Since the hypothesis of Theorem 1 is a more restricted form of the hypothesis of Theorem 2, it would be proper to prove Theorem 2 and then establish the extra property required in Theorem 1. The reduction from soluble to nilpotent-by-finite will be achieved using two intermediate steps; these are dealt with in the following lemmas.

LEMMA 2.1. *The wreath product of a cyclic group of order p with the infinite cyclic group is not in the class $SP(U,V)$ where U,V are as in the statement of Theorem 2.*

LEMMA 2.2. *Let $G = \langle A,t\rangle$ where A is a torsion-free abelian group of finite rank on which $\langle t\rangle$ acts rationally irreducibly. If $G \in SP(U,V)$ where U,V are as in the statement of Theorem 2, then for some positive integer k, $\langle t^k\rangle$ acts trivially on A.*

LEMMA 2.3. *If $G = \langle A,t\rangle$, where $A \trianglelefteq G$ and is abelian of finite rank, and $G \in SP(U,V)$ where U,V are as in the statement of Theorem 2, then for some $\ell > 0$, $\langle A,t^\ell\rangle$ has a non-trivial centre.*

LEMMA 2.4. *Let $G = \langle A,t\rangle$ where A is a torsion-free abelian group of finite rank on which $\langle t\rangle$ acts rationally irreducibly. If $G \in SP(U,V)$ where U,V are as in the statement of Theorem 1, then $\langle t\rangle$ acts trivially on A.*

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Proof of Theorem 2: By hypothesis, $X = \{x_1, \ldots, x_n\}$ $u = u(X) = x_{11}x_{12}\ldots x_{1m}$ where $x_{1i} \in X$ for all $i = 1, \ldots, m$ and $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. Let G be a finitely generated soluble group such that for any n-tuple $(H_1, \ldots, H_n)$ of subgroups of G, there is a permutation $\sigma \neq 1$ in S_n such that

$$u(H_1, \ldots, H_n) = H_{11}H_{12}\ldots H_{1m} = u\big(H_{\sigma(1)}, \ldots, H_{\sigma(n)}\big) = H_{\sigma(11)}\ldots H_{\sigma(1m)}.$$

We need to show that G is nilpotent-by-finite, and we proceed by induction on the solubility length of G. If G is abelian then there is nothing to prove. Let G be soluble of length d and assume that the result holds for soluble groups of smaller length. Since the class $SP(U,V)$ is subgroup and quotient closed, we may suppose that G has a normal abelian subgroup A such that G/A is nilpotent-by-finite. In particular G is abelian by polycyclic. If G does not have finite rank then it has a section isomorphic to the wreath product of a cyclic group of prime order p and the infinite cyclic group. This is not possible by Lemma 2.1. Hence we conclude that G has finite rank.

As G is finitely generated abelian-by-polycyclic, it satisfies the maximal condition for normal subgroups. If G is not nilpotent-by-finite, then let B be a maximal normal subgroup of G such that G/B is not nilpotent-by-finite. Now we replace G by G/B and hence assume that every proper quotient of G is nilpotent-by-finite.

Let T be the torsion subgroup of A. Then T has finite rank and is of bounded exponent since G satisfies the maximal condition for normal subgroups. Thus T is finite, and $C = C_G(T)$, the centraliser of T in G, is of finite index in G. If $T \neq 1$ then G/T is nilpotent-by-finite and hence C/T is nilpotent-by-finite. Since $T \leqslant Z(C)$, the centre of C, then C and hence G would be nilpotent-by-finite. Thus we assume $T = 1$ and hence A is torsion-free, and by passing to a suitable subgroup of finite index in G, if necessary, we may assume further that G/A is a finitely generated torsion-free nilpotent group. Thus there exists a finite set $T = \{t_1, \ldots, t_r\}$ of elements in G such that $G = \langle A, T\rangle$ and

$$A = G_0 \leqslant \langle G_0, t_1\rangle = G_1 \leqslant \ldots \leqslant \langle G_{r-1}, t_r\rangle = G_r = G$$

is a central series from A to G with torsion-free factors.

If $r = 1$ then $G = \langle A, t_1\rangle$. By Lemma 2.3 $Z\Big(\langle A, t_1^{\ell_1}\rangle\Big) \neq 1$ for some $\ell_1 > 0$ and hence $D = A \cap Z\Big(\langle A, t_1^{\ell_1}\rangle\Big)$ is a non-trivial normal subgroup of G. By our choice of G, G/D is nilpotent-by-finite and hence G is nilpotent-by-finite.

