Proceedings of the Edinburgh Mathematical Society (2012) **55**, 781–796
DOI:10.1017/S0013091510000076

RECONSTRUCTING SCHEMES FROM THE DERIVED CATEGORY

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(Received 11 January 2010)

Abstract We generalize Bondal and Orlov's Reconstruction Theorem for a Gorenstein scheme X and a projective morphism $X \to T$ whose (relative) dualizing sheaf is either T-ample or T-antiample.

Keywords: derived categories; equivalences; reconstruction; linear functors

2010 *Mathematics subject classification:* Primary 18E30
Secondary 14F05

Introduction

This paper deals with the problem of reconstructing a scheme X from its derived category $D(X)$ (where $D(X)$ is some kind of derived category such as $D^b_c(X)$ or $D_{\text{perf}}(X)$). Does an equivalence $\varPhi\colon D(X) \xrightarrow{\sim} D(Y)$ induce an isomorphism $X \simeq Y$? The answer is positive with great generality for the category D_{perf} if we assume that the equivalence respects $\otimes$-products [**3**, **4**]. If $\varPhi$ does not preserve $\otimes$-products, the answer is more difficult. As a negative answer, there are examples of non-isomorphic varieties with equivalent D^b_c: abelian varieties and $K3$ surfaces [**15**–**18**], varieties connected by some kinds of flops [**6**, **9**, **11**] and many others where Y is a moduli space of certain kind of sheaves on X [**5**, **8**, **10**]. On the contrary, one has the fundamental result of Bondal and Orlov [**7**] that states that if X is a smooth projective variety over a field k with ample or antiample canonical sheaf and the equivalence $\varPhi$ is k-linear and graded, then $X \simeq Y$. So X is determined by the k-linear and graded structure of $D^b_c(X)$. Not even the triangulated structure of $D^b_c(X)$ is required. The aim of this paper is to extend this result in two directions.

Firstly, we shall replace the smoothness condition on X by a Gorenstein condition: X is a Gorenstein, connected and equidimensional projective k-scheme. Note that X may be non-irreducible or even non-reduced. With this hypothesis on X we shall prove that the Bondal and Orlov result (Theorem 1.15) still holds.

Theorem. *Let X be a connected equidimensional Gorenstein projective k-scheme with ample canonical or antiample canonical sheaf. If $\mathcal{D} = D_{\text{perf}}(X)$ (respectively, $D^b_c(X)$) is* *equivalent as a graded category to* $D_{\rm perf}(X')$ *(respectively,* $D^b_c(X')$*) for some other proper* k*-scheme* X'*, then* X *is isomorphic to* X'*.*

Again, only the k-linear graded structure of $D_{\rm perf}(X)$ (or $D^b_c(X)$) is involved. If we consider the triangulated structure (i.e. we assume that Φ is exact), this result is obtained in [**2**] by different methods.

The original proof of Bondal and Orlov is based on identifying (categorically) the objects of $D^b_c(X)$ that are isomorphic to the skyscraper sheaf of a closed point (up to translations). From this, one can identify invertible sheaves and then the topology of X. Finally, the Serre functor allows us to construct the ring structure. In our case, we shall identify what we have called Gorenstein 0-cycles, Z_x, supported at a closed point x. By this we mean a zero-dimensional closed subscheme $Z_x \subset X$ supported at a closed point x and such that $\mathcal{O}_{Z_x}$ is perfect (as an $\mathcal{O}_X$-module) and Z_x is Gorenstein. These objects also allow us to identify invertible sheaves and the proof then works as in the smooth case.

As in [**7**], one can apply this reconstruction theorem to calculate the group of exact autoequivalences of $D^b_c(X)$. One obtains the same result as Bondal and Orlov (see Corollary 1.17 and [**2**]).

Secondly, we shall give a relative version of Bondal and Orlov result (Theorem 2.10). By this we mean the following. Assume that X is a Gorenstein scheme (i.e. its local rings are Gorenstein) and let $f\colon X \to T$ be a proper morphism. Then $D^b_c(X)$ has a T-linear structure (this roughly means that one can multiply objects of $D^b_c(X)$ by objects of $D_{\rm perf}(T)$; see Definition 2.1 for details). Assume also that the relative dualizing complex $f^!\mathcal{O}_T$ is an invertible sheaf $\omega_{X/T}$, placed at degree $\dim X - \dim T$ (this holds, for example, for any $f\colon X \to T$ if T is Gorenstein). Then we shall prove that if $\omega_{X/T}$ is either T-ample or T-antiample, then X is determined (as a T-scheme) by $D^b_c(X)$ with its T-linear structure. The precise statement is as follows.

Theorem. *Let* X *be a Gorenstein scheme, let* $f\colon X \to T$ *be a proper morphism of finite Tor-dimension and let* T *be a Cohen–Macaulay scheme. Let* $D_{X/T}$ *be the relative dualizing complex. Assume that* $D_{X/T} \simeq \omega_{X/T}[n]$*,* $n \in \mathbb{Z}$*, where* $\omega_{X/T}$ *is an invertible sheaf which is either* T*-ample or* T*-antiample. If* $X' \to T$ *is another Gorenstein* T*-scheme and one has a* T*-linear equivalence* $D_{\rm perf}(X) \xrightarrow{\sim} D_{\rm perf}(X')$*, then* $X \simeq X'$ *(isomorphism of* T*-schemes).*

Note that the definition of the T-linear structure of $D^b_c(X)$ does not require the triangulated structure of either $D^b_c(X)$ or $D_{\rm perf}(T)$.

As in the absolute case, one can apply this reconstruction theorem to calculate the group of exact T-linear autoequivalences of $D^b_c(X)$ (see Corollary 2.12). Let us see some other immediate applications of this relative reconstruction theorem.

(a) Let X be a projective Gorenstein k-scheme of dimension n. Assume that the canonical sheaf ω_X is neither ample nor antiample (i.e. we are not under the Bondal and Orlov hypothesis). Choose a finite and flat morphism $X \to \mathbb{P}^n$, which induces a $\mathbb{P}^n$-linear structure on $D_{\rm perf}(X)$. Since it is an affine morphism, the relative dualizing sheaf $\omega_{X/\mathbb{P}^n}$ is $\mathbb{P}^n$-ample. Then X is determined, as a scheme over $\mathbb{P}^n$, by the $\mathbb{P}^n$-linear structure of $D_{\mathrm{perf}}(X)$ (or $D^b_c(X)$). Of course, natural questions arise: how many $\mathbb{P}^n$-linear structures admit $D_{\mathrm{perf}}(X)$? Can one recognize the $\mathbb{P}^n$-linear structures coming from finite and flat morphisms $X \to \mathbb{P}^n$?

