5

Local Methods in the Theory of Twisted Sums

Just as there is a local theory of (quasi-) Banach spaces, nowadays an essential part of (quasi-) Banach space theory, we also have a local theory of extensions of (quasi-) Banach spaces. In this chapter we explain what it means and how it can be used. Following the usage of Banach space theory (and in contrast with its usage in most other areas in mathematics), 'local' refers to finite-dimensional objects. Accordingly, let us think about those exact sequences that split at the finite-dimensional level (i.e., *locally*). Such as? Such as the sequence $0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$, which may not split, but no matter which finite-dimensional subspace of ℓ_∞/c_0 one chooses, the resulting pullback sequence 2-splits thanks to Sobczyk's theorem (but not thanks to Sobczyk's theorem, really). And the same occurs no matter which finite-dimensional subspace of ℓ_∞ one adds to c_0 (this time due to Sobczyk's theorem, for real). The paramount examples of locally split sequences are those that involve $\mathscr{L}_\infty$ or $\mathscr{L}_1$ spaces, to the point that those classes can be characterised by the facts that all exact sequences of Banach spaces $0 \longrightarrow \mathscr{L}_\infty \longrightarrow \cdot \longrightarrow \cdot \longrightarrow 0$ and $0 \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \mathscr{L}_1 \longrightarrow 0$ split locally.

The material of the chapter is divided into three sections: The first contains the definition and characterisations of locally split sequences (including the just-mentioned characterisations of $\mathscr{L}_\infty$ and $\mathscr{L}_1$ spaces) and their connections with the extension and lifting of operators. The second presents the uniform boundedness theorem for exact sequences. The general form of a uniform boundedness principle is that 'if something happens then it happens uniformly'. In this case, *what* happens is

$$\operatorname{Ext}(X, Y) = 0,$$

and the meaning of *uniformly* depends on the way one interprets $\operatorname{Ext}(X, Y)$, leading to different formulations of the principle. In its plain Banach space form it says that if all extensions of X by Y split then there is a constant λ such

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that Y is λ-complemented in every enlargement Z such that $Z/Y \approx X$. In its not-so-intuitive quasilinear form it says that if all quasilinear maps $X \longrightarrow Y$ are trivial then their distances to linear maps can be controlled by the quasilinearity constant. Oddly, the plain Banach space form is the more difficult to prove, while obtaining the quasilinear form is really simple. There are other two forms: the projective and injective forms. All of them will be explained and relentlessly compared. From the applications side, the different forms of the principle will be used to show that $\operatorname{Ext}(X, Y) \neq 0$ implies that also $\operatorname{Ext}(X', Y') \neq 0$ when X' has the same local structure as X and Y' has the same local structure as Y, in a sense to be specified. For instance, X' could be an ultrapower of X or its bidual. From here one can easily obtain that $\operatorname{Ext}(A, B) \neq 0$ (or not) for many pairs X, Y of spaces, both classical and exotic, or that $\operatorname{Ext}(X, Y) \neq 0$ if X, Y are B-convex Banach spaces, or that whenever X is an $\mathscr{L}_\infty$-space and Y is an $\mathscr{L}_1$-space, $\operatorname{Ext}_{\mathbf{B}}(X, Y) \neq 0$.

The third section of the chapter studies concrete applications of the local approach to Banach spaces. In this study, the Maleficent role, that of a powerful uninvited guest, is played by the bounded approximation property, which, with its ability of decomposing infinite-dimensional objects into finite-dimensional pieces in a controlled way, allows obtaining global information when only local information is given to us. In this line, we present general forms of Lusky's result (kernels of projective presentations of BAP spaces have the BAP), its dual version of Figiel, Johnson and Pełczyński, solutions to the duality or ultrapower splitting problem and applications to the extension or lifting of operators, all of which casts new light on the often overlooked homological nature of the BAP.

5.1 Local Splitting

We want to consider short exact sequences with trivial behaviour at the finite-dimensional level. There are at least two ways to interpret 'finite-dimensional level' in this sentence: by looking at the middle twisted sum or, alternatively, by looking at the quotient space. The following result says that it makes no difference.

Lemma 5.1.1 *Let $0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0$ be a short exact sequence of quasi-Banach spaces. The following are equivalent:*

(i) *There is a constant λ such that if F is a finite-dimensional subspace of Z, then there is $\tau_F \in \mathfrak{L}(F, Y)$ with $\|\tau_F\| \leq \lambda$ such that $\jmath\, \tau_F(f) = f$ for every $f \in F \cap \jmath\,[Y]$.*

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(ii) *There is a constant μ such that if G is a finite-dimensional subspace of X then there is an operator $s_G\colon G \longrightarrow Z$ with $\|s_G\| \le \mu$ such that $\rho s_G = \mathbf{1}_G$.*

If the sequence is isometrically exact and Z is a p-Banach space then (ii) *implies* (i) *with $\lambda = (1+\mu^p)^{1/p}$, while* (i) *implies* (ii) *for any $\mu > (1+\lambda^p)^{1/p}$.*

Proof It suffices to prove the last statement. There is no loss of generality in assuming that Y is a subspace of Z and $X = Z/Y$, in which case the condition on τ_F becomes $\tau_F f = f$ for $f \in F \cap Y$. We prove (ii) $\Longrightarrow$ (i): given a finite-dimensional subspace $F \subset Z$, set $G = \rho[F]$ and let $s_G\colon G \longrightarrow Z$ be an operator with $\|s_G\| \le \mu$ such that $\rho s_G = \mathbf{1}_G$. The operator $\tau_F\colon F \longrightarrow Y$ given by $\tau_F(f) = f - s_G\rho(f)$ has norm at most $(1+\mu^p)^{1/p}$ and is the identity on $F \cap Y$. To prove (i) $\Longrightarrow$ (ii), suppose G is a finite-dimensional subspace of X. Fixing $\varepsilon > 0$, an elementary compactness argument shows that there is a finite-dimensional $F \subset \rho^{-1}[G] \subset Z$ such that for every $g \in G$, there is $f \in F$ with $\rho(f) = g$ and $\|f\| \le (1+\varepsilon)\|g\|$. The restriction of ρ to F allows us to identify G and $F/(F \cap Y)$ as sets and thus $\|g\|_G \le \|g\|_{F/(F\cap Y)} \le (1+\varepsilon)\|g\|$ for every $g \in G$. If $\tau_F\colon F \longrightarrow Y$ is the operator provided by (i), the map $f \mapsto f - \tau_F(f)$ is an operator $F \longrightarrow Y$ with norm at most $(1+\lambda^p)^{1/p}$ that vanishes on $F \cap Y$; so, it induces an operator $s_G\colon F/(F\cap Y) \longrightarrow Z$ such that $\rho s_G = \mathbf{1}_G$. Clearly, $\|s_G\colon F/(F\cap Y) \longrightarrow Z\| \le (1+\lambda^p)^{1/p}$, and so $\|s_G\colon G \longrightarrow Z\| \le (1+\varepsilon)(1+\lambda^p)^{1/p}$, as required. □

Definition 5.1.2 An exact sequence splits locally if it satisfies the equivalent conditions of the preceding lemma. A subspace $Y \subset Z$ is locally complemented when the sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow Z/Y \longrightarrow 0$ splits locally.

The operators τ_F appearing in Lemma 5.1.1 are called local projections; the operators s_G will be called local sections. The constant λ appearing in Lemma 5.1.1 (i) shall be referred to as the local splitting constant of the sequence or the local complementation constant of the subspace, depending on the context, and it is clear what should be understood by a locally λ-split sequence and a locally λ-complemented subspace. The constant μ has no specific name. Local splitting can also be described neatly in quasilinear terms.

Lemma 5.1.3 *An exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus_\Phi X \longrightarrow X \longrightarrow 0$ splits locally if and only if there is a constant M such that for every finite-dimensional $G \subset X$ there is a linear map $L\colon G \longrightarrow Y$ such that $\|\Phi|_G - L\| \le M$.*

The reader should not have any difficulty in writing down a complete proof just following a simple pattern of quiet and curiosity. It is clear that complemented subspaces are locally complemented and that 'to be a locally

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complemented subspace' is a transitive relation. Natural situations in which local complementation appears include:

Proposition 5.1.4

(a) *Every Banach space is locally 1^+-complemented in its bidual.*
(b) *Every p-Banach space is locally 1^+-complemented in its ultrapowers.*
(c) *Given an ultrafilter $\mathfrak{U}$ on a set I, the space $c_0^{\mathfrak{U}}(I, X_i)$ is locally complemented in $\ell_\infty(I, X_i)$.*
(d) *If Y is a locally complemented subspace of a p-Banach space Z then Z/Y is isomorphic to a locally complemented subspace of some ultrapower of Z.*
(e) *The Pełczyński–Lusky sequence $0 \longrightarrow c_0(\mathbb{N}, X_n) \longrightarrow c(\mathbb{N}, X_n) \longrightarrow X \longrightarrow 0$ splits locally when (X_n) an increasing sequence of finite-dimensional subspaces of the p-Banach space $X = \overline{\bigcup_n X_n}$.*