Now suppose we have established the result for the case $r < d$ and suppose $r = d$. Then G_{d-1} is nilpotent-by-finite and $G = \langle G_{d-1}, t_d\rangle$. Let $H = \langle A, G_{d-1}^{\ell}\rangle$ for some suitable $\ell > 0$ so that H is nilpotent. Let $Y = A \cap Z(H)$ then Y is normal in $\langle H, t_d\rangle$ which is of finite index in G. Moreover $Z\Big(\langle Y, t_d^{\ell_1}\rangle\Big) \neq 1$ for some $\ell_1 > 0$ by

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Lemma 2.3, so that $D_1 = Y \cap Z\Big(\langle Y, t_d^{\ell_1}\rangle\Big)$ is a non-trivial subgroup of G contained in the centre of $\langle H, t_d^{\ell_1}\rangle$ which is of finite index in G. We may replace $\langle H, t_d^{\ell_1}\rangle$ by its normal interior in G, if necessary; it still contains A and hence D_1. Now $\langle H, t_d^{\ell_1}\rangle / D_1$ is nilpotent-by-finite, $D_1 \leqslant Z\Big(\langle H, t_d^{\ell_1}\rangle\Big)$ and $\langle H, t_d^{\ell_1}\rangle$ is of finite index in G. Thus G is nilpotent-by-finite, as required. ∎

Proof of Theorem 1: Since the hypotheses of Theorem 2 are satisfied by the group of Theorem 1, we may assume G to be finitely generated nilpotent-by-finite. Let T be the maximal finite normal subgroup of G. Since we wish to show that G is finite-by-nilpotent we may look at G/T, if necessary, and hence assume that G has no non-trivial finite normal subgroup. Let F be the Fitting subgroup of G. If $F \neq G$ then pick any $t \in G \backslash F$ such that $t^p \in F$. Clearly it is sufficient to show that $\langle F, t\rangle$ is nilpotent for G/F is finite and soluble, we can reach G from F by a subnormal series with factors of prime order. Thus we assume $G = \langle F, t\rangle$, $t^p \in F$ and F is torsion-free.

Let H be the hypercentre of G. Then $H \cap F$ is isolated in F. This may be seen by first checking it for $Z(G) \cap F$ and then by taking the quotient of G by this subgroup, and using induction. Observe that if $H \nleqslant F$ then $G = HF$ and G is nilpotent. So assume $H \leqslant F$. Next we look at G/H. If G/H is nilpotent then so is G. So we assume $H = 1$. Let A be a non-trivial normal subgroup of G of least Hirsch length and $A \leqslant Z(F)$. Since $\langle A, t\rangle \in SP(U, V)$, $\langle t\rangle$ acts trivially on A by Lemma 2.4. Thus $A \leqslant Z(G)$ contradicting the assumption that $Z(G) = 1$. This concludes the proof that if $G \in SP(U, V)$ then G is finite-by-nilpotent.

Now suppose that G is a finitely generated finite-by-nilpotent group. For any subgroup L of G let $\gamma(L)$ denote the nilpotent residual of L. Thus $\gamma(L)$ is the intersection of the terms of the lower central series of L. Let $F = \gamma(G)$. It is finite by hypothesis and G/F is nilpotent of class c_1 for some $c_1 > 0$. Thus $\gamma(L) = \gamma_c(L)$ for all $L \leqslant G$ where $c = |F| + c_1$. We show, by induction on $|\gamma\langle H, K\rangle| = s$, that $(HK)^{d_s} = (KH)^{d_s} = \langle H, K\rangle$ for all subgroups H, K of G where $d_1 = (4r)^c$, $r =$ rank of G; $d_i = d_{i-1} + 2i(i + d_1)$, $i > 1$. In particular $(HK)^d = (KH)^d = \langle H, K\rangle$ for all H, K where $d = d_f$, and $f = |F|$.

By Proposition 2 of [7], $\Gamma(HK)^t = \Gamma(KH)^t = \langle K, H\rangle$ where $\Gamma = \gamma\langle K, H\rangle$, $t = (4r)^c$, and r is the rank of G. Thus if $\Gamma = 1$ then $d_1 = t$ will suffice.

For any $a \in (HK)^t$, $a = gb$ for some $g \in \Gamma$ and $b \in (KH)^t$ so that $ab^{-1} = g \in \Gamma \cap (HK)^{2t}$. If $\Gamma \cap (KH)^{2t} = 1$, then $a = b$ and $(HK)^t = (KH)^t$. This implies $\langle H, K\rangle = (HK)^t$, and again $d_1 = t$ suffices.