(b) Let

$$\begin{array}{ccc} X & & X' \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

be two birational proper morphisms between Gorenstein schemes. Let $K \in D^b_c(X \times_Y X')$ be an object of finite homological dimension over X (this is a technical condition that ensures that the associated integral functor maps bounded complexes to bounded complexes). The associated integral functor

$$\Phi_K \colon D^b_c(X) \to D^b_c(X')$$

is a Y-linear functor. If either X or X' has a Y-ample or a Y-antiample relative dualizing sheaf and Φ_K is an equivalence, then $X \simeq X'$ as Y-schemes. For example, if X is the blow-up of Y along a regular centre, then $\omega_{X/Y}$ is Y-antiample.

Conventions

Throughout the paper a scheme means a connected and equidimensional scheme. The latter means that all the irreducible components of X have the same dimension. All schemes are also assumed to be quasi-compact and quasi-separated.

1. Absolute case

In this section k is a field and we deal with k-linear graded categories with finite Homs.

1.1. Serre functor

Let $\mathcal{D}$ be a k-linear category. For any $P, Q \in D$ we shall denote

$$\mathrm{Hom}^\bullet(P, Q) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(P, Q).$$

Definition 1.1. An object $P \in \mathcal{D}$ is called a *perfect object* if $\mathrm{Hom}^\bullet(P, Q)$ is finite dimensional for any $Q \in \mathcal{D}$. We denote $\mathcal{D}_{\mathrm{perf}}$ the faithful subcategory of perfect objects. It is a triangulated subcategory.

Any autoequivalence $\Phi \colon \mathcal{D} \to \mathcal{D}$ preserves perfectness, i.e. induces an autoequivalence $\Phi \colon \mathcal{D}_{\mathrm{perf}} \to \mathcal{D}_{\mathrm{perf}}$.

Definition 1.2. A covariant functor $S \colon \mathcal{D}_{\mathrm{perf}} \to \mathcal{D}_{\mathrm{perf}}$ is called a *Serre functor* if it is a category equivalence and there are bi-functorial isomorphisms

$$\varphi_{A,B} \colon \mathrm{Hom}_{\mathcal{D}}(A, B)^* \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(B, S(A))$$

for any $A, B \in \mathcal{D}_{\mathrm{perf}}$.

Any autoequivalence $\Phi\colon \mathcal{D}_{\mathrm{perf}} \to \mathcal{D}_{\mathrm{perf}}$ commutes with a Serre functor. Any Serre functor is graded and exact. Moreover, if it exists, it is unique up to a graded natural isomorphism.

Example 1.3. Let X be a proper k-scheme of dimension n and let $\mathcal{D} = D^b_c(X)$. Then $P \in \mathcal{D}$ is a perfect object if and only if P is a perfect complex, i.e. $\mathcal{D}_{\mathrm{perf}} = D_{\mathrm{perf}}(X)$. For this, see the proof of [**12**, Lemma 1.2].

Assume in addition that X is Gorenstein. The dualizing complex is isomorphic to $\omega_X[n]$, where ω_X is an invertible sheaf (the canonical sheaf), n is the dimension of X and the functor

$$(\cdot) \otimes \omega_X[n] \tag{1.1}$$

is a Serre functor.

Remark 1.4. For any scheme X the subcategory $D_{\mathrm{perf}}(X)$ of $D^b_c(X)$ can be recognized categorically: an object $P \in D^b_c(X)$ belongs to $D_{\mathrm{perf}}(X)$ if and only if for any Q there exists an integer n_Q such that $\mathrm{Hom}^n(P, Q) = 0$ for any $n > n_Q$.

Proposition 1.5. *Let X be a proper k-scheme of dimension n. Then $D_{\mathrm{perf}}(X)$ has a Serre functor if and only if X is Gorenstein.*

Proof. For any $P, Q \in D_{\mathrm{perf}}(X)$ one has by duality

$$\mathrm{Hom}(P, Q)^* = \mathrm{Hom}\left(Q, P \overset{\boldsymbol{L}}{\otimes} D_X\right),$$

where D_X is the dualizing complex of X over k. If $S\colon D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(X)$ is a Serre functor, then the identity $S(P) \to S(P)$ gives a morphism

$$S(P) \to P \overset{\boldsymbol{L}}{\otimes} D_X.$$

This morphism becomes an isomorphism after taking $\mathrm{Hom}(Q, \cdot)$ for any perfect Q. Hence, it is an isomorphism:

$$S(P) = P \overset{\boldsymbol{L}}{\otimes} D_X.$$

In particular, D_X is perfect. Since S is an autoequivalence, one must have

$$D_X \overset{\boldsymbol{L}}{\otimes} D_X^{\vee} \simeq \mathcal{O}_X.$$

By [**1**, Theorem 1.5.2], $D_X \simeq \mathcal{L}[r]$, $r \in \mathbb{Z}$, where $\mathcal{L}$ is an invertible sheaf. Hence, X is Gorenstein. □

1.2. Gorenstein 0-cycles

Definition 1.6. Let X be a scheme and let $x \in X$ be a closed point. A Gorenstein 0-cycle supported at x is a closed subscheme $Z_x \hookrightarrow X$ supported at x such that

(i) Z_x is a Gorenstein zero-dimensional scheme,

(ii) $\mathcal{O}_{Z_x}$ is an $\mathcal{O}_X$-module of finite Tor-dimension, i.e. $\mathcal{O}_{Z_x} \in D_{\mathrm{perf}}(X)$.

Remark 1.7. Let X be a Gorenstein scheme.

1. Let $Z \hookrightarrow X$ be a closed subscheme supported at a closed point $x \in X$ and let n be the dimension of the local ring of $\mathcal{O}_X$ at x. Then Z is Gorenstein if and only if

$$\mathcal{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) \simeq \mathcal{O}_Z$$

   In particular, any local complete intersection (l.c.i.) zero cycle is Gorenstein (see [**12**] for the definition of an l.c.i. zero cycle).

2. For any closed point $x \in X$ there exists a Gorenstein 0-cycle Z_x supported at x. In fact it is enough to take an l.c.i. zero cycle supported at x (which exists because X is Cohen–Macaulay [**12**, Lemma 1.9]).

3. The set

$$\Omega = \{\mathcal{O}_{Z_x} \text{ for all closed points } x \in X \text{ and all Gorenstein 0-cycles } Z_x \text{ supported on } x\}$$

   is a spanning class for $D^b_c(X)$. This follows from [**13**, Lemma 3.4].