Proof (a) is contained in the principle of local reflexivity. To check (b), let X be a p-Banach space , let $\delta\colon X \longrightarrow X_{\mathfrak{U}}$ be the canonical inclusion and let F be a finite-dimensional subspace of $X_{\mathfrak{U}}$. Choose $x_1, \ldots, x_m \in X$ and $f^1, \ldots, f^n \in F$ such that $\delta(x_1), \ldots, \delta(x_m)$ is a basis of $F \cap \delta[X]$ and $\delta(x_1), \ldots, \delta(x_m), f^1, \ldots, f^n$ is a basis of F. For each $1 \le k \le n$, we fix a bounded family $(f_i^k)_i$ in X such that $[(f_i^k)] = f^k$. Now, for each $i \in I$, we define an operator $T_i\colon F \longrightarrow X$ taking $T_i(\delta(x_j)) = x_j$ for $1 \le j \le m$ and $T_i(f^k) = f_i^k$ for $1 \le k \le n$. Clearly, each T_i is a local projection in the sense that $\delta T_i = \delta$ on $\delta[X] \cap F$ and $\|T_i\| \to 1$ along $\mathfrak{U}$. Therefore, for each fixed $\varepsilon > 0$, there are (many) $i \in I$ such that $\|T_i\| \le 1 + \varepsilon$. To prove (c), we will find local sections for the quotient map of the sequence

$$0 \longrightarrow c_0^{\mathfrak{U}}(I, X_i) \longrightarrow \ell_\infty(I, X_i) \xrightarrow{[\cdot]} [X_i]_{\mathfrak{U}} \longrightarrow 0$$

Let G be a finite-dimensional subspace of $[X_i]_{\mathfrak{U}}$ and let $g^1, \ldots, g^n$ be a basis of G. For each $1 \le k \le n$, let (g_i^k) be a representative of g^k. For each $i \in I$, we define an operator $u_i\colon G \longrightarrow X_i$ by setting $u_i(g) = g_i^k$ and extending linearly on the rest of G. The definition of the ultrapower p-norm guarantees that for each $\varepsilon > 0$, the set $I_\varepsilon = \{i \in I\colon u_i \text{ is an } \varepsilon\text{-isometry from } G \text{ into } X_i\}$ belongs to $\mathfrak{U}$. If we fix $\varepsilon \in (0, 1)$ and define $s_G\colon G \longrightarrow \ell_\infty(I, X_i)$ as $s_G(f)(i) = u_i(f)1_{I_\varepsilon}(i)$ then $[\cdot] \circ s_G = \mathbf{1}_G$ and $\|s_G\| \le 1 + \varepsilon$. To obtain (d), put $X = Z/Y$ and let $\pi\colon Z \longrightarrow X$ be the natural quotient map. For each $G \in \mathscr{F}(X)$, let $s_G\colon G \longrightarrow Z$ be a local section with $\|s_G\| \le \mu$. Let $\mathfrak{U}$ be an ultrafilter refining the order filter on $\mathscr{F}(X)$ and define $S : X \longrightarrow Z_{\mathfrak{U}}$ by $S(x) = [s_G(x1_G(x))]$; here 1_G is the characteristic function of G so that the Gth entry of the family that defines $S(x)$ is $s_G(x)$ if $x \in G$, and zero otherwise. This S is well defined, is linear and satisfies $\|S\| \le \mu$; moreover, the composition $\pi_{\mathfrak{U}} S : X \longrightarrow Z_{\mathfrak{U}} \longrightarrow X_{\mathfrak{U}}$ agrees with the diagonal embedding, which implies that S is an embedding (actually $\|S(x)\| \ge \|x\|$).

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Finally, $S[X]$ is locally complemented in $Z_{\mathcal{U}}$ by (b). The proof of (e), which somehow is contained in that of (c), is simple: given a finite-rank operator $\tau\colon Y \longrightarrow X$, there is no loss of generality in assuming that $\tau[Y] \subset X_N$ for some N. The operator $T\colon Y \longrightarrow c(\mathbb{N}, X_n)$ given by $T(y) = (0, \dots, 0, \tau(y), \tau(y), \dots)$ provides an equal norm lifting of τ. □

When X is a Banach space, the dual sequence $0 \longrightarrow X^* \longrightarrow c(\mathbb{N}, X_n)^* \longrightarrow \ell_1(\mathbb{N}, X_n^*) \longrightarrow 0$ splits since the operator $s((x_n^*))(x_n) = \sum x_n^*(x_n)$ is a section of the quotient map. Thus, if only we could be now aware of Corollary 5.1.8, we would get an alternative proof for $p = 1$. The following simple result yields quantitative estimates connecting local and global splitting of exact sequences when either the subspace or the quotient space is finite-dimensional.

Lemma 5.1.5 *Let Z be a p-Banach space, Y a subspace of Z and $\varepsilon > 0$.*

(a) *If Y is locally λ-complemented in Z and Z/Y is finite-dimensional then Y is $((2 + \lambda^p)^{1/p} + \varepsilon)$-complemented in Z.*
(b) *If Y is finite-dimensional and the quotient map $Z \longrightarrow Z/Y$ has local sections bounded by μ then it has a section with norm at most $(2+\mu^p)^{1/p}+\varepsilon$.*

Proof We just prove (a) since (b) is analogous. By Lemma 5.1.1 the sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow Z/Y \longrightarrow 0$ has local (hence global) sections of norm $(1 + \lambda^p + \varepsilon^p)^{1/p}$ for every $\varepsilon > 0$. This yields a (global) projection of Z onto Y of norm at most $(2 + \lambda^p + \varepsilon^p)^{1/p}$, and the result follows. □

The stability properties of locally split sequences are simple: local splitting is preserved under isomorphisms of sequences, pullback and pushout. The first assertion is obvious and the other two are easy to check. Consider, for instance, a pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\alpha"'] & Z \arrow[r, "\rho"] \arrow[d, "\overline{\alpha}"'] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & \mathrm{PO} \arrow[r, "\overline{\rho}"] & X \arrow[r] & 0
\end{tikzcd}$$

If G is a finite-dimensional subspace of X and $s_G\colon G \longrightarrow Z$ is a local section for ρ then $\overline{\alpha} s_G\colon F \longrightarrow \mathrm{PO}$ is a local section for $\overline{\rho}$. As for pullback diagrams,

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u, "\underline{\beta}"] & B \arrow[r] \arrow[u, "\beta"'] & 0
\end{tikzcd}$$

if F is a finite-dimensional subspace of PB then the composition of $\underline{\beta}$ with a local projection $\tau\colon \underline{\beta}[F] \longrightarrow Y$ yields a local projection $\tau\underline{\beta}\colon F \longrightarrow Y$.

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Under what conditions does a locally split sequence split? There is a rather satisfactory answer to this question.

Proposition 5.1.6 *Every locally split sequence in which the subspace is an ultrasummand splits.*

Proof What we will actually prove is that every operator from a locally complemented subspace $Y \subset Z$ to an ultrasummand extends to Z. Let λ be the local complementation constant, let A be an ultrasummand and let $u\colon Y \longrightarrow A$ be an operator. For each finite-dimensional $F \subset Z$, let $\tau_F\colon F \longrightarrow Y$ be a local projection with $\|\tau_F\| \leq \lambda$. We define a mapping $\tilde{u}\colon Z \longrightarrow \ell_\infty(\mathscr{F}(Z), A)$ by $\tilde{u}(x)_F = u(\tau_F(x1_F(x)))$ that, most likely, is not linear. Pick an ultrafilter $\mathcal{U}$ refining the Fréchet filter on $\mathscr{F}(Z)$. Define a mapping $U\colon Z \longrightarrow A_{\mathcal{U}}$ as the composition $[\cdot] \circ \tilde{u}\colon Z \longrightarrow \ell_\infty(\mathscr{F}(Z), A) \longrightarrow A_{\mathcal{U}}$. This is indeed an operator: it is obviously homogeneous and bounded by λ. Besides, if $x_1, x_2 \in Z$, we have, as long as $F \in \mathscr{F}(Z)$ contains both x_1 and x_2,

$$\tilde{u}(x_1 + x_2)_F = u(\tau_F(x_1 + x_2)) = u\tau_F(x_1) + u\tau_F(x_2) = \tilde{u}(x_1)_F + \tilde{u}(x_2)_F,$$

hence U is additive. Moreover, the restriction of U to Y is just the composition of u with the diagonal embedding of A into $A_{\mathcal{U}}$. Thus, if $P\colon A_{\mathcal{U}} \longrightarrow A$ is a projection, then the composition PU is an extension of u to Z. □

In Banach spaces, the local splitting notion interacts particularly well with duality:

Lemma 5.1.7 *Let Y be a subspace of a Banach space Z. The following are equivalent:*

(i) *Y is locally λ^+-complemented in Z.*
(ii) *There is a linear extension operator $E\colon Y^* \longrightarrow Z^*$ such that $\|E\| \leq \lambda$.*
(iii) *The inclusion $Y \longrightarrow Y^{**}$ has a λ-extension to Z.*
(iv) *Y^{**} is λ-complemented in Z^{**}.*