If $\Gamma_1 = \Gamma \cap (HK)^{2t} \neq 1$, then for each integer $m \geqslant 1$ let $\Gamma_{m+1} = \Gamma_m \cup \Gamma_m^{HK}$ so that $\Gamma_m \subseteq (HK)^{2t+2m}$. Observe that $\Gamma_m = \Gamma_{m+1}$ implies $\langle \Gamma_m\rangle = \langle \Gamma_m^H\rangle = \langle \Gamma_m^K\rangle$. Since $\Gamma_m \subseteq \Gamma$ and $|\Gamma| = s$, $\Gamma_s = \Gamma_{s+1}$. Also note that $\langle \Gamma_m\rangle \subseteq \Gamma_m^s$. Thus the normal closure

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N of Γ_1 in $\langle H,K\rangle$ lies in $(HK)^{\lambda}$ where $\lambda = \lambda_s = 2(s^2+ts)$.

Now NH and NK both lie in $(HK)^{\lambda}$ and $(NHNK)^m \subseteq (HK)^{\lambda+m}$ for all $m > 0$. Rank of $\langle H,K\rangle/N$ is no greater than r, $\gamma(\langle H,K\rangle/N) = \gamma_c(\langle H,K\rangle/N)$ and $|\gamma(\langle H,K\rangle/N)| < |\gamma(\langle H,K\rangle)|$. Thus by the induction hypothesis, $N(HK)^{d'} = N(KH)^{d'} = \langle H,K\rangle$ where $d' = t+\lambda_2+\cdots+\lambda_{s-1} = d_{s-1}$. Since $(HK)^{\lambda} \geqslant N$, we obtain $(HK)^{d_s} = (KH)^{d_s} = \langle H,K\rangle$ where $d_s = d_{s-1}+\lambda_s$.

Now that we have shown that for a finitely generated finite-by-nilpotent group G there is an integer d such that $(HK)^d = (KH)^d$ for all subgroups H,K of G, we let $u = u(x,y) = (xy)^d$, $v = v(x,y) = (yx)^d$ then $G \in SP(u,v)$. This completes the proof of the second part of the theorem. ∎

Proof of Lemma 2.1: Let G be the wreath product of a cyclic group of order p and an infinite cyclic group $\langle t\rangle$. Then we can identify each element of G by a pair $(f(t),\ t^{\alpha})$ where $f(t) \in \mathbb{F}_p\langle t\rangle$, the additive group of the group ring of the infinite cyclic group $\langle t\rangle$ over the field $\mathbb{F}_p$ of p elements, and $\alpha \in \mathbb{Z}$. The product of two such elements is then given by the rule: $\big(f(t),\ t^{\alpha}\big)\big(g(t), t^{\beta}\big) = \big(f(t)+t^{\alpha}\cdot g(t),\ t^{\alpha+\beta}\big)$. The elements of the base group correspond to those pairs where $\alpha = 0$ and the elements of the top group correspond to those pairs where $f(t) = 0$.

We are given $X = \{x_1,\dots,x_n\}$, $u = u(X) = x_{11}x_{12}\dots x_{1m}$; $x_{1i} \in X$, $x_{1i} \neq x_{1i+1}$, $i = 1,\dots,m-1$ and we are required to show that there exist subgroups $H_1,\dots,H_n$ of G such that $H_{11}H_{12}\dots H_{1m} \neq H_{\phi(11)}H_{\phi(12)}\cdots H_{\phi(1m)}$ for any $\phi \neq 1$ in S_n.

Take $H_i = \langle h_i\rangle$ where $h_i = ((1-t^{\alpha_i})f_i, t^{\alpha_i})$; $f_i = f_i(t)$ and α_i are to be chosen appropriately. Note that $h_i^k = \big((1-t^{k\alpha_i})f_i,\ t^{k\alpha_i}\big)$, and a general element of $u(H_1,\dots,H_n)$ is $h_{11}^{k_1}\dots h_{1m}^{k_m} =$

$$\big((1-t^{k_1\alpha_{11}})f_{11}, t^{k_1\alpha_{11}}\big)\dots\big((1-t^{k_m\alpha_{1m}})f_{1m}, t^{k_m\alpha_{1m}}\big)$$
$$\big((1-t^{k_1\alpha_{11}})f_{11}+t^{\lambda_1}(1-t^{k_2\alpha_{12}})f_{12}+\cdots+t^{\lambda_{m-1}}(1-t^{k_m\alpha_{1m}})f_{1m},\ t^{\lambda_m}\big)$$

where $\lambda_i = k_1\alpha_{11}+\cdots+k_i\alpha_{1i}$, $\quad i = 1,\dots,m$. Partition the set $\{1,\dots,m\}$ as the union $S_1\cup\cdots\cup S_n$ where $S_i = \{j \mid x_{ij} = x_i\}$. Then a general element of $u(H_1,\dots,H_n)$ is of the form $\big(\sum_{i=1}^{n}(f_i\sum_{j\in S_i}(t^{\lambda_{j-1}}-t^{\lambda_j})), t^{\lambda_m}\big)$ with the understanding that $\lambda_0 = 0$. Likewise $u(H_{\sigma(1)},\dots,H_{\sigma(n)})$ consists of elements of the form $\Big(\sum_{i=1}^{n}\big(f_{\phi(i)}\sum_{j\in S_i}(t^{\mu_{j-1}}-t^{\mu_j})\big),\ t^{\mu_m}\Big)$ where $\mu_i = \ell_1\alpha_{\phi(11)}+\cdots+\ell_i\alpha_{\phi(1i)}$, $i = 1,\dots,m$ and $\mu_0 = 0$. If σ denotes the inverse

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of ϕ, then we may write these elements as $\left(\sum_{i=1}^{n}\left(f_i \sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})\right),\ t^{\mu_m}\right)$.