1.3. Reconstruction of a Gorenstein *k*-scheme from the derived category of perfect complexes

For any object $P \in \mathcal{D}$ we shall define $\mathcal{O}_P = \mathrm{Hom}(P, P)$. It is a finite k-algebra, possibly non-commutative. The product is given by composition: $f \cdot g = f \circ g$.

For any $P, Q \in \mathcal{D}$ the set $\mathrm{Hom}(P, Q)$ has a natural structure of right $\mathcal{O}_P$-modules and another one of left $\mathcal{O}_Q$-modules:

$$f \cdot g = f \circ g, \quad h \cdot f = h \circ f, \quad f \in \mathrm{Hom}(P, Q), \quad g \in \mathcal{O}_P, \quad h \in \mathcal{O}_Q.$$

In particular, $\mathrm{Hom}^n(P, P)$ is a left and right $\mathcal{O}_P$-module.

Example 1.8. Let Z be a zero-dimensional closed subscheme of a k-scheme X. Then

$$\mathrm{Hom}_{D^b_c(X)}(\mathcal{O}_Z, \mathcal{O}_Z) = \mathcal{O}_Z$$

(since Z is zero dimensional, we identify $\mathcal{O}_Z$ with its global sections). That is, if we take $P = \mathcal{O}_Z$, then $\mathcal{O}_P = \mathcal{O}_Z$.

Definition 1.9. An object $P \in \mathcal{D}_{\mathrm{perf}}$ is called a *Gorenstein* 0*-cycle object* of codimension s, if

(i) $S(P) \simeq P[s]$,

(ii) $\mathrm{Hom}^{<0}(P,P) = 0$,

(iii) $\mathcal{O}_P := \mathrm{Hom}^0(P,P)$ is a commutative local k-algebra,

(iv) $\mathrm{Hom}^s(P,P)$ is a monogenerated right $\mathcal{O}_P$-module.

Proposition 1.10. *Let X be a Gorenstein proper k-scheme of dimension n with ample canonical or anticanonical sheaf. Then an object $P \in D_{\mathrm{perf}}(X)$ is a Gorenstein* 0*-cycle object if and only if $P \simeq \mathcal{O}_{Z_x}[r]$, $r \in \mathbb{Z}$, is isomorphic (up to translation) to the structure sheaf of a Gorenstein* 0*-cycle supported at a closed point $x \in X$.*

Remark 1.11. Since X has an ample invertible sheaf, it is projective.

Proof. The structure sheaf of a Gorenstein 0-cycle supported at a closed point obviously satisfies conditions (i)–(iv) of Definition 1.9.

Suppose now that $P \in D_{\mathrm{perf}}(X)$ satisfies (i)–(iv). Let $\mathcal{H}^i$ be the cohomology sheaves of P. If follows immediately from (i) that $s = n$ and $\mathcal{H}^i \otimes \omega_X \simeq \mathcal{H}^i$. Since ω_X is either an ample or an antiample sheaf, we conclude that $\mathcal{H}^i$ are finite length sheaves, i.e. their supports consist of isolated closed points. Sheaves with the support in different points are homologically orthogonal, therefore any such object decomposes into the direct sum of those which have the support of all cohomology sheaves in a single point. By (iii), the object P is indecomposable; hence, all $\mathcal{H}^i$ have their support in a single point. Now, condition (ii) implies that all but one of the cohomology sheaves are trivial. Indeed, suppose that $\mathcal{H}^i \neq 0$, $\mathcal{H}^j \neq 0$, $i < j$ and $\mathcal{H}^m = 0$ for $m < i$ or $m > j$. Then $\mathrm{Hom}^{i-j}(P,P) = \mathrm{Hom}(\mathcal{H}^i, \mathcal{H}^j) \neq 0$. Hence, $P \simeq M[r]$, where M is a coherent $\mathcal{O}_X$-module of finite Tor-dimension supported at a closed point x. The surjection $M \to M/\mathfrak{m}_x M$ induces a surjection (of right $\mathcal{O}_P$-modules)

$$\mathrm{Hom}^n(M,M) \to \mathrm{Hom}^n(M, M/\mathfrak{m}_x M) \to 0$$

By (iv), one concludes that $\mathrm{Hom}^n(M, M/\mathfrak{m}_x M)$ is a monogenerated $\mathcal{O}_P$-module. Then $M/\mathfrak{m}_x M$ is a one-dimensional $k(x)$-vector space. By Nakayama, M is monogenerated, i.e. $M \simeq \mathcal{O}_X/\mathcal{I}$, with $\mathcal{I}$ an $\mathfrak{m}_x$-primary ideal. To conclude, we have to prove that $\mathcal{O}_X/\mathcal{I}$ is Gorenstein. We have to prove that $\mathcal{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X) \simeq \mathcal{O}_X/\mathcal{I}$. The surjection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$ induces a surjection

$$\mathcal{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X) \to \mathcal{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X/\mathcal{I}) \to 0$$

which is an isomorphism because they have the same dimension as k-vector spaces. Finally, $\mathcal{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X/\mathcal{I})$ is isomorphic to $\mathcal{O}_X/\mathcal{I}$ because it is a monogenerated $\mathcal{O}_X/\mathcal{I}$-module (by (iv)) and has the same dimension as $\mathcal{O}_X/\mathcal{I}$. This completes the proof. □

Definition 1.12. An object $L \in \mathcal{D}$ is called invertible if, for any Gorenstein 0-cycle object $P \in \mathcal{D}$, there exists $s \in \mathbb{Z}$ such that

(i) $\mathrm{Hom}^s(L, P) \simeq \mathcal{O}_P$ (isomorphism of left $\mathcal{O}_P$-modules),

(ii) $\mathrm{Hom}^i(L, P) = 0$ for $i \neq s$.