Proof (i) $\Longrightarrow$ (ii) Assume Y is a locally λ^+-complemented subspace of Z. We introduce an order on $\mathscr{F}(Z) \times (0, \infty)$ by declaring $(G, \varepsilon) \leq (F, \delta)$ if $G \subset F$ and $\varepsilon \geq \delta$ and let $\mathcal{U}$ be an ultrafilter containing the order filter. Now, given $F \in \mathscr{F}(Z)$ and $\varepsilon > 0$, let us choose a local projection $P_{(F,\varepsilon)}\colon F \longrightarrow Y$ with $\|P_{(F,\varepsilon)}\| \leq \lambda + \varepsilon$ and define $E\colon Y^* \longrightarrow Z^*$ by $E(y^*)(x) = \lim_{\mathcal{U}(F,\varepsilon)} \langle y^*, P_{(F,\varepsilon)}(x)\rangle$. Such E is obviously linear, $E(y^*)$ extends y^* and $\|E\| \leq \lambda$. (ii) $\Longrightarrow$ (iii) The λ-extension is $E^*|_Z$. (iii) $\Longrightarrow$ (ii) If $T\colon Z \longrightarrow Y^{**}$ extends the inclusion $Y \longrightarrow Y^{**}$ then $T^*|_{Y^{**}}$ is an extension operator. (ii) $\Longrightarrow$ (iv) $E^*\colon Z^{**} \longrightarrow Y^{**}$ is a projection. (iv) $\Longrightarrow$ (i) is obvious: Y is locally 1^+-complemented in Y^{**}, which in turn is λ-complemented in Z^{**}. Hence Y is λ^+-complemented in Z^{**} and so in Z. □

The immediate consequence is:

Corollary 5.1.8 *A short exact sequence of Banach spaces splits locally if and only if the dual sequence splits if and only if the bidual sequence splits.*

A striking application is that in any pullback diagram of Banach spaces

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & X \arrow[r] & \mathscr{L}_\infty \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] & E \arrow[r] \arrow[u] & \ell_2(I) \arrow[r] \arrow[u, "\tau"'] & 0
\end{tikzcd} \tag{5.1}$$

the lower sequence splits locally because the lower sequence in the dual diagram does:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{L}_1 \arrow[r] \arrow[d, "\tau^*"'] & X^* \arrow[r] \arrow[d] & Y^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ell_2(I) \arrow[r] & E^* \arrow[r] & Y^* \arrow[r] & 0
\end{tikzcd}$$

Indeed, the operator τ^* is 2-summing by Grothendieck's theorem, and thus it extends anywhere. The next result reveals the operator ideal fabric out of which the local splitting notion is made.

Proposition 5.1.9 *Let Y be a subspace of a Banach space Z. The following are equivalent:*

(i) *Y is locally complemented in Z.*
(ii) *There is a constant $\lambda > 0$ such that every finite-rank operator $\tau\colon Y \longrightarrow E$ with values in a Banach space has an λ-extension to Z.*
(iii) *For every Banach space E, every approximable, compact or weakly compact operator $\tau\colon Y \longrightarrow E$ can be extended to an operator $T\colon Z \longrightarrow E$ of the same type.*

Proof The equivalence (i) $\Longleftrightarrow$ (ii) follows from the duality formula $\mathfrak{L}(A, B^*) = \mathfrak{L}(B, A^*)$. Thus, each finite-rank operator $v\colon E \longrightarrow Y^*$ corresponds to a finite-rank operator $u\colon Y \longrightarrow E^*$ in such a way that $\langle v(e), y\rangle = \langle u(y), e\rangle$ for every $e \in E$ and every $y \in Y$. Moreover, $V \in \mathfrak{L}(E, Z^*)$ is a lifting of v if and only if its mate $U \in \mathfrak{L}(Z, E^*)$ is an extension of u, as illustrated in the diagrams

$$\begin{tikzcd}
Y \arrow[rr, "\text{inclusion}"] \arrow[dr, "u=v^*|_Y"'] & & Z \arrow[dl, "U \text{ extends } u"] \\
& E^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y^* = Z^*/Y^\perp & & Z^* \arrow[ll, "\text{restriction}"'] \\
& E \arrow[ul, "v=u^*|_X"] \arrow[ur, "V \text{ lifts } v"'] &
\end{tikzcd}$$

If Y is locally complemented in Z then the dual sequence splits, and thus operators v lift 'uniformly' to Z^*, which means that operators u extend uniformly to

Z, which is (ii). Conversely, if (ii) holds then the dual sequence splits locally, hence it splits by Proposition 5.1.6, and Y is locally complemented in Z by the preceding corollary. (ii) implies the approximable assertion of (iii) by a limit process. We prove that (i) implies (iii) for compact operators: given a compact $\tau\colon Y \longrightarrow E$, for each finite-dimensional $F \subset Z$, we consider a 'local projection' $P_F\colon F \longrightarrow Y$ with $\|P_F\| \leq \lambda$ and the net of operators $\tau P_F\colon F \longrightarrow Y \longrightarrow E$. Pick an ultrafilter $\mathfrak{U}$ refining the order filter on $\mathscr{F}(Z)$ and form the operator $T(z) = \lim_{\mathfrak{U}(F)} \tau(P_F(z))$, taking advantage of the relative compactness of $\tau[\lambda B_Y]$. The proof for weakly compact τ is the same, using the weak topology instead of the norm topology. The reason no uniform bound appears in (iii) is that the operator ideals $\mathfrak{G}, \mathfrak{K}, \mathfrak{W}$ are closed in the operator norm; the fact that all operators in the ideal can be extended automatically entails the existence of a constant controlling the norm of the extension ... depending a priori on the target space E. To obtain a 'universal constant', consider, for instance, the case of compact operators. Let $\tau_i\colon Y \longrightarrow E_i$ be any family of compact operators indexed by I. Form the Banach space $E = \ell_1(I, E_i)$ and let λ be the associated constant so that each compact operator $\tau\colon Y \longrightarrow E$ has a compact λ-extension to Z. Since each τ_i can be seen as a compact operator in $\mathfrak{L}(Y, E)$, each τ_i has a compact λ-extension $T_i\colon Z \longrightarrow E$. Composition with the obvious projection $\pi_i\colon E \longrightarrow E_i$ does the trick. It is now clear that any version of (iii) implies (ii). The ideal $\mathfrak{F}$ is, however, not closed, and thus only the existence of a uniform bound for the extension guarantees local splitting. □

Ultrapowers of Exact Sequences

The connections between an exact sequence and its ultrapowers are like a spider's web: almost invisible, stronger than it seems and difficult to get rid of. One might shrewdly conjecture that ultrapowers of locally split sequences that split locally must split. But this is not a riddle in which only the sound of words matters: it is true that a sequence splits locally if and only if its ultrapowers do, but there are locally split sequences without trivial ultrapowers. Examples include any exact sequence $0 \longrightarrow \mathsf{G} \longrightarrow C(K) \longrightarrow \cdot \longrightarrow 0$, since no ultrapower of the Gurariy space G is complemented in a $\mathscr{C}$-space [22, Section 3.3.4] and because of the Foiaş–Singer sequence which remains non-trivial after applying ultrapowers (Lemma 10.5.5).

Locally Injective and Locally Projective Spaces

Local splitting suggests that injectivity and projectivity can also be localised.

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Definition 5.1.10 A p-Banach space Y is said to be locally injective if every exact sequence of p-Banach spaces $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow \cdot \longrightarrow 0$ splits locally.

It should be almost clear that every locally injective space Y is locally λ-injective for some λ, with the meaning that every isometrically exact sequence $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow \cdot \longrightarrow 0$ of p-Banach spaces λ-splits locally: the 'right-downwards' diagonal sequence mentioned in Section 2.15.3 allows us to obtain from sequences $0 \longrightarrow Y \longrightarrow Z_n \longrightarrow X_n \longrightarrow 0$ that do not n-split locally an exact sequence $0 \longrightarrow Y \longrightarrow \mathrm{PO} \longrightarrow \ell_p(\mathbb{N}, X_n) \longrightarrow 0$ that cannot, obviously, split locally, in contradiction with the local injectivity of Y. Locally injective spaces are easy to characterise but difficult to find. Easy to characterise:

Lemma 5.1.11 *A p-Banach space Y is locally λ-injective if and only if every operator $\tau\colon F \longrightarrow Y$ defined on a subspace of a finite-dimensional p-Banach space E has a λ-extension $T\colon E \longrightarrow Y$.*

Proof Suppose not. If E is finite-dimensional, the diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, "\tau"] & E \arrow[r] \arrow[d] \arrow[dl, dotted] & E/F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & \mathrm{PO} \arrow[r] & E/Y \arrow[r] & 0
\end{tikzcd}$$

conveys the idea that local λ-splitting of the lower sequence yields a λ-extension for τ. Conversely, if Y has the property in the statement and $E \subset Z$ is finite-dimensional, a λ-extension of the inclusion $Y \cap E \longrightarrow Y$ to E provides the required local projection. □