Now
$$(\sum_{i=1}^{n}(f_i \sum_{j\in S_i}(t^{\lambda_{j-1}} - t^{\lambda_j})), t^{\lambda_m}) = (\sum_{i=1}^{n}(f_i \sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})),\ t^{\mu_m})$$

implies $\lambda_m = \mu_m$ and

$$(1) \qquad \sum_{i=1}^{n} f_i(\sum_{j\in S_i}(t^{\lambda_{j-1}} - t^{\lambda_j})) = \sum_{i=1}^{n} f_i(\sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})).$$

Let $p_1,\dots,p_n$ be distinct primes, each greater than m. Put $p = p_1\dots p_n$, $k_i = 1$ and $\alpha_i = p/p_i$, $i = 1,\dots,n$. Then for each $i > 0$, $\lambda_i = \alpha_{11} + \dots + \alpha_{1i}$. Let $f_i = t^{p^i}$. Note that $\lambda_j < p$ for all j and they are all distinct. Also note that $f_i t^{\lambda_j} = f_{i'} t^{\mu_{j'}}$ implies $p^i + \lambda_j = p^{i'} + \mu_{j'}$. Hence $\mu_{j'} \equiv \lambda_j \mod p$ so that $\mu_{j'} \not\equiv \lambda_i \mod p$ for any $i \neq j$. Thus each λ_j is congruent modulo p to precisely one $\mu_{j'}$.

Now $1 \in S_{\sigma(k)}$ for some k. Thus $f_k(t^{\mu_0} - t^{\mu_1})$ is a term on the right hand side of (1). Since $\mu_0 = 0$ and the only λ_i equal to zero is λ_0, $f_k(t^{\lambda_0} - t^{\lambda_1})$ appears on the left hand side of (1). In particular $1 \in S_k$. Since $S_1,\dots,S_n$ partition the set $\{1,\dots,m\}$ and $1 \in S_k \cap S_{\sigma(k)}$, it follows that $\sigma(k) = k$. Hence μ_1 and λ_1 are both congruent to zero $\mod p/p_k$; it follows that $\mu_1 = \lambda_1$.

Suppose, by way of induction, that we have established that $\mu_j = \lambda_j$ for all $j < e$. Then $\mu_e - \mu_{e-1} = \ell_e \alpha_{\phi(1e)}$ which is congruent to zero mod all primes p_i except possibly one namely. $p_{\phi(1e)}$. Now we look at $\{\lambda_j - \lambda_{e-1},\quad j = e,\dots,m\}$. $\lambda_e - \lambda_{e-1} = \alpha_{1e}$ is congruent to zero mod all primes p_i, $p_i \neq p_{1e}$. For each of the other $\lambda_j - \lambda_{e-1}$, we can find at least two primes amongst $\{p_1,\dots,p_n\}$ such that $\lambda_j - \lambda_{e-1}$ is not congruent to zero mod either of them. Hence $0 \not\equiv \alpha_{1e} \equiv \ell_e \alpha_{\phi(1e)} \mod p_{1e}$. But $\alpha_{\phi(1e)} \neq \alpha_{1e}$ implies $\alpha_{\phi(1e)} \equiv 0 \mod p_{1e}$. Thus $\alpha_{\phi(1e)} = \alpha_{1e}$, and $x_{1e} = x_{\phi(1e)} = x_{k'}$, say. Thus $\sigma(k') = k'$ and $e \in S_{k'}$. Thus $f_{k'}(t^{\mu_{e-1}} - t^{\mu_e}) = f_{k'}(t^{\lambda_{e-1}} - t^{\lambda_e})$ and $\lambda_e = \mu_e$.

It is now clear that $\phi(j) = j$ for all $j = 1,\dots,n$ and hence ϕ is the identity permutation of the set $\{1,\dots,n\}$ as required. ∎

Proof of Lemma 2.2: We are given a group $G = \langle A, t\rangle$ where A is torsion-free abelian of finite rank on which $\langle t\rangle$ acts rationally irreducibly. Let us assume, if possible, that $[A,t] \neq 1$. Then $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}\langle t\rangle$- module and by Schur's Lemma, the centraliser ring $\Gamma = \mathrm{End}_{\mathbb{Q}\langle t\rangle} V$ is a division ring of finite dimension over $\mathbb{Q}$. The image of $\langle t\rangle$ in $\mathrm{End}_{\mathbb{Q}} V$ clearly lies in and spans Γ so that Γ is an algebraic number field. Moreover, regarded as a Γ-space, V is one-dimensional. Thus we may consider A to be an additive subgroup of $\mathbb{Q}(\tau)$ for some algebraic number τ and the action of conjugation by t as multiplication by τ.