Proposition 1.13. *Let X be a Gorenstein scheme. Assume that all Gorenstein 0-cycle objects have the form $\mathcal{O}_{Z_x}[s]$ for some Gorenstein 0-cycle Z_x supported at a closed point x, $s \in \mathbb{Z}$. Then an object $L \in \mathcal{D}$ is invertible if and only if $L \simeq \mathcal{L}[t]$ for some invertible sheaf $\mathcal{L}$ on X, $t \in \mathbb{Z}$.*

Proof. The converse is obvious. Now let $\mathcal{H}^i$ be the cohomology sheaves of an invertible object L. Consider the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}^p(\mathcal{H}^{-q}, \mathcal{O}_{Z_x}) \Longrightarrow \mathrm{Ext}^{p+q}(L, \mathcal{O}_{Z_x})$$

and let $\mathcal{H}^{q_0}$ be the non-trivial cohomology sheaf with maximal index. Then for any closed point $x \in X$ in the support of $\mathcal{H}^{q_0}$, $\mathrm{Hom}(\mathcal{H}^{q_0}, \mathcal{O}_{Z_x}) \neq 0$. Moreover, $\mathrm{Hom}(\mathcal{H}^{q_0}, \mathcal{O}_{Z_x})$ and $\mathrm{Ext}^1(\mathcal{H}^{q_0}, \mathcal{O}_{Z_x})$ are intact by the differentials of the spectral sequence. Therefore, in view of the definition of an invertible object, we conclude that, for *any* point x from the support of $\mathcal{H}^{q_0}$,

(a) $\mathrm{Hom}(\mathcal{H}^{q_0}, \mathcal{O}_{Z_x}) \simeq \mathcal{O}_{Z_x}$,

(b) $\mathrm{Ext}^1(\mathcal{H}^{q_0}, \mathcal{O}_{Z_x}) = 0$.

Since X is connected, and due to Lemma 1.14, $\mathcal{H}^{q_0}$ is invertible. It follows that $\mathrm{Ext}^i(\mathcal{H}^{q_0}, \mathcal{O}_{Z_x}) = 0$ for $i > 0$. Hence, $\mathrm{Hom}(\mathcal{H}^{q_0-1}, \mathcal{O}_{Z_x})$ are intact by differentials of the spectral sequence. This means that $\mathrm{Hom}(\mathcal{H}^{q_0-1}, \mathcal{O}_{Z_x}) = 0$ for any $x \in X$, i.e. $\mathcal{H}^{q_0-1} = 0$. Repeating this argument for $\mathcal{H}^q$ with smaller q, we easily see that all $\mathcal{H}^q$, except $q = q_0$, are trivial. This proves the proposition. □

Lemma 1.14. *Let $\mathcal{O}$ be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finite $\mathcal{O}$-module. Let I be an $\mathfrak{m}$-primary ideal such that $\mathcal{O}/I$ is a zero-dimensional Gorenstein ring. Then $M \simeq \mathcal{O}$ if and only if*

(a) $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O}/I) \simeq \mathcal{O}/I$,

(b) $\mathrm{Ext}^1_{\mathcal{O}}(M, \mathcal{O}/I) = 0$.

Proof. Let us define $\bar{\mathcal{O}} = \mathcal{O}/I$. For any $\bar{\mathcal{O}}$-module N let us define $N^\vee = \mathrm{Hom}_{\bar{\mathcal{O}}}(N, \bar{\mathcal{O}})$. Since $\bar{\mathcal{O}}$ is Gorenstein, the natural map $N \to N^{\vee\vee}$ is an isomorphism (N of finite type).

Now $(M/IM)^\vee \simeq \bar{\mathcal{O}}$ by (a). Hence, $M/IM \simeq (M/IM)^{\vee\vee} \simeq \bar{\mathcal{O}}$. One has then an epimorphism $\mathcal{O} \xrightarrow{f} M$ that becomes an isomorphism after taking $\mathrm{Hom}_{\mathcal{O}}(\,,\bar{\mathcal{O}})$. Let K be the kernel of f. By (b), one obtains that $\mathrm{Hom}_{\mathcal{O}}(K, \bar{\mathcal{O}}) = 0$; that is, $(K/IK)^\vee = 0$. Then $K/IK = 0$ and by Nakayama $K = 0$. □

Theorem 1.15. *Let X be a connected equidimensional Gorenstein projective k-scheme with ample canonical or antiample canonical sheaf. If $\mathcal{D} = D_{\text{perf}}(X)$ is equivalent as a graded category to $D_{\text{perf}}(X')$ for some other proper k-scheme X', then X is isomorphic to X'.*

Proof. While saying that two isomorphism classes of objects, one in $D_{\text{perf}}(X)$ and the other in $D_{\text{perf}}(X')$, are equal, we mean that the former is taken to the latter by the primary equivalence $D_{\text{perf}}(X) \xrightarrow{\sim} D_{\text{perf}}(X')$. First note that X' is Gorenstein by Proposition 1.5. As in [**7**], we proceed in several steps.

Step 1. Denote by $\mathcal{G}_{\mathcal{D}}$ the set of isomorphism classes of the Gorenstein 0-cycle objects in $\mathcal{D}$, and by $\mathcal{G}_X$ the set of isomorphism classes of objects in $D_{\text{perf}}(X)$:

$$\mathcal{G}_X := \{\mathcal{O}_{Z_x}[r],\ x \in X,\ Z_x \text{ is a Gorenstein 0-cycle supported at } x,\ r \in \mathbb{Z}\}.$$

By Proposition 1.10, $\mathcal{G}_{\mathcal{D}} \simeq \mathcal{G}_X$. Obviously, $\mathcal{G}_{X'} \subset \mathcal{G}_{\mathcal{D}}$.

Now we shall identify two 0-cycle Gorenstein objects whenever they have the same support. Let us define

$$\mathcal{P}_{\mathcal{D}} = \mathcal{G}_{\mathcal{D}}/\sim,$$

where $P \sim Q$ if $\operatorname{Hom}^\bullet(P, Q) \neq 0$. Analogously, $\mathcal{P}_X = \mathcal{G}_X/\sim$. One still has $\mathcal{P}_{X'} \subset \mathcal{P}_{\mathcal{D}}$. Suppose that there is an object $[P]$ in $\mathcal{P}_{\mathcal{D}}$ which is not contained in $\mathcal{P}_{X'}$. Since two different objects in $\mathcal{P}_{\mathcal{D}}$ are mutually orthogonal, it follows that $P \in D_{\text{perf}}(X')$ is orthogonal to the structure sheaf of any Gorenstein 0-cycle $\mathcal{O}_{Z_{x'}}$, $x' \in X'$. Hence, P is zero. Therefore, $\mathcal{P}_{X'} = \mathcal{P}_{\mathcal{D}} = \mathcal{P}_X$. We conclude that $\mathcal{G}_{X'} = \mathcal{G}_{\mathcal{D}}$. Let $P \in \mathcal{G}_{\mathcal{D}}$. Since $\mathcal{P}_{X'} = \mathcal{P}_{\mathcal{D}}$, there exists a Gorenstein 0-cycle $Z_{x'_0}$, $x'_0 \in X'$, such that $[P] = [\mathcal{O}_{Z_{x'_0}}]$. Hence, for any $x' \in X'$, $x' \neq x'_0$, P is orthogonal to $\mathcal{O}_{Z_{x'}}$. Hence, $P \in D_{\text{perf}}(X')$ is supported at the single point x'_0. As shown in the proof of Proposition 1.10, P is isomorphic (up to translation) to the structure sheaf of a Gorenstein 0-cycle of X' supported at x'_0. Hence, $P \in \mathcal{G}_{X'}$.