And difficult to find, at least for $p < 1$: we just obtained one in 2.13.1. The Banach space tree of locally injective spaces is much leafier thanks to the following two facts: (a) the existence of enough injectives and (b) that locally complemented subspaces of $\mathscr{L}_\infty$-spaces are $\mathscr{L}_\infty$-spaces, whose proof can be found in [331] (a streamlined version for (b) is in [22, Appendix 2]). One has:

Proposition 5.1.12 *A Banach space is an $\mathscr{L}_\infty$-space if and only if it is locally injective. In particular, an $\mathscr{L}_{\infty,\lambda}$-space X is $(2+\lambda)^+$-complemented in every superspace Z such that Z/X is finite-dimensional.*

Proof The 'if' part is an obvious consequence of the Hahn–Banach theorem. The other implication is clear since X has to be locally λ-complemented in some $\ell_\infty(I)$. The coda is a consequence of the Hahn–Banach argument above: $\mathscr{L}_{\infty,\lambda}$-spaces are locally λ-complemented in every superspace and therefore Lemma 5.1.5 applies. □

The projective case is much more controversial. Observe why:

Definition 5.1.13 A p-Banach space X is locally projective if every exact sequence of p-Banach spaces $0 \longrightarrow \cdot \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ splits locally. A p-Banach space X is finitely λ-projective if for every finite-dimensional space F and every subspace $E \subset F$, every operator $\tau : X \longrightarrow F/E$ has a lifting $T\colon X \longrightarrow F$ with $\|T\| \leq \lambda\|\tau\|$. We say that a p-Banach space is finitely projective if it is finitely λ-projective for some λ.

It is plain that locally projective implies finitely projective. What catches us unprepared is that any space with trivial dual is finitely projective. So it seems that the right notion to consider is local projectivity, which will lead us to Kalton's definition of $\mathscr{L}_p$-spaces for $0 < p \leq 1$ in the next section. We have already seen that $\mathscr{L}_\infty$-spaces are the locally injective Banach spaces, and thus it could not be strange that $\mathscr{L}_1$-spaces are the locally projective Banach spaces. But it is important to remark that this depends on a non-trivial fact from [331]: a Banach space is an $\mathscr{L}_\infty$-space if and only if its dual is an $\mathscr{L}_1$-space. With this in hand, the situation for Banach spaces becomes neat:

Proposition 5.1.14 *In the category of Banach spaces, the following are equivalent:*

(i) *X is locally projective.*
(ii) *X is finitely projective.*
(iii) *X is an $\mathscr{L}_1$-space.*

Proof Keep in mind the proof of Proposition 5.1.9: how the identity $\mathfrak{L}(A, B^*) = \mathfrak{L}(B, A^*)$ works, how an extension $U\colon F \longrightarrow X^*$ of an operator $u\colon E \longrightarrow X^*$ from a subspace $E \subset F$ of a finite-dimensional space corresponds to a lifting $V \in \mathfrak{L}(X, E^*)$ of the corresponding operator $v = u^*|_X \in \mathfrak{L}(X, F^*)$ and the diagrams

$$\begin{tikzcd} E \arrow[rr, "\text{inclusion}"] \arrow[dr, "u"'] & & F \arrow[dl, "U \text{ extends } u"] \\ & X^* & \end{tikzcd} \qquad \begin{tikzcd} E^* = F^*/E^\perp & & F^* \arrow[ll, "\text{restriction}"'] \\ & X \arrow[ul, "v=u^*|_X"] \arrow[ur, "V \text{ lifts } v"'] & \end{tikzcd}$$

This and Lemma 5.1.11 render almost obvious that X is finitely projective if and only if X^* is locally injective since $\langle U(f), z\rangle = \langle u(f), z\rangle$ for every $f \in E$ and every $z \in X$ if and only if $\langle V(z), f\rangle = \langle v(z), f\rangle$ for every $f \in E$ and every $z \in X$. Additionally, $\|u\| = \|v\|$ and $\|U\| = \|V\|$. □

We can now round off Lindenstrauss' lifting:

5.1.15 Lindenstrauss' lifting and its converse *A Banach space X is an $\mathscr{L}_1$-space if and only if* $\mathrm{Ext}_{\mathbf{B}}(X, U) = 0$ *for every ultrasummand U.*

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Proof The 'only if' part is Lindenstrauss' lifting. The converse is a combination of $\mathrm{Ext}_{\mathbf{B}}(X, A^*) = 0$ for every Banach space A; Corollary 4.4.5, which yields $\mathrm{Ext}_{\mathbf{B}}(A, X^*) = \mathrm{Ext}_{\mathbf{B}}(X, A^*) = 0$ for every Banach space A; and the fact that X^* is therefore injective. Thus, X^* is an $\mathscr{L}_\infty$-space and X must be an $\mathscr{L}_1$-space. □

And obtain the following companion to Proposition 5.1.12:

Proposition 5.1.16 *A Banach space X is an $\mathscr{L}_1$-space if and only if there is a constant μ such that every isometrically exact sequence of Banach spaces $0 \longrightarrow F \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ in which F is finite-dimensional admits a linear section bounded by μ.*

Proof Every short exact sequence of Banach spaces whose quotient is an $\mathscr{L}_{1,\mu}$-space has local sections of norm μ^+. Thus, the only if part is clear from Lemma 5.1.5 (b). To prove the other implication, we shall establish that X^* is an $\mathscr{L}_\infty$-space. If $0 \longrightarrow X^* \longrightarrow \cdot \longrightarrow G \longrightarrow 0$ is an isometrically exact sequence of Banach spaces with G finite-dimensional, let $0 \longrightarrow G^* \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ be the exact sequence arising from the formula $\mathrm{Ext}_{\mathbf{B}}(G, X^*) = \mathrm{Ext}_{\mathbf{B}}(X, G^*)$. Since the latter sequence admits a μ-section, the former admits a μ-projection. Hence, X^* is an $\mathscr{L}_\infty$-space and X is an $\mathscr{L}_1$-space. □

A final quantitative observation: if X is locally λ-complemented in, say, $\ell_\infty(I)$, its bidual X^{**} is λ-complemented in $\ell_\infty(I)^{**}$. But we only know that a λ-complemented subspace of a $\mathscr{C}$-space is an $\mathscr{L}_{\infty,\mu}$-space for some μ (not necessarily λ) (see [22, Lemma A.12]), except when $\lambda = 1^+$, in which case we know that a 1^+-complemented subspace of a 1-injective space is an $\mathscr{L}_{\infty,1^+}$-space; see [466]. The principle of local reflexivity then yields in all cases that if X^{**} is an $\mathscr{L}_{\infty,\mu}$-space then X is an $\mathscr{L}_{\infty,\mu^+}$-space. Thus, Lindenstrauss spaces, which are the $\mathscr{L}_{\infty,1^+}$-spaces, coincide with the locally 1^+-injective spaces.

For dessert, a few enjoyable consequences for 3-space properties:

Corollary 5.1.17 *Being an $\mathscr{L}_\infty$-space or an $\mathscr{L}_1$-space are* 3*-space properties of Banach spaces. Moreover, given an exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ of Banach spaces, if Y and Z are $\mathscr{L}_\infty$-spaces then so is X, and if Z and X are $\mathscr{L}_1$-spaces then so is Y.*

The two remaining cases? Both false: Bourgain's examples (2.9) and (2.10) show that Z, X can be both $\mathscr{L}_\infty$-spaces, but Y not, and that Y, Z can be both $\mathscr{L}_1$-spaces, and X not.

The $\mathscr{L}_p$-Spaces for $0 < p < 1$

We now study the locally projective p-Banach spaces. We know that the spaces $\ell_p(I)$ are locally projective; the next most likely candidates are the $L_p(\mu)$ spaces:

Lemma 5.1.18 *Let X be a p-Banach space. Assume that there is $\lambda \geq 1$ and a system of finite-dimensional subspaces $(X_i)_{i\in I}$, directed by inclusion, with $d(X_i, \ell_p^n) \leq \lambda$ for $n = \dim(X_i)$ and such that $X = \overline{\bigcup_{i\in I} X_i}$. Then X is locally λ^+-projective among p-Banach spaces.*