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Let $h_i = b_i(1-\tau^{\alpha_i})t^{\alpha_i}$ for suitable integer α_i and $b_i(1-\tau^{\alpha_i}) \in A$. Let $H_i = \langle h_i \rangle$. Note that $h_i^k = b_i(1-\tau^{k\alpha_i})t^{k\alpha_i}$.

As in Lemma 2.1, we are given $u = u(x_1, \dots, x_n) = x_{11}x_{12}\dots x_{1m}$; $x_{1i} \in \{x_1, \dots, x_n\}, x_{1i} \neq x_{1i+1}$, $i = 1, \dots, m-1$; and we need to show that with proper choice of b_i and α_i, subgroups $H_1, \dots, H_n$ can be found such that $H_{11}\dots H_{1m} \neq H_{\phi(11)}\dots H_{\phi(1m)}$ for any $\phi \neq 1$ in S_n. Now

$$\begin{aligned} h_{11}^{k_1}\dots h_{1m}^{k_m} &= b_{11}(1-\tau^{k_1\alpha_{11}})t^{k_1\alpha_{11}}\dots b_{1m}(1-\tau^{k_m\alpha_{1m}})t^{k_m\alpha_{1m}} \\ &= b_{11}(1-\tau^{\lambda_1}) + b_{12}(\tau^{\lambda_1}-\tau^{\lambda_2}) + \dots + b_{1m}(\tau^{\lambda_{m-1}}-\tau^{\lambda_m})t^{\lambda_m} \end{aligned}$$

where $\lambda_i = k_1\alpha_{11} + \dots + k_i\alpha_{1i}, \quad i = 1, \dots, m.$

We shall put $\lambda_0 = 0$ and write $1 = \tau^0 = \tau^{\lambda_0}$.

Thus a general element of $u(H_1, \dots, H_n)$ has the form

$$\sum_{i=1}^{n}\Big(b_i\sum_{j\in S_i}(\tau^{\lambda_{j-1}}-\tau^{\lambda_j})\Big)t^{\lambda_m}$$

where $S_i = \{j;\ x_{1j} = x_i\}$ so that $\{1, \dots, m\}$ is the disjoint union of $S_1, \dots, S_n$. Likewise the general element of $u(H_{\phi(1)}\dots H_{\phi(n)})$ has the form

$$\sum_{i=1}^{n}\Big(b_i\sum_{j\in S_{\sigma(i)}}(\tau^{\mu_{j-1}}-\tau^{\mu_j})\Big)t^{\mu_m}$$

where $\mu_i = \ell_1\alpha_{\phi(1)} + \dots + \ell_i\alpha_{\phi(1i)}, \quad i = 1, \dots, m$; $\mu_0 = 0$ and $\sigma = \phi^{-1}$. This is shown in the same way as in the proof of Lemma 2.1. In particular $\mu_m = \lambda_m$ and

$$(2) \qquad \sum_{i=1}^{n} b_i\Big(\sum_{j\in S_i}(\tau^{\lambda_{j-1}}-\tau^{\lambda_j}) - \sum_{j\in S_{\sigma(i)}}(\tau^{\mu_{j-1}}-\tau^{\mu_j})\Big) = 0.$$

Now we return to pick b_i and α_i appropriately. For each integer $r > 1$, pick primes $p_{r1}, \dots, p_{rn}$ to satisfy $2^r < p_{r1}$ and $p_{ri}^2 < p_{ri+1}$, $i = 1, \dots, n-1$. Put $q_r = p_{r1}\dots p_{rn}$, $b_{ri}(y) = y^{q_r^i}$ and $\alpha_{ri} = q_r/p_{ri}$, $i = 1, \dots, n$. To make the notation simpler, we shall write b_i for b_{ri} and α_i for α_{ri}, where there is no ambiguity. Since there are infinitely many choices of q_r and each choice of q_r determines the sequence $H_1, \dots, H_n$ of subgroups which in turn corresponds to some permutation $\phi \neq 1$ such that $u(H_1, \dots, H_n) = u(H_{\phi(1)}, \dots, H_{\phi(n)})$, there is an infinite number of choices of r such that q_r correspond to the same permutation ϕ.