Step 2. Denote by $\mathcal{L}_{\mathcal{D}}$ the set of isomorphism classes of invertible objects in $\mathcal{D}$; denote by $\mathcal{L}_X$ the set of isomorphism classes of objects in $D_{\text{perf}}(X)$ defined by

$$\mathcal{L}_X := \{L[r],\ L \text{ is an invertible sheaf on } X,\ r \in \mathbb{Z}\}.$$

By Step 1, both X and X' satisfy the assumptions of Proposition 1.13. It follows that $\mathcal{L}_X = \mathcal{L}_{\mathcal{D}} = \mathcal{L}_{X'}$.

Step 3. Let us fix some invertible object L_0 in $\mathcal{D}$, which is an invertible sheaf on X. By Step 2, L_0 can be regarded, up to translation, as an invertible sheaf on X'. Moreover, changing, if necessary, the equivalence $D_{\text{perf}}(X) \xrightarrow{\sim} D_{\text{perf}}(X')$ by the translation functor, we can assume that L_0, regarded as an object on X', is a genuine invertible sheaf (the same precision as in [**7**] can be taken). Obviously, by Step 1, the set $g_{\mathcal{D}} \subset \mathcal{G}_{\mathcal{D}}$,

$$g_{\mathcal{D}} := \{P \in \mathcal{G}_{\mathcal{D}},\ \operatorname{Hom}(L_0, P) \simeq \mathcal{O}_P\},$$

coincides with both sets

$$g_X = \{\mathcal{O}_{Z_x},\ x \in X,\ Z_x \text{ is a Gorenstein 0-cycle supported at } x\}$$

and

$$g_{X'} = \{\mathcal{O}_{Z_{x'}},\ x' \in X',\ Z_{x'} \text{ is a Gorenstein 0-cycle supported at } x'\}.$$

If we define $p_{\mathcal{D}} := g_{\mathcal{D}}/\sim$ (analogously p_X, $p_{X'}$) we obtain $p_X = p_{\mathcal{D}} = p_{X'}$; this gives us a pointwise identification of X and X'.

Step 4. Now let l_X (respectively, $l_{X'}$) be the subset in $\mathcal{L}_{\mathcal{D}}$ of isomorphism classes of invertible sheaves on X (respectively, on X').

They can be recognized from the graded category structure as follows:

$$l_{X'} = l_X = l_{\mathcal{D}} := \{L \in \mathcal{L}_{\mathcal{D}} \colon \operatorname{Hom}(L, P) \simeq \mathcal{O}_P \text{ for any } P \in g_{\mathcal{D}}\}.$$

For $\alpha \in \operatorname{Hom}(L_1, L_2)$, where $L_1, L_2 \in l_{\mathcal{D}}$, and $P \in g_{\mathcal{D}}$, denote by α_P^* the induced morphism

$$\operatorname{Hom}(L_2, P) \to \operatorname{Hom}(L_1, P)$$

and by W_α the subset of those objects $P \in g_{\mathcal{D}}$ for which α_P^* is an isomorphism. Note that if $P \in W_\alpha$ and $Q \sim P$, then $Q \in W_\alpha$. Hence, $U_\alpha := W_\alpha/\sim$ is a subset of $p_{\mathcal{D}}$. By [**14**], an algebraic variety has an ample system of invertible objects. This means that U_α, where α runs over all elements in $\operatorname{Hom}(L_1, L_2)$, and L_1 and L_2 run over the elements in $l_{\mathcal{D}}$, constitute a basis for the Zariski topologies of both X and X'. It follows that the topologies of X and X' coincide.

Step 5. Now the rest of the proof is identical to that in [**7**]; we indicate it briefly. Since codimensions of all Gorenstein 0-cycle objects are equal to the dimensions of X and X', we have $\dim X = \dim X'$. Twisting by the canonical sheaf on X and X' induces equal transformation on the set $l_{\mathcal{D}}$. Let $L_i = S^i L_0[-ni]$, i.e. $\{L_i\}$ it is the orbit of L_0 with respect to twisting by the canonical sheaf on X or X'. Then, since ω_X is either ample or antiample, the set of all U_α, where α runs over all elements in $\operatorname{Hom}(L_i, L_j)$, $i, j \in \mathbb{Z}$, is the basis for the Zariski topology on X, and hence on X'. This means that $\omega_{X'}$ is also ample or antiample.

For all pairs (i, j) there are natural isomorphisms

$$\operatorname{Hom}(L_i, L_j) \simeq \operatorname{Hom}(L_0, L_{j-i})$$

that induce a ring structure in the graded algebra A over k with graded components $A_i = \operatorname{Hom}(L_0, L_i)$. If we denote by B (respectively, B') the graded algebra with graded components $B_i = \operatorname{Hom}_X(\mathcal{O}_X, \omega_X^{\otimes i})$ (respectively, $B'_i = \operatorname{Hom}_X(\mathcal{O}_{X'}, \omega_{X'}^{\otimes i})$), we have isomorphisms of graded algebras

$$B \simeq A \simeq B'$$

and then

$$X \xrightarrow{\sim} \operatorname{Proj} B \xrightarrow{\sim} \operatorname{Proj} B' \xrightarrow{\sim} X'.$$

$\square$

Remark 1.16. Since any graded equivalence $D^b_c(X) \to D^b_c(X')$ induces an equivalence between perfect objects, one also obtains Theorem 1.15 for the category $D^b_c(X)$.

The same methods as in [**7**] yield the following.

Corollary 1.17. *Let X be a connected equidimensional Gorenstein projective k-scheme with ample canonical or anticanonical sheaf. Then the group $\operatorname{Aut} D^b_c(X)$ of isomorphism classes of exact autoequivalences $D^b_c(X) \to D^b_c(X)$ is the semi-direct product of its subgroups $\operatorname{Pic} X \oplus \mathbb{Z}$ and $\operatorname{Aut} X$, $\mathbb{Z}$ being generated by the translation functor:*

$$\operatorname{Aut} D^b_c(X) \simeq \operatorname{Aut} X \ltimes (\operatorname{Pic} X \oplus \mathbb{Z}).$$

2. Relative case

Now we shall relativize the ampleness or antiampleness of the canonical sheaf. We consider a Gorenstein scheme X together with a proper morphism $X \to T$ such that the relative canonical sheaf $\omega_{X/T}$ is either T-ample or T-antiample. We shall then see that X is determined by the T-linear structure of $D^b_c(X)$.