Proof We must check that every quotient map $\rho\colon Z \longrightarrow X$ from a p-Banach space Z has uniformly bounded local sections. By replacing the p-norm of Z by an equivalent one, if necessary, we can assume that ρ maps the open ball of Z onto that of X. Let $G \subset X$ be a finite-dimensional subspace. If $G \subset X_i$ for some $i \in I$ then there is a local section $s\colon G \longrightarrow Z$ with $\|s\| \leq \lambda$, as follows from the lifting property of ℓ_p^n and the hypothesis on X_i. Otherwise, a perturbation argument is needed. Let $g_1, \ldots, g_k$ be a normalised basis of G and take $C > 0$ such that $\|(c_n)\|_p \leq C\|\sum_{1\leq n\leq k} c_n g_n\|$ for all $c_k \in \mathbb{K}$. Fix $\varepsilon > 0$ and take X_i large enough so that for each $1 \leq n \leq k$, there is $f_n \in X_i$ such that $\|f_n - g_n\| < \varepsilon/C$. Let $F = [f_1, \ldots, f_k]$, and let $\ell \in \mathfrak{L}(F, Z)$ be a local section, with $\|\ell\| \leq \lambda$. For each $1 \leq n \leq k$, take $z_n \in Z$ such that $\rho(z_n) = g_n - f_n$ with $\|z_n\| < \varepsilon/C$ and define a linear map $s\colon G \longrightarrow Z$ by $sg_n = \ell f_n + z_n$. It is clear that $\rho s = \mathbf{1}_G$. It only remains to obtain a (uniform) bound for $\|s\|$. Take $g = \sum_{n\leq k} c_n g_n$, with $\|g\| \leq 1$. Then, if $f = \sum_{n\leq k} c_n f_n$, we have $\|g - f\|^p \leq \sum_n |c_n|^p \|f_n - g_n\|^p \leq \varepsilon^p$. In particular, $\|f\|^p \leq 1 + \varepsilon^p$. Hence

$$\|sg\|^p \leq \|Lf\|^p + \sum_n |c_n|^p \|z_n\|^p \leq \|L\|^p(1 + \varepsilon^p) + \varepsilon^p. \qquad \square$$

The spaces $L_p(\mu)$ satisfy the hypothesis of the lemma (think of the simple functions), and therefore they are locally projective in $p\mathbf{B}$. The same is true of their locally complemented subspaces:

Lemma 5.1.19 *Every locally complemented subspace of a locally projective space is locally projective.*

Proof Let Y be a locally complemented subspace of a locally projective p-Banach space Z. Let $\pi\colon \ell_p(I) \longrightarrow Z$ and $\varpi\colon \ell_p(J) \longrightarrow Y$ be quotient maps. The following diagram, whose lower rows are apparently unrelated,

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$$\begin{tikzcd}
0 \arrow[r] & \kappa_p(Z) \arrow[r] & \ell_p(I) \arrow[r, "\pi"] & Z \arrow[r] & 0 \\
0 \arrow[r] & \kappa_p(Z) \arrow[r] \arrow[u, equal] & \pi^{-1}[Y] \arrow[r, "\pi"] \arrow[u] & Y \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \kappa_p(Y) \arrow[r] & \ell_p(J) \arrow[r, "\varpi"] & Y \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

should be looked at very suspiciously, as though it were something sticky we stepped in on board the Nostromo. Our plan is then to show that the lowest row is a pushout of the middle one, which only requires that we lift π through ϖ. For each finite-dimensional $G \subset Z$, we pick a local section $s_G \in \mathfrak{L}(G, Y)$ with $\|s_G\| \leq M$ for some M independent of G. Let $\mathfrak{U}$ be an ultrafilter refining the order filter on $\mathscr{F}(Z)$. We form a diagram from the 'diagonal' embedding $\delta\colon Z \longrightarrow \mathscr{F}(Z)_{\mathfrak{U}}$ given by $\delta(z) = [z1_G(z)]$, the ultraproduct operators $[s_G]_{\mathfrak{U}}\colon \mathscr{F}(Z)_{\mathfrak{U}} \longrightarrow Y_{\mathfrak{U}}$ and $\varpi_{\mathfrak{U}}\colon (\ell_p(J))_{\mathfrak{U}} \longrightarrow Y_{\mathfrak{U}}$, a lifting $L\colon \ell_p(I) \longrightarrow Y_{\mathfrak{U}}$ of $[s_G]_{\mathfrak{U}}\delta$ through $\varpi_{\mathfrak{U}}$ that exists because $\ell_p(I)$ is projective plus the two unlabelled diagonal embeddings:

$$\begin{tikzcd}
& \ell_p(I) \arrow[r, "\pi"] \arrow[ddl, "L"'] & Z \arrow[dr, "\delta"] & \\
& \ell_p(J) \arrow[r, "\varpi"] \arrow[dl] & Y \arrow[u, "\text{inclusion}"'] \arrow[dr] & \mathscr{F}(Z)_{\mathfrak{U}} \arrow[d, "{[s_G]_{\mathfrak{U}}}"] \\
(\ell_p(J))_{\mathfrak{U}} \arrow[ur, dotted, bend left, "P"] \arrow[rrr, "\varpi_{\mathfrak{U}}"] & & & Y_{\mathfrak{U}}
\end{tikzcd}$$

Since $\ell_p(J)$ is an ultrasummand, there is a projection P along the diagonal embedding, and thus the restriction of PL to $\pi^{-1}[Y]$ yields the much sought map. □

And that is all:

Lemma 5.1.20 *Every locally projective p-Banach space is isomorphic to a locally complemented subpace of some $L_p(\mu)$.*

Proof This is an easy consequence of Proposition 5.1.4(d) once we are told that every ultrapower of $L_p(\mu)$ is isometric to some $L_p(\nu)$ [427, Proposition 3.3]. Thus, if X is locally projective, then it is locally complemented in an ultrapower of some $\ell_p(I)$ just considering a quotient map $\pi\colon \ell_p(I) \longrightarrow X$. □

We have thus obtained the following characterisation of local projectivity:

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Theorem 5.1.21 *In the category of p-Banach spaces, the following are equivalent:*

(i) *X is locally projective.*
(ii) *Some (all) projective presentations of X split locally.*
(iii) *X is isomorphic to a locally complemented subspace of some $L_p(\mu)$.*

Proof The implication (i) $\Longrightarrow$ (ii) is trivial; the converse, used implicitly in the proof of Lemma 5.1.19, follows from the fact that all extensions of X are pushouts of any projective presentation and that pushouts preserve the local triviality of sequences. (ii) $\Longrightarrow$ (iii) is Lemma 5.1.20. (iii) $\Longrightarrow$ (ii) is Lemma 5.1.18 plus Lemma 5.1.19. □

Statement (iii) corresponds to Kalton's definition of $\mathscr{L}_p$-spaces when $0 < p < 1$ in [255]. The definition also makes sense for $1 \leq p \leq \infty$ and would provide an alternative to the more popular definition of Lindenstrauss and Rosenthal, which, in turn, *could* be 'extended' to $0 < p < 1$. Both definitions are equivalent for $p = 1, 2, \infty$. However, when $p \in (1, \infty), p \neq 2$, a 'Kalton $\mathscr{L}_p$-space', would simply be a complemented subspace of some $L_p(\mu)$, while an $\mathscr{L}_p$-space is a complemented subspace of some $L_p(\mu)$ that is not isomorphic to an infinite-dimensional Hilbert space. The grafting of Lindenstrauss–Rosenthal's definition to Kalton's zone is more problematic since it seems to be unknown if even L_p satisfy it. Theorem 5.1.21 and Proposition 5.1.6 yield:

5.1.22 Lindenstrauss' p-lifting *Let* $p \in (0, 1]$. *For every p-Banach ultrasummand U and every $\mathscr{L}_p$-space X, we have* $\operatorname{Ext}_{p\mathbf{B}}(X, U) = 0$.

We do not know if this characterises the $\mathscr{L}_p$-spaces for $0 < p < 1$. It can be proved that a p-Banach space having the BAP is a (necessarily discrete, see below) $\mathscr{L}_p$-space if $\operatorname{Ext}_{p\mathbf{B}}(X, U) = 0$ for every p-Banach ultrasummand U.

$\mathscr{L}_p$-Subspaces of ℓ_p for $0 < p \leq 1$

In classifying $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces, one of the basic questions one can ask is how many isomorphism types there are. For $\mathscr{L}_\infty$-spaces, the first answer that occurs would be 'a lot', since there are uncountably many non-isomorphic separable $\mathscr{C}$-spaces based on ordinals, just for starters. A more thoughtful answer would then be 'a hell of a lot' since more and more exotic types of $\mathscr{L}_\infty$-spaces keep coming into life. The question for $\mathscr{L}_1$-spaces was posed by Lindenstrauss, who partially solved it in [327], showing that there are infinitely many. Johnson and Lindenstrauss showed later [226] that there is actually a continuum of non-isomorphic $\mathscr{L}_1$-subspaces of ℓ_1. The existence of infinitely many isomorphism types can be achieved through the following partial converse of the Lindenstrauss–Rosenthal theorem, due to Lindenstrauss:

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Lemma 5.1.23 *$\kappa(\mathscr{L}_1) \simeq \kappa(\mathscr{L}_1')$ if and only if $\mathscr{L}_1 \simeq \mathscr{L}_1'$.*

Proof If $\kappa(\mathscr{L}_1)$ and $\kappa(\mathscr{L}_1')$ are isomorphic, the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \kappa(\mathscr{L}_1) \arrow[r] \arrow[d, equal] & \ell_1 \arrow[r] & \mathscr{L}_1 \arrow[r] & 0 \\ 0 \arrow[r] & \kappa(\mathscr{L}_1') \arrow[r] & \ell_1 \arrow[r] & \mathscr{L}_1' \arrow[r] & 0 \end{tikzcd}$$

are semi-equivalent since $\mathrm{Ext}_{\mathbf{B}}(\mathscr{L}_1, \ell_1) = \mathrm{Ext}_{\mathbf{B}}(\mathscr{L}_1', \ell_1) = 0$. The diagonal principles therefore yield $\mathscr{L}_1 \times \ell_1 \simeq \mathscr{L}_1' \times \ell_1$. Since every $\mathscr{L}_1$-space contains ℓ_1 complemented, $\mathscr{L}_1 \simeq \mathscr{L}_1 \times \ell_1 \simeq \mathscr{L}_1' \times \ell_1 \simeq \mathscr{L}_1'$. □