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If, for some value of r, we have the following stronger version of (2):

$$\sum_{i=1}^{n} b_i(y)L_i(y) - \sum_{i=1}^{n} b_i(y)M_i(y) = 0$$

where

$$L_i(y) = \sum_{j\in S_i} \left(y^{\lambda_{j-1}} - y^{\lambda_j}\right), \quad M_i(y) = \sum_{j\in S_{\sigma(i)}} \left(y^{\mu_{j-1}} - y^{\mu_j}\right)$$

and y is an indeterminant; then $\mu_j = \lambda_j$ for all j and $\phi = 1$. This is seen using arguments similar to those in the proof of Lemma 2.1. Thus we may suppose that for every r,

$$P(y) = \sum_{i=1}^{n} b_i(y)L_i(y) - \sum_{i=1}^{n} b_i(y)M_i(y)$$

is not zero but $P(\tau) = 0$. If $Q(y)$ is any non-trivial segment of $P(y)$ such that $Q(\tau) = 0$, then $Q'(y) = P(y) - Q(y)$ is a segment of $P(y)$ with $Q'(\tau) = 0$. Moreover one or both of $Q(y)$ or $Q'(y)$ contains at least as many monomials from $\sum_{i=1}^{n} b_i(y)L_i(y)$ as from $\sum_{i=1}^{n} b_i(y)M_i(y)$. Let $Q(y)$ be such a segment of $P(y)$ of shortest length. Thus

I. $Q(y) \neq 0$

II. $Q(\tau) = 0$

III. $Q(y)$ contains at least as many monomials from $\sum_{i=1}^{n} b_i(y)L_i(y)$ as from $\sum_{i=1}^{n} b_i(y)M_i(y)$ and

IV. No proper segment has properties I and II.

We write $Q(y) = Q_1(y) - Q_2(y)$ where $Q_1(y)$ is a segment $\sum_{i=1}^{u} \pm y^{\lambda_i'}$ of $\sum_{i=1}^{n} b_i(y)L_i(y)$, $Q_2(y)$ a segment $\sum_{i=1}^{v} \pm y^{\mu_i'}$ of $\sum_{i=1}^{n} b_i(y)M_i(y)$ and we may suppose that there is no term in $Q_1(y)$ equal to any term in $Q_2(y)$. If $Q_1(y)$ has only one term in it, then $Q(y) = \pm y^{\lambda}$ or $\pm y^{\lambda} \pm y^{\mu}$ where $0 \neq \lambda$ and $\mu \neq \lambda$. In both cases $Q(\tau) = 0$ implies τ is a root of unity and $[t^k, A] = 1$ for some $k > 0$, as required.

We may therefore assume that $Q_1(y) = \sum_{i=1}^{u} \pm y^{\lambda_i'}$ has more than one term; $\lambda_i' = q^{i'} + \lambda_{j_i}$ and $0 \leqslant \lambda_1' < \cdots < \lambda_u'$. Similarly $Q_2(y) = \sum_{i=1}^{\nu} \pm y^{\mu_i'}$ where $\mu_1' \leqslant \ldots \leqslant \mu_v'$. Let

$$\nu_1 = \min\{\lambda_1', \mu_1'\} \quad \text{and} \quad \nu_2 = \max\{\lambda_u', \mu_v'\}.$$

Then $y^{-\nu_1} \cdot Q(y)$ is a polynomial of degree $\nu_2 - \nu_1$ with non-zero constant term. Moreover $\nu_2 - \nu_1 \geqslant \lambda_u' - \lambda_1' > 2^r$. Thus the degree of the polynomial increases with r,

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and hence there are infinitely many expressions

$$1 = \sum_{i=1}^{u+v} \varepsilon_i \tau^{\gamma_i}$$

where $\gamma_i \geqslant 0$, $\varepsilon_i \in \{-1,0,1\}$ and no subsum of the right hand side of the equation is zero. But this is not possible by Theorem 1 of [9] which we state below for convenience. This completes the proof. ∎

THEOREM. (Van Der Poorten) *Let K be a field of characteristic zero and H a finitely generated subgroup of the multiplicative group of K. Then for each integer $m > 0$ there are only finitely many relations $u_1 + \cdots + u_m = 1$ with each $u_i \in H$ and no subsum of the left hand side is zero.*

A result very similar to the above was proved by Evertse in [3]. One can avoid using the above deep result and use Lemma 1 and 2 of [7] and modify the argument slightly to get the required contradiction.