2.1. *T*-linear structure

Definition 2.1. Let T be a scheme. A T-linear structure on a graded category $\mathcal{D}$ is a bigraded functor

$$\begin{aligned} D_{\mathrm{perf}}(T) \times \mathcal{D} &\to \mathcal{D}, \\ (\mathcal{E}, P) &\mapsto \mathcal{E} \otimes P, \end{aligned}$$

satisfying functorial isomorphisms:

1. $\phi_P \colon \mathcal{O}_T \otimes P \simeq P$;
2. $\psi_{\mathcal{E}_1,\mathcal{E}_2,P} \colon \mathcal{E}_1 \otimes (\mathcal{E}_2 \otimes P) \simeq \left(\mathcal{E}_1 \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_T} \mathcal{E}_2\right) \otimes P$.

Definition 2.2. A T-linear category is a graded category endowed with a T-linear structure. A T-linear functor $F \colon \mathcal{D} \to \mathcal{D}'$ between T-linear categories is a functor satisfying a bifunctorial isomorphism $F(\mathcal{E} \otimes P) \simeq \mathcal{E} \otimes F(P)$, $\mathcal{E} \in D_{\mathrm{perf}}(T)$, $P \in \mathcal{D}$, which is compatible with ϕ_P and $\psi_{\mathcal{E}_1,\mathcal{E}_2,P}$ in the obvious sense.

2.2. Local homomorphisms: $\mathbb{R}\boldsymbol{\mathcal{H}}\mathbf{om}^{\bullet}_{T}$

Definition 2.3. Let P and Q be two objects of a T-linear category $\mathcal{D}$. We shall denote by $\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P, Q)$ the object of $D_{\mathrm{perf}}(T)$, if it exists, satisfying

$$\operatorname{Hom}_{D_{\mathrm{perf}}(T)}(\mathcal{E}, \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P, Q)) = \operatorname{Hom}_{\mathcal{D}}(\mathcal{E} \otimes P, Q).$$

Taking $\mathcal{E} = \mathcal{O}_T[-i]$, one obtains

$$H^i(T, \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P, Q)) = \operatorname{Hom}^i_{\mathcal{D}}(P, Q),$$

where $H^i(T, \mathcal{E}) := H^i \mathbb{R}\Gamma(T, \mathcal{E})$. We shall define $\mathcal{H}\mathrm{om}^i_T(P, Q) = \mathcal{H}^i(\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P, Q))$.

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Composition

The identity $\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q) \to \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q)$ gives a morphism in $\mathcal{D}$:

$$\epsilon_P \colon \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q) \otimes P \to Q.$$

Then one can define a composition

$$\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(Q,R) \otimes_{\mathcal{O}_T} \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q) \to \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,R),$$

which corresponds to the composition

$$\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(Q,R) \otimes_{\mathcal{O}_T} \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q) \otimes P \xrightarrow{1\otimes\epsilon_P} \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(Q,R) \otimes Q \xrightarrow{\epsilon_Q} R.$$

Taking cohomology, one obtains morphisms

$$\mathcal{H}\mathrm{om}^{i}_T(Q,R) \otimes_{\mathcal{O}_T} \mathcal{H}\mathrm{om}^{j}_T(P,Q) \to \mathcal{H}\mathrm{om}^{i+j}_T(P,R).$$

Example 2.4. Let $f \colon X \to T$ be a T-scheme. Then $D_{\mathrm{perf}}(X)$ and $D^b_c(X)$ have a natural T-linear structure: for any $\mathcal{E} \in D_{\mathrm{perf}}(T)$ and $P \in D_{\mathrm{perf}}(X)$ (respectively, $P \in D^b_c(X)$) one defines

$$\mathcal{E} \otimes P := \boldsymbol{L}f^*\mathcal{E} \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_X} P.$$

If f has finite Tor-dimension (e.g. either T regular or f flat) and $P,Q \in D_{\mathrm{perf}}(X)$, then $\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q)$ exists. Indeed,

$$\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q) := \mathbb{R}f_*\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_{\mathcal{O}_X}(P,Q).$$

The hypothesis of finite Tor-dimension ensures that $\mathbb{R}f_*\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_{\mathcal{O}_X}(P,Q)$ is perfect.

Let $X \to T$ and $X' \to T$ be two T-schemes and let $K \in D^b_c(X \times_T X')$ be an object of finite homological dimension over X (see [**13**, Definition 2.1]). One has the integral functor

$$\Phi_K \colon D^b_c(X) \to D^b_c(X'),$$
$$M \mapsto \mathbb{R}q_*\Big(\boldsymbol{L}p^*M \overset{\boldsymbol{L}}{\otimes} K\Big),$$

where p and q are the projections from $X \times_T X'$ to X and X', respectively. Then Φ_K is T-linear (as a consequence of the projection formula).

Any T-linear autoequivalence $\Phi \colon \mathcal{D} \to \mathcal{D}$ preserves $\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T$, i.e.

$$\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q) \simeq \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(\Phi(P),\Phi(Q)).$$

2.3. *T*-Serre functor

Definition 2.5. Let $\mathcal{D}$ be a T-linear category such that $\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q)$ exists for any P and Q. A *T-Serre functor* is an autoequivalence $S_T \colon \mathcal{D} \to \mathcal{D}$ satisfying a bifunctorial isomorphism

$$\mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(P,Q)^{\vee} \xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_T(Q,S_T(P)), \quad P,Q \in \mathcal{D},$$

where we define $\mathcal{E}^{\vee} = \mathbb{R}\mathcal{H}\mathrm{om}^{\bullet}_{\mathcal{O}_T}(\mathcal{E},\mathcal{O}_T)$ for any $\mathcal{E} \in D_{\mathrm{perf}}(T)$.

Taking $\operatorname{Hom}(\mathcal{E}, \cdot)$, $\mathcal{E} \in D_{\mathrm{perf}}(T)$, in the above isomorphism, one obtains

$$\operatorname{Hom}\left(\mathcal{E} \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_T} \mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T(P,Q), \mathcal{O}_T\right) \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{E} \otimes Q, S_T(P)).$$

In particular,

$$\operatorname{Hom}_{\mathcal{D}}(Q, S_T(P)) = \operatorname{Hom}_{D_{\mathrm{perf}}(T)}(\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T(P,Q), \mathcal{O}_T).$$

Any T-linear autoequivalence $\Phi\colon \mathcal{D} \to \mathcal{D}$ commutes with a T-Serre functor. A T-Serre functor, if it exists, is unique up to a natural isomorphism.