Proposition 5.1.24 *There are infinitely many separable non-isomorphic $\mathscr{L}_1$-spaces.*

Proof An infinite sequence of non-isomorphic separable $\mathscr{L}_1$-spaces is given by $\kappa^n(L_1)$, where we inductively define $\kappa^{n+1}(X) = \kappa(\kappa^n(X))$ for $n \geq 1$. If $\kappa^m(L_1) \simeq \kappa^n(L_1)$ for $m > n$ then $\kappa^{m-n}(L_1) \simeq L_1$, which is impossible since L_1 is not a subspace of ℓ_1. □

The same idea works for $0 < p < 1$ and produces the family of (Kalton) $\mathscr{L}_p$-spaces mentioned after Corollary 5.3.5. Call an $\mathscr{L}_p$-space *discrete* when it has the BAP. The $\ell_p(I)$ spaces are prototypes, and since the kernels of projective presentations of $\mathscr{L}_p$-spaces are locally complemented and the BAP passes to locally complemented subspaces (obvious and explicitly stated in Proposition 5.3.2 (b)), one has:

Lemma 5.1.25 *Let $0 < p < 1$. If X is an $\mathscr{L}_p$-space and $\pi\colon \ell_p(I) \longrightarrow X$ is a quotient map then $\ker \pi$ is a discrete $\mathscr{L}_p$-space.*

Now, given a separable p-Banach space X, the isomorphism type of $\ell_p \times \kappa_p(X)$ is well defined by Corollary 2.7.4. So we can pretend to be working with the sequence $(\kappa_p^n(X))_{n\geq 1}$ of subspaces of ℓ_p defined by $\kappa_p^1(X) = \kappa_p(X)$ and $\kappa_p^{n+1}(X) = \kappa_p(\kappa_p^n(X))$. We have:

Proposition 5.1.26 *The spaces $\kappa_p^n(L_p)$ are pairwise non-isomorphic discrete $\mathscr{L}_p$-spaces.*

Indeed, observe that $\mathfrak{L}(L_p, \kappa_p^n(L_p)) = 0$ for all n and use the same argument as before, taking into account that $\mathrm{Ext}_{p\mathbf{B}}(\mathscr{L}_p, \ell_p) = 0$. In [255, Section 7], Kalton produces a continuum of mutually non-isomorphic $\mathscr{L}_p$-spaces with trivial dual and observes that they lead, as one might guess, to a continuum of non-isomorphic discrete $\mathscr{L}_p$-subspaces of ℓ_p.

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5.2 Uniform Boundedness Principles for Exact Sequences

We thus arrive at the second milestone of the chapter: the statement of uniform boundedness principles for exact sequences of (quasi-) Banach spaces. A number of results of the type 'if something happens then it happens uniformly' have already been encountered. In the case we are considering now, *what* happens is $\mathrm{Ext}(X, Y) = 0$. The first two possible interpretations of that fact directly follow from the open mapping theorem.

Projective form. If Ext is interpreted via projective presentations as $\mathrm{Ext}^{\mathrm{proj}}_{p\mathbf{B}}$ then $\mathrm{Ext}(X, Y) = 0$ means that once a projective presentation $0 \longrightarrow \kappa_p(X) \longrightarrow \mathcal{P} \longrightarrow X \longrightarrow 0$ has been chosen, all operators $\kappa_p(X) \longrightarrow Y$ extend to $\mathcal{P}$, and 'uniformly' in this context refers to the ratio between the norm of the extension and the norm of the operator.

Injective form. If Ext is interpreted via injective presentations as $\mathrm{Ext}^{\mathrm{inj}}_{\mathbf{B}}$ then $\mathrm{Ext}(X, Y) = 0$ means that once an injective presentation $0 \longrightarrow Y \longrightarrow \mathcal{J} \longrightarrow c\kappa(Y) \longrightarrow 0$ has been chosen, all operators $X \longrightarrow c\kappa(Y)$ lift to $\mathcal{J}$, and 'uniformly' refers to the ratio between the norm of the lifting and the norm of the operator.

Although clean and simple, these forms are not very manageable because, in general, one does not have explicit projective or injective presentations. Also, as explained in Section 4.4, while $\mathrm{Ext}^{\mathrm{proj}}$ behaves very well with pushouts, it does not with pullbacks; $\mathrm{Ext}^{\mathrm{inj}}$ exhibits the opposite behaviour. Understanding Ext in its basic form as the space of exact sequences makes $\mathrm{Ext}(X, Y) = 0$ mean that all sequences $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ split, and the principle should then say that all of them split 'uniformly'. And here the problematic point arises: it is necessary to find a way to measure the 'degree of exactness' and the 'degree of splitting' of the sequence, all of which will be done very soon in Lemmata 5.2.3 and 5.2.4. We thus pass to the:

Quasilinear form. Interpreting Ext in the naturally equivalent form Q_{LB}, we know by now that $\mathsf{Q}_{\mathsf{LB}}(X, Y) = 0$ means that each quasilinear map $\Phi\colon X \longrightarrow Y$ admits a linear map $L\colon X \longrightarrow Y$ at finite distance. Now 'uniformly' refers to the ratio between the distance $D(\Phi)$ to linear maps and the quasilinearity constant $Q(\Phi)$ of the map.

5.2.1 Uniform boundedness principle for quasilinear maps *Let X be a quasinormed space and Y be a quasi-Banach space. If every quasilinear map from X to Y is trivial then there is a constant K such that, whenever $\Phi\colon X \longrightarrow Y$ is a quasilinear map, there is a linear map $L\colon X \longrightarrow Y$ such that $\|\Phi - L\| \leq KQ(\Phi)$.*

The proof, Theorem 3.6.5, derives from the fact that the semiquasinorms $D(\cdot)$ and $Q(\cdot)$ on $\mathsf{Q}(X, Y)$ are comparable, become norms on $\mathsf{Q}_{\mathsf{L}}(X, Y)$ and make

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this space complete when all the elements of $\mathsf{Q}(X, Y)$ are trivial. The p-normed version was Theorem 3.6.8.

5.2.2 Uniform boundedness principle for p-linear maps *Let X be a p-normed space and Y a p-Banach space. If every p-linear map from X to Y is trivial then there is a constant K such that, whenever $\Phi\colon X \longrightarrow Y$ is a p-linear map, there is a linear map $L\colon X \longrightarrow Y$ such that $\|\Phi - L\| \le KQ^{(p)}(\Phi)$.*

Observe that the case $p = 1$ of 5.2.2 is definitely not 5.2.1. The bonuses accrued from working with quasilinear maps are the usual ones: both pullbacks and pushouts, as well as the vector space operations, are easy to handle. So let us move forward by focusing on applications, specialisations and so on of the quasilinear form of the uniform boundedness principle. To begin with, observe that Theorems 3.6.5 and 3.6.8 were followed by their quantifications through the parameters $K[\cdot,\cdot]$ and $K^{(p)}[\cdot,\cdot]$. To properly do the same with the uniform principles, we need to keep track of the concavity constants involved:

Lemma 5.2.3 *Let X, Y be quasi-Banach spaces such that $\operatorname{Ext}(X, Y) = 0$ and let $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ be an isometrically exact sequence. If Δ is the modulus of concavity of Z then for every $\varepsilon > 0$, there is a linear section $S : X \longrightarrow Z$ for the quotient map such that $\|S\| \le \Delta(1 + 2\Delta^2 K[X, Y]) + \varepsilon$ and a projection $P\colon Z \longrightarrow Y$ such that $\|P\| \le (1 + 2\Delta)\max(\Delta, 2\Delta^2 K[X, Y]) + \varepsilon$.*

Proof The proof is just a combination of two known results. Fix $\varepsilon > 0$ and observe that $\Delta_Y, \Delta_X \le \Delta$. Use Proposition 3.3.7 to obtain a quasilinear $\Phi\colon X \longrightarrow Y$ with $Q(\Phi) \le 2(1 + \varepsilon)\Delta^2$, yielding a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "\imath"] \arrow[d, equal] & Y \oplus_\Phi X \arrow[r, "\pi"] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "\jmath"] & Z \arrow[r, "\rho"] & X \arrow[r] & 0
\end{tikzcd}$$

with $\|u\| \le (1 + \varepsilon)\Delta$ and $\|u^{-1}\| \le (2 + \varepsilon)\Delta + 1$. Use now Lemma 3.3.4 to get that any linear map $L\colon X \longrightarrow Y$ at finite distance from Φ produces a section s of π with $\|s\colon X \longrightarrow Y \oplus_\Phi X\| \le 1 + \|\Phi - L\|$ as well as a projection $p\colon Y \oplus_\Phi X \longrightarrow Y$ with $\|p\| \le \max(\Delta, \|\Phi - L\|)$. Clearly, $S = us$ is a section of ρ, and $P = pu^{-1} : Z \longrightarrow Y$ is a projection along $\jmath$. Choosing L such that $\|\Phi - L\| \le K[X, Y]Q(\Phi)$, one obtains:

$$\|S\| \le \|u\|\,\|s\| \le (1 + \varepsilon)\Delta(1 + 2(1 + \varepsilon)\Delta^2 K[X, Y]);$$
$$\|P\| \le \|u^{-1}\|\,\|p\| \le (1 + (2 + \varepsilon)\Delta)\max(\Delta, 2(1 + \varepsilon)\Delta^2 K[X, Y]). \qquad \square$$

We now state and prove a version of the preceding result for p-Banach spaces where the concavity of the middle space is controlled through the

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Banach–Mazur distance: p-Banach spaces have modulus of concavity at most $2^{1/p-1}$, and thus a space at distance λ from a p-Banach space has modulus of concavity not larger than $\lambda 2^{1/p-1}$.