PROOF OF LEMMA 2.3: If the torsion subgroup of A is non-trivial then it has a non-trivial normal subgroup A_1 of exponent p for some prime p. This is finite since A has finite rank, hence it is centralised by t^{ℓ} for some $\ell > 0$ and A_1 lies in the centre of $\langle A, t^{\ell}\rangle$. We may thus assume that A is torsion-free. Let D be a non-trivial subgroup of A of least rank subject to $D \trianglelefteq G$. Lemma 2.2 applies to $\langle D, t\rangle$ and we conclude that $\langle D, t^k\rangle$ is abelian for some $k > 0$. Hence D lies in the centre of $\langle A, t^k\rangle$. ∎

PROOF OF LEMMA 2.4: As the hypothesis of Theorem 1 is stronger than that of Theorem 2, Lemma 2.2 and its proof applies. We follow the proof of Lemma 2.2 and reach the situation where we may assume A to be an additive subgroup of $\mathbb{Q}(\tau)$ for some algebraic number τ and the action of t under conjugation is that of multiplication by τ. Furthermore we may assume τ to be a primitive kth root of unity and we need to show that $\tau = 1$.

Let $h_i = \big(k^i(1-\tau), t^{-1}\big)$, and $H_i = \langle h_i\rangle$. Observe that $h_i^{\lambda} = \big(k^i(1-\tau^{\lambda}), t^{-\lambda}\big)$ and $h_i^k = t^{-k}$. Let $X = (H_1 \ldots H_n)^r$ and suppose that for some $\phi \neq 1$ in S_n, $X = \big(H_{\phi(1)} \ldots H_{\phi(n)}\big)^r$. If $\phi(1) \neq 1$, then $\langle H_1, H_{\phi(1)}\rangle \subseteq X$. But this is not possible since X is the union of a finite number of cosets of $\langle t^k\rangle$ whereas $\langle H_1, H_{\phi(1)}\rangle$ contains the subgroup generated by $\big(k^{\phi(1)} - k\big)(1-\tau)$, an infinite cyclic subgroup of $Q(\tau)$, not contained in any finite union of cosets of $\langle t^k\rangle$. Hence $\phi(1) = 1$ and similarly $\phi(n) = n$.

For any permutation π in S_n, a typical element x of $H_{\pi(1)} \ldots H_{\pi(n)}$ has the form

$$\begin{aligned} x &= h_{\pi(1)}^{\lambda_1} \cdots h_{\pi(n)}^{\lambda_n} \\ &= \Big(k^{\pi(1)}(1-\tau^{\lambda_1}) + k^{\pi(2)}(1-\tau^{\lambda_2})\tau^{\lambda_1} + \cdots + k^{\pi(n)}(1-\tau^{\lambda_n})\tau^{\lambda_1+\cdots+\lambda_{n-1}}, t^{-\mu}\Big) \end{aligned}$$

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where $\mu = \lambda_1 + \cdots + \lambda_n$ and λ_i are arbitrary integers. In turn these elements may be written as

$$\left(k^{\pi(1)}(1-\tau^{\alpha_1}) + k^{\pi(2)}(\tau^{\alpha_1} - \tau^{\alpha_2}) + \cdots + \ldots k^{\pi(n)}(\tau^{\alpha_{n-1}} - \tau^{\alpha_n}),\ t^{-\alpha_n}\right)$$

where α_i are arbitrary integers. In particular $t^{-\alpha_0}x$ has the form $(h, t^{-\alpha_n})$ where

$$\begin{aligned} h &= k^{\pi(1)}(\tau^{\alpha_0} - \tau^{\alpha_1}) + k^{\pi(2)}(\tau^{\alpha_1} - \tau^{\alpha_2}) + \cdots + k^{\pi(n)}(\tau^{\alpha_{n-1}} - \tau^{\alpha_n}) \\ &= (k^{\pi(1)}\tau^{\alpha_0} + \tau^{\alpha_1}(k^{\pi(2)} - k^{\pi(1)}) + \cdots + \tau^{\alpha_{n-1}}(k^{\pi(n)} - k^{\pi(n-1)}) - \tau^{\alpha_n}k^{\pi(n)}. \end{aligned}$$

Thus, if α_0 is a given fixed integer, the real part of h is maximised by choosing $\alpha_i \equiv 0 (\mathrm{mod}\ k)$ if $\pi(i+1) > \pi(i)$, $\alpha_i \equiv q (\mathrm{mod}\ k)$ where $q = [k/2]$ if $\pi(i+1) < \pi(i)$; $i = 1, \ldots n-1$ and $\alpha_n \equiv q (\mathrm{mod}\ k)$. If π is the identity permutation then this value is $k\cos(2\pi\alpha_0)/(k) + (k^n - k) - k^n\cos(2\pi q)/k$.

On the other hand if $\pi \neq 1$ and $\pi(1) = 1$, $\pi(n) = n$, then the maximum real part of the value of h is

$$k\cos\frac{2\pi\alpha_0}{k} + (k^n - k) - k^n\cos\frac{2\pi q}{k} + \sum\left(k^{\pi(i)} - k^{\pi(i+1)}\right)\left(1 - \cos\frac{2\pi q}{k}\right)$$

where the sum is over all values of i such that $\pi(i) > \pi(i+1)$. This value is clearly greater than the value obtained for the identity permutation π.