Proposition 2.6. *Let $f\colon X \to T$ be a proper morphism of finite Tor-dimension. Let $D_{X/T} := f^!\mathcal{O}_T$ be the dualizing complex of X over T. Then $D_{\mathrm{perf}}(X)$ has a T-Serre functor if and only if $D_{X/T} \simeq \omega_{X/T}[d]$, $d \in \mathbb{Z}$, $\omega_{X/T}$ is an invertible sheaf on X (the relative canonical sheaf). Moreover, if X and T are Cohen–Macaulay, then*

$$S_T \simeq (\cdot) \otimes \omega_{X/T}[n],$$

where $n = \dim X - \dim S$.

Proof. First recall that $\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T(P,Q) = \mathbb{R}f_*\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_{\mathcal{O}_X}(P,Q)$. Relative duality yields an isomorphism

$$\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T(P,Q)^\vee \xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T(\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_{\mathcal{O}_X}(P,Q), D_{X/T}).$$

Since P is perfect,

$$\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T(\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_{\mathcal{O}_X}(P,Q), D_{X/T}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T\left(Q, P \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_X} D_{X/T}\right).$$

Hence, if $D_{X/T} \simeq \omega_{X/T}[d]$, then $(-) \otimes_{\mathcal{O}_X} \omega_{X/T}[d]$ is a Serre functor. Conversely, if S_T is a T-Serre functor, one has

$$\operatorname{Hom}(Q, S_T(P)) = \operatorname{Hom}(\mathbb{R}\mathcal{H}\mathrm{om}^\bullet_T(P,Q), \mathcal{O}_T) = \operatorname{Hom}\left(Q, P \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_X} D_{X/T}\right).$$

Hence, the identity $S_T(P) \to S_T(P)$ gives a morphism

$$S_T(P) \to P \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_X} D_{X/T}.$$

One proves as in Proposition 1.5 that $D_{X/T} \simeq \omega_{X/T}[d]$. Finally, let us see that if X and T are Cohen–Macaulay, then $d = \dim X - \dim T$. Let $x \in X$ be a closed point and let $t = f(x)$. One has

$$\operatorname{Hom}^i(k(x), \mathcal{O}_X) = \operatorname{Hom}^i(k(x), \omega_{X/T}) = \operatorname{Hom}^i(k(x), D_{X/T}[-d]) = \operatorname{Hom}^{i-d}(k(t), \mathcal{O}_T).$$

Hence, $\operatorname{depth}\mathcal{O}_{X,x} = d + \operatorname{depth}\mathcal{O}_{T,t}$. Since X and T are Cohen–Macaulay (and equidimensional), $\dim X = \operatorname{depth}\mathcal{O}_{X,x} = d + \operatorname{depth}\mathcal{O}_{T,t} = d + \dim T$. □

Proposition 2.7. *One has a natural isomorphism*

$$\mathbb{R}\mathcal{H}om^\bullet_T(P,Q) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om^\bullet_T(S_T(P), S_T(Q)).$$

Proof. Indeed,

$$\mathbb{R}\mathcal{H}om^\bullet_T(P,Q) \simeq \mathbb{R}\mathcal{H}om^\bullet_T(P,Q)^{\vee\vee} \simeq \mathbb{R}\mathcal{H}om^\bullet_T(Q,S_T(P))^\vee \simeq \mathbb{R}\mathcal{H}om^\bullet_T(S_T(P), S_T(Q)).$$

□

2.4. Reconstruction of a T-scheme from the T-linear category of $D_{\mathrm{perf}}(X)$

Now we shall show that a T-scheme X can be uniquely reconstructed from the T-linear category $D_{\mathrm{perf}}(X)$, provided that X is Gorenstein and the relative dualizing sheaf $\omega_{X/T}$ is either T-ample or T-antiample.

First we proceed to reconstruct Gorenstein 0-cycles of X from the T-linear structure of $D_{\mathrm{perf}}(X)$.

Definition 2.8. Let $\mathcal{D}$ be a T-linear category. A 0-cycle Gorenstein object $P \in \mathcal{D}$ of relative codimension s is an object satisfying the following:

1. $S_T(P) \simeq P[s]$;
2. $\mathrm{Hom}^{<0}_{\mathcal{D}}(P,P) = 0$;
3. $\mathcal{H}om^0_T(P,P)$ is supported at a closed point $t \in T$;
4. $\mathrm{Hom}^0_{\mathcal{D}}(P,P)$ is a commutative and local algebra;
5. $\mathrm{Hom}^{s+\dim T}_{\mathcal{D}}(P,P)$ is a monogenerated right $\mathcal{O}_P$-module.

Proposition 2.9. *Let X be a Gorenstein scheme, let $f\colon X \to T$ be a proper morphism of finite Tor-dimension. Let $D_{X/T} = f^!\mathcal{O}_T$ be the relative dualizing complex. Assume that $D_{X/T} \simeq \omega_{X/T}[n]$, $n = \dim X - \dim T$, $\omega_{X/T}$, an invertible sheaf which is either T-ample or T-antiample. Then an object $P \in D_{\mathrm{perf}}(X)$ is a Gorenstein 0-cycle object if and only if $P \simeq \mathcal{O}_{Z_x}[r]$, $r \in \mathbb{Z}$, is isomorphic (up to translation) to the structure sheaf of a Gorenstein 0-cycle supported at a closed point $x \in X$.*

Proof. $S_T(P) \simeq P[s]$ implies that $s = \dim X - \dim T$ and that the support of P is finite over T. Condition 3 of Definition 2.8 implies that P is supported at the fibre of t. Hence, the support of P is a finite number of closed points on the fibre of s. The rest of the proof is like the absolute case (Proposition 1.10). □

Once we have identified Gorenstein 0-cycles of X, Proposition 1.13 identifies invertible sheaves. Now we can proceed as in the absolute case to give a relative reconstruction theorem.

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Theorem 2.10. *Let X be a Gorenstein scheme, let $f\colon X \to T$ be a proper morphism of finite Tor-dimension and let T be a Cohen–Macaulay scheme. Let $D_{X/T}$ be the relative dualizing complex. Assume that $D_{X/T} \simeq \omega_{X/T}[n]$, $n \in \mathbb{Z}$, where $\omega_{X/T}$ is an invertible sheaf which is either T-ample or T-antiample. If $X' \to T$ is another Gorenstein T-scheme and one has a T-linear equivalence*

$$D_{\mathrm{perf}}(X) \xrightarrow{\sim} D_{\mathrm{perf}}(X'),$$

then $X \simeq X'$ (an isomorphism of T-schemes).