Lemma 5.2.4 *Let X, Y be p-Banach spaces. If every exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ in which Z is isomorphic to a p-Banach space splits, then there are increasing functions $\mu, \nu\colon [1, \infty) \longrightarrow \mathbb{R}$ such that every isometrically exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ of quasi-Banach spaces in which Z is λ-isomorphic to a p-Banach space admits a linear section S of the quotient map such that $\|S\| \leq \mu(\lambda)$ and a projection $P\colon Z \longrightarrow Y$ such that $\|P\| \leq \nu(\lambda)$.*

Proof The hypothesis implies that $K^{(p)}[X, Y]$ is finite. The proof now goes as before: if $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is isometrically exact and Φ is a quasilinear map obtained as the difference of a homogeneous section $B\colon X \longrightarrow Z$ with $\|B\| \leq 1 + \varepsilon$ and a linear section of the quotient map, then $Q^{(p)}(\Phi) = Q^{(p)}(B) \leq 2^{1/p}(1 + \varepsilon)\lambda$, provided Z is λ-isomorphic to a p-normed space, since in that case, for every finite set of points $z_1, \ldots, z_n \in Z$, we have $\left\|\sum_{i\leq n} z_i\right\|^p \leq \lambda^p \sum_{i\leq n} \|z_i\|^p$. Now, proceed as before, replacing $K[X, Y]$ by $K^{(p)}[X, Y]$ and $Q(\Phi)$ by $Q^{(p)}(\Phi)$, taking into account that $\Delta_Z \leq \lambda 2^{1/p-1}$ when necessary. □

The assumption on the Banach–Mazur distance is necessary by the following argument. On one hand, $\mathrm{Ext}_{\mathbf{B}}(\ell_1, \mathbb{K}) = 0$. On the other hand, using the n-dimensional versions $\varrho_n\colon \ell_1^n \longrightarrow \mathbb{K}$ of Ribe's map (Proposition 3.2.3), we obtain isometrically exact sequences $0 \longrightarrow \mathbb{K} \longrightarrow Z(\varrho_n) \longrightarrow \ell_1^n \longrightarrow 0$ in which $\Delta(Z(\varrho_n)) \leq 2$ and, since the sequences are necessarily trivial, $Z(\varrho_n) \simeq \mathbb{R} \oplus \ell_1^n$. However, $\|s\| \geq \frac{1}{2}\log n$ for any linear section. In particular, the Banach–Mazur distance between $Z(\rho_n)$ and the Banach spaces is larger than $\frac{1}{2}\log n$.

The Ext Form of the Vector-Valued Sobczyk Theorem

Now that we have the language, we can state (compare with 2.14.8):

5.2.5 Vector-valued Sobczyk theorem *Let X be a separable Banach space and let (E_n) be a sequence of Banach spaces. If $\mathrm{Ext}_{\mathbf{B}}(X, E_n) = 0$ uniformly on n then $\mathrm{Ext}_{\mathbf{B}}(X, c_0(\mathbb{N}, E_n)) = 0$ in either of the following situations:*

(a) *X has the BAP.*
(b) *The sequence (E_n) has the joint-UAP.*

The proof of (a) is just the proof of 2.14.8. And the same is true for (b) since the hypothesis forces $\ell_\infty(\mathbb{N}, E_n)/c_0(\mathbb{N}, E_n)$ to have the BAP: since the space

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$\ell_\infty(\mathbb{N}, E_n)$ has the λ-UAP and the sequence $0 \longrightarrow c_0(\mathbb{N}, E_n) \longrightarrow \ell_\infty(\mathbb{N}, E_n) \longrightarrow Q \longrightarrow 0$ splits locally, $Q^{**} \simeq \ell_\infty(\mathbb{N}, E_n)^{**}/c_0(\mathbb{N}, E_n)^{**}$ has the λ-UAP, as well as Q. Observe that everything in this plot works because of the UAP and falls flat for the mere BAP; the Pełczyński–Lusky sequence explains why. A remarkable instance of 5.2.5 follows; the next section has another.

Corollary 5.2.6 *Let X be a separable Banach space. If (E_n) is a sequence of $\mathscr{L}_{\infty,\lambda}$-spaces and* $\operatorname{Ext}_{\mathbf{B}}(X, E_n) = 0$ *uniformly on n then* $\operatorname{Ext}_{\mathbf{B}}(X, c_0(\mathbb{N}, E_n)) = 0$. *In particular, if each E_n is λ-separably injective then $c_0(\mathbb{N}, E_n)$ is separably injective.*

Why doesn't the BAP occur in this formulation? Well, the long version of the explanation is in Section 10.1. The short version is: because it is! Admittedly, it is disguised by Propositions 5.3.1 and 4.2.10: every separable Banach space is a twisted sum of two spaces with the BAP and, therefore, a space X is separably injective if and only if $\operatorname{Ext}_{\mathbf{B}}(S, X) = 0$ for all separable spaces S with the BAP. Now apply the proof 2.14.8 to those BAP spaces. Quantitative estimates are not so easy to obtain because of the required BAP decomposition. The result that $c_0(\mathbb{N}, E_n)$ is $f(\lambda)$-separably injective when all the spaces E_n are λ-separably injective has been independently obtained by Rosenthal [416] using operator techniques, by Johnson and Oikhberg [230], using M-ideals, by Cabello Sánchez [61] using a topological approach and by Castillo and Moreno [114] with non-linear techniques. Each of them comes with its own estimate for $f(\lambda)$: Rosenthal obtains $f(\lambda) = \lambda(1+\lambda)^+$, Johnson and Oikhberg get $f(\lambda) = 2\lambda^2$ and Cabello Sánchez obtains $f(\lambda) = 3\lambda^2$, while Castillo and Moreno get $f(\lambda) = 6\lambda^+$.

$\mathscr{L}_\infty$ and $\mathscr{L}_p$ Spaces, $0 < p \leq 1$ Revisited

The second instance of 5.2.5 is as follows:

Corollary 5.2.7 *If X is an $\mathscr{L}_1$-space and all E_n are μ-complemented in their biduals then* $\operatorname{Ext}_{\mathbf{B}}(X, c_0(\mathbb{N}, E_n)) = 0$.

There is also a version for extension of operators:

Corollary 5.2.8 *Let Y be a subspace of the Banach space Z such that Z/Y is an $\mathscr{L}_{1,\lambda}$-space and let E_n be Banach spaces μ-complemented in their biduals. Any operator $\tau\colon Y \longrightarrow c_0(\mathbb{N}, E_n)$ admits a $\lambda\mu(1+\lambda)$-extension to Z.*

Proof The proof of Theorem 3.7.1 in combination with Lemma 3.5.4 says that the components $Y \longrightarrow E_n$ have $\lambda\mu$-extensions to Z. Corollary 2.14.7 yields that τ admits a $\lambda\mu(1+\lambda)$-extension. □

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Combining this with the decomposition provided by Proposition 5.3.1 (every subspace of c_0 is a twisted sum of two spaces $c_0(\mathbb{N}, F_n)$ with all F_n finite-dimensional) plus the fact that, for fixed X, properties of the form $\operatorname{Ext}(X, \cdot) = 0$ are 3-space properties, yields part (a) in the following:

Proposition 5.2.9

(a) *A separable Banach space X is an $\mathscr{L}_1$-space if and only if* $\operatorname{Ext}_{\mathbf{B}}(X, H) = 0$ *for every subspace H of c_0.*
(b) *A Banach space X is an $\mathscr{L}_\infty$-space if and only if* $\operatorname{Ext}_{\mathbf{B}}(H^*, X) = 0$ *for every subspace $H \subset c_0$.*

Proof Part (b) follows by duality: the dual of a subspace of c_0 is a twisted sum of two spaces having the form $\ell_1(\mathbb{N}, F_n)$ with F_n finite-dimensional and, consequently, that it is enough to prove that $\operatorname{Ext}_{\mathbf{B}}(\ell_1(\mathbb{N}, F_n), X) = 0$. This amounts to saying that all exact sequences $0 \longrightarrow X \longrightarrow \cdot \longrightarrow F \longrightarrow 0$ in which F is finite-dimensional split uniformly, which is precisely the characterisation of $\mathscr{L}_\infty$-spaces given in Proposition 5.1.12. □

One might wonder about minimal classes $\mathscr{V}$ of Banach spaces that can replace the class of subspaces of c_0 in the characterisation (a) of $\mathscr{L}_1$-spaces above: X is an $\mathscr{L}_1$-space if and only if $\operatorname{Ext}_{\mathbf{B}}(X, V) = 0$ for all $V \in \mathscr{V}$. Let us show that the class of reflexive spaces is a valid choice:

Proposition 5.2.10 *A Banach space X is an $\mathscr{L}_1$-space if and only if* $\operatorname{Ext}_{\mathbf{B}}(X, C_r^{(1)}) = 0$ *for some (all)* $1 \le r \le \infty$ *or* $r = 0$.

Proof Lindenstrauss' lifting 5.1.15 yields one implication since the spaces $C_r^{(1)}$ are ultrasummands for all $1 \le r < \infty$. The case $r = \infty$ merely rewrites Proposition 5.1.16 (b). As for the other implication, since $C_r^{(1)}$ contains isometric 1-complemented copies of all the F_n and (F_n) is dense in $\mathscr{F}^{(1)}$, we have

$$\sup_{F \in \mathscr{F}^{(1)}} K^{(1)}[X, F] = \sup_n K^{(1)}[X, F_n] \le K^{(1)}[X, C_2^{(1)}] < \infty.$$

To conclude, invoke Proposition 5.1.16 (b) again. All this proves the case $r \neq 0$. For $r = 0$, use the same argument combined with Corollary 5.2.7. □

Even if $\operatorname{Ext}_{\mathbf{B}}(X, C_2^{(1)}) = 0$ implies that X is an $\mathscr{L}_1$-space, Hilbert spaces alone do not suffice since $\mathcal{B}^*$ has the KPP. Whether the class of super-reflexive spaces suffices is open and difficult. The p-versions of those results are simple, up to a point: $\operatorname{Ext}_{p\mathbf{B}}(L_p, C_q^{(r)}) = 0$ for all $p \le q, r \le \infty$ because ℓ_r is an r-Banach space ultrasummand when $r \le 1$, as well as $C_q^{(r)}$ (same proof). Readers who dare to go off-limits should inspect Section 10.1.

Extensions of Spaces with the Same Local Structure

We now present a technique to show that there exist non-trivial twisted sums of two spaces X, Y provided the existence of non-trivial twisted sums of other spaces X', Y' such that X and X' (resp. Y and Y') have the same local structure, in a sense to be determined next. Let $\mathscr{E}$ be a family of quasi-Banach spaces.

Definition 5.2.11 A quasi-Banach space X is said to be λ-locally $\mathscr{E}$ if every finite-dimensional subspace of X is contained in another finite-dimensional subspace $F \subset X$ such that $d(F, E) \leq \lambda$ for some $E \in \mathscr{E}$. We say that X is locally $\mathscr{E}$ if it is λ-locally $\mathscr{E}$ for some $\lambda \geq 1$. The space X is said to contain the class $\mathscr{E}$ uniformly (complemented) if there is λ such that every element of $\mathscr{E}$ is λ-isomorphic to some (λ-complemented) subspace of X.

We have already encountered examples of these notions: for instance, the $\mathscr{L}_p$-spaces are the locally ℓ_p^n spaces and the B-convex spaces contain ℓ_2^n uniformly complemented. Much more sophisticated is Bourgain's example [49]: the space $\ell_\infty(L_1)$ is locally $\ell_\infty^n(\ell_1^m)$. As for more general examples, we have:

Lemma 5.2.12 *Let X be a Banach space.*

(a) *If X^{**} is λ-locally $\mathscr{E}$ then X is λ^+-locally $\mathscr{E}$.*
(b) *Suppose $\mathscr{X}$ is a net of finite-dimensional subspaces of X whose union is dense in X. Then X is 1^+-locally $\mathscr{X}$.*

Proof (a) is obvious from the principle of local reflexivity. (b) seems obvious, but it is not: the complete proof can be found in Lacey [316, Theorem 6, p. 168] and uses local convexity in an essential way. □

Time to launch the idea of *uniform* splitting for families.

Definition 5.2.13 Given two families of quasi-Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, we write $\operatorname{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ to mean $\operatorname{Ext}(X, Y) = 0$ for every $X \in \mathscr{X}, Y \in \mathscr{Y}$. We shall say that $\operatorname{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ uniformly if

$$K[\mathscr{X}, \mathscr{Y}] = \sup\{K[X, Y] \colon X \in \mathscr{X}, Y \in \mathscr{Y}\} < \infty.$$

If $\mathscr{X}$ and $\mathscr{Y}$ consist of p-Banach spaces only, we write $\operatorname{Ext}_{p\mathbf{B}}(\mathscr{X}, \mathscr{Y}) = 0$ to mean $\operatorname{Ext}_{p\mathbf{B}}(X, Y) = 0$ for every $X \in \mathscr{X}, Y \in \mathscr{Y}$ and say $\operatorname{Ext}_{p\mathbf{B}}(\mathscr{X}, \mathscr{Y}) = 0$ uniformly if

$$K^{(p)}[\mathscr{X}, \mathscr{Y}] = \sup\{K^{(p)}[X, Y] : X \in \mathscr{X}, Y \in \mathscr{Y}\} < \infty.$$

The uniform splitting is much stronger than $\operatorname{Ext}(\mathscr{X}, \mathscr{Y}) = 0$. For instance, $\operatorname{Ext}(\mathscr{F}, \mathscr{F}) = 0$ rather obviously, but the splitting is not uniform by far. It is clear that to get $\operatorname{Ext}(\mathscr{X}, Y) = 0$ uniformly, it is sufficient that $\operatorname{Ext}(X, Y) = 0$

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for some space X containing $\mathscr{X}$ be uniformly complemented, and the same for the other variable. Now, Lemma 5.2.3 forces uniform splitting for families to behave well; that is, assertion (a) below holds:

Proposition 5.2.14 *Let $\mathscr{X}, \mathscr{Y}$ be families of quasi-Banach spaces and let $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ be an isometrically exact sequence with $X \in \mathscr{X}, Y \in \mathscr{Y}$ in which Z has modulus of concavity Δ_Z, and let $\varepsilon > 0$.*

(a) *If* $\operatorname{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ *uniformly then there is a linear section $S : X \longrightarrow Z$ of the quotient map such that $\|S\| \leq \Delta(1+2\Delta^2 K[\mathscr{X}, \mathscr{Y}])+\varepsilon$ and a projection $P\colon Z \longrightarrow Y$ such that $\|P\| \leq (1 + 2\Delta) \max(\Delta, 2\Delta^2 K[\mathscr{X}, \mathscr{Y}]) + \varepsilon$.*
(b) *If the families $\mathscr{X}, \mathscr{Y}$ have uniformly bounded moduli of concavity and there is a function f such that for every exact sequence as above there is a linear section $S : X \longrightarrow Z$ such that $\|S\| \leq f(\Delta_Z)$ (or a linear projection $P\colon Z \longrightarrow Y$ such that $\|P\| \leq f(\Delta_Z)$) then* $\operatorname{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ *uniformly.*
(c) *If $\mathscr{X}, \mathscr{Y}$ are formed by p-Banach spaces then* $\operatorname{Ext}_{p\mathbf{B}}(X, Y) = 0$ *uniformly if and only if there is a function f such that whenever $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is an isometrically exact sequence in which Z is λ-isomorphic to a p-Banach space, there is a projection $P\colon Z \longrightarrow Y$ (resp. a section $S : X \longrightarrow Z$) bounded by $f(\lambda)$.*

Proof We prove (b). Take $\Delta \geq \Delta_A$ for all A in $\mathscr{X}, \mathscr{Y}$, pick $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$ and let $\Phi\colon X \longrightarrow Y$ be a quasilinear map with $Q(\Phi) \leq 1$. Lemma 3.3.9 yields $\Delta_{Y \oplus_\Phi X} \leq 2\Delta^2$. By hypothesis, the sequence $0 \longrightarrow Y \longrightarrow Y \oplus_\Phi X \longrightarrow X \longrightarrow 0$ admits a linear section $S : X \longrightarrow Y \oplus_\Phi X$ with $\|S\| \leq f(\Delta)$ which necessarily has the form $S(x) = (L(x), x)$ for some linear map $L\colon X \longrightarrow Y$. Then, $\|S\| = \|\Phi - L\| + 1$, and thus $\|\Phi - L\| \leq f(\Delta)$, which yields $K[\mathscr{X}, \mathscr{Y}] \leq f(\Delta) < \infty$. □

One more estimate is necessary before practical results can be harvested:

Lemma 5.2.15 *Let X and Y be quasi-Banach spaces. If X' is λ-isomorphic to a λ'