Now the general element of $\left(H_{\pi(1)} \ldots H_{\pi(n)}\right)^r$ is $(h, t^{-\alpha})$ where h is expressible in the form

$$\sum_{i=1}^{r} k^{\pi(1)}\tau^{\alpha_{i0}} + \tau^{\alpha_{i1}}\left(k^{\pi(2)} - k^{\pi(1)}\right) + \cdots + \tau^{\alpha_{in-1}}\left(k^{\pi(n)} - k^{\pi(n-1)}\right) - \tau^{\alpha_{in}}k^{\pi(n)}$$

where $\alpha_{10} = 0$, $\alpha_{in} = \alpha_{i+10}$, $i = 1, \ldots, r-1$, $\alpha_{rn} = \alpha$. By picking the values for α_{ij} to maximise the real part of h as above, it is clear that the value achieved when $\pi \neq 1$ is greater than for $\pi = 1$. Thus $\left(H_{\phi(1)} \ldots H_{\phi(n)}\right)^r$, $\phi(1) = 1$, $\phi(n) = n$, $\phi \neq 1$, contains elements not contained in $(H_1 \ldots H_n)^r$. This completes the proof. □

3. Example

Let $U = \{u\}$ where $u = u(x_1, \ldots, x_4) = x_1x_4x_2x_3x_2x_3x_4x_1$ and $V = \{u_\sigma \mid \sigma \in S_4\}$. Then the infinite dihedral group G is in $SP(U, V)$.

Consider $u(H_1, H_2, H_3, H_4)$ for given subgroups H_1, H_2, H_3, H_4 of G. If H_1 or H_4 is normal in G then $u(H_1, H_2, H_3, H_4) = u(H_4, H_2, H_3, H_1)$. If H_2 or H_3 is normal in G then $u(H_1, H_2, H_3, H_4) = u(H_1, H_3, H_2, H_4)$. So assume none of the H_i's is normal in G. If for some i, H_i is not of order two, then it contains a subgroup K_i normal

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in G and of index two in H_i. Moreover $u(H_{\sigma(1)} \dots H_{\sigma(4)}) = K_i u(H_{\sigma(1)} \dots H_{\sigma(4)})$ for all $\sigma \in S_n$, and we may replace G by G/K_i and each of H_j by $H_j K_i / K_i$. Thus the essential case to be considered is one where each H_i is of order two.

Now $G = \langle a, t \rangle$ where $a^t = a^{-1}$ and $t^2 = 1$. $H_i = \langle a^{\lambda_i} t \rangle$, $i = 1, 2, 3, 4$. We will show that the set $L = H_4 H_2 H_3 H_2 H_3 H_4$ equals the set $R = H_4 H_3 H_2 H_3 H_2 H_4$. From this it follows that $u(H_1, H_2, H_3, H_4) = H_1 L H_1 = H_1 R H_1 = u(H_1, H_3, H_2, H_4)$.

Now $a^\lambda \in H_2 H_3 H_2 H_3$ if and only if $\lambda = 0$, $\lambda_2 - \lambda_3$, $\lambda_3 - \lambda_2$ or $2\lambda_2 - 2\lambda_3$. $a^\lambda t \in H_2 H_3 H_2 H_3$ if and only if $\lambda = \lambda_2, \lambda_3, 2\lambda_2 - \lambda_3$ or $2\lambda_3 - \lambda_2$. Hence $H_2 H_3 H_2 H_3 \backslash H_3 H_2 H_3 H_2$ consists of $a^{2\lambda_2 - 2\lambda_3}$ only and $H_3 H_2 H_3 H_2 \backslash H_2 H_3 H_2 H_3$ consists of $a^{2\lambda_3 - 2\lambda_2}$ only. But $H_4\ a^{2\lambda_2 - 2\lambda_3} H_4$ consists of a^λ where $\lambda \in \{2\lambda_2 - 2\lambda_3,\ 2\lambda_3 - 2\lambda_2\}$ and $a^\lambda t$ where $\lambda \in \{\lambda_4 - 2\lambda_2 + 2\lambda_3,\ \lambda_4 + 2\lambda_2 - 2\lambda_3\}$. From the symmetry between λ_2 and λ_3 above it is clear that $H_4\ a^{2\lambda_2 - 2\lambda_3} H_4 = H_4\ a^{2\lambda_3 - 2\lambda_2} H_4$. Thus the sets L and R are equal and $G \in SP(U, V)$.

We have not tried to analyse conditions on words u for which $D_\infty \notin SP(U, V)$ where $U = \{u\}$ and $V = \{u_\sigma \mid \sigma \in S_n\}$.

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