Proof. We follow the same steps as in the proof of Theorem 1.15. We mention the necessary changes to be made.

Step 1. The same arguments as in the proof of Theorem 1.15 give an identification $\mathcal{G}_X = \mathcal{G} = \mathcal{G}_{X'}$ (note that X' is assumed to be Gorenstein).

Step 2. This remains unchanged.

Step 3. We obtain again an identification $\mathcal{P}_X = \mathcal{P} = \mathcal{P}_{X'}$. Let us denote by $\mathcal{P}_T$ the set of closed points of T. We have a map

$$\begin{aligned} f_X\colon \mathcal{P}_X &\to \mathcal{P}_T, \\ [P] &\mapsto \operatorname{supp}(\mathcal{H}om^0_T(P,P)). \end{aligned}$$

Analogously for X', $f_{X'}\colon \mathcal{P}_{X'} \to \mathcal{P}_T$. The identification $\mathcal{P}_X = \mathcal{P}_{X'}$ is compatible with f_X and $f_{X'}$. We denote $f\colon \mathcal{P}_{\mathcal{D}} \to \mathcal{P}_T$.

Step 4. One again obtains $l_X = l_{\mathcal{D}} = l_{X'}$. For each affine open subset V of T and each $\alpha \in \Gamma(V, \mathcal{H}om_T(L_1, L_2))$, we define

$$\begin{aligned} W_{(V,\alpha)} = \{P \in \mathcal{G}_{\mathcal{D}} \colon & f([P]) \in V \\ & \text{and } \alpha^*_P\colon \mathcal{H}om_T(L_2,P)|_V \to \mathcal{H}om_T(L_1,P)|_V \text{ is an isomorphism}\} \end{aligned}$$

and $U_{(V,\alpha)} = W_{(V,\alpha)}/{\sim}$, which is contained in $\mathcal{P}_{\mathcal{D}}$. These $U_{(V,\alpha)}$, where α runs over all elements in $\Gamma(V, \mathcal{H}om_T(L_1, L_2))$, L_1 and L_2 run over all elements in $l_{\mathcal{D}}$ and V runs over all affine open subsets of T, constitute a basis for the Zariski topologies of both X and X'.

Step 5. Since codimensions of all Gorenstein 0-cycle objects are equal to the relative dimensions of X and X', we have $\dim X = \dim X'$. Twisting by the relative canonical sheaf on X and X' induce equal transformation on the set $l_{\mathcal{D}}$. Let $L_i = S^i_T L_0[-ni]$, i.e. $\{L_i\}$ is the orbit of L_0 with respect to twisting by the relative canonical sheaf on X or X'. Then, since $\omega_{X/T}$ is either (relatively) ample or antiample, the set of all $U_{(V,\alpha)}$, where α runs over all elements in $\Gamma(V, \mathcal{H}om_T(L_i, L_j))$, $i, j \in \mathbb{Z}$, is a basis for the Zariski topology on $X_V = f_X^{-1}(V)$ for sufficiently small V, and hence on X'_V. This means that $\omega_{X'/T}$ is also (relatively) ample or antiample.

By Proposition 2.7, for all pairs (i,j) there are natural isomorphisms

$$\mathcal{H}om_T(L_i, L_j) \simeq \mathcal{H}om_T(L_0, L_{j-i}),$$

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which induce a ring structure in the graded $\mathcal{O}_T$-algebra $\mathcal{A}$ with graded components $\mathcal{A}_i = \mathcal{H}\mathrm{om}_T(L_0, L_i)$. If we denote by $\mathcal{B}$ (respectively, $\mathcal{B}'$) the graded algebra with graded components $\mathcal{B}_i = \mathcal{H}\mathrm{om}_T(\mathcal{O}_X, \omega_{X/T}^{\otimes i}) \simeq f_{X*}\omega_{X/T}^{\otimes i}$ (respectively, $\mathcal{B}'_i = \mathcal{H}\mathrm{om}_T(\mathcal{O}_{X'}, \omega_{X'/T}^{\otimes i}) \simeq f_{X'*}\omega_{X'/T}^{\otimes i}$), we have isomorphisms of graded $\mathcal{O}_T$-algebras

$$\mathcal{B} \simeq \mathcal{A} \simeq \mathcal{B}'$$

and then

$$X \xrightarrow{\sim} \operatorname{Proj}\mathcal{B} \xrightarrow{\sim} \operatorname{Proj}\mathcal{B}' \xrightarrow{\sim} X',$$

an isomorphism of T-schemes. □

Remark 2.11. If T is a Gorenstein and proper scheme over a field k, then the Gorenstein assumption on X' is unnecessary. Indeed, the transitivity of the dualizing complex gives an isomorphism

$$D_{X'/k} \simeq D_{X'/T} \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_{X'}} \boldsymbol{L}g^* D_{T/k},$$

with $g\colon X' \to T$ the structure morphism. Now $D_{X'/T}$ is an invertible sheaf (up to translations) because $D_{\mathrm{perf}}(X')$ has a T-Serre functor, following the equivalence $D_{\mathrm{perf}}(X) \simeq D_{\mathrm{perf}}(X')$. Moreover, $D_{T/k}$ is an invertible sheaf because T is Gorenstein. Hence, $D_{X'/k}$ is an invertible sheaf, i.e. X' is Gorenstein.

Assume now that T is either affine or regular. One can easily adapt the methods of [**7**] to obtain the group of exact T-linear autoequivalences $D_c^b(X) \to D_c^b(X)$, as follows.

Corollary 2.12. *Let $X \to T$ be as in the preceding theorem and assume that T is either affine or regular. Then the group $\operatorname{Aut}_T D_c^b(X)$ of isomorphism classes of exact T-linear autoequivalences $D_c^b(X) \to D_c^b(X)$ is the semi-direct product of its subgroups $\operatorname{Pic} X \oplus \mathbb{Z}$ and $\operatorname{Aut}_T X$, $\mathbb{Z}$ being generated by the translation functor:*

$$\operatorname{Aut}_T D_c^b(X) \simeq \operatorname{Aut}_T X \ltimes (\operatorname{Pic} X \oplus \mathbb{Z}).$$

Acknowledgements. This work was supported by Research Projects MTM2006-04779 (MEC) and SA001A07 (JCYL).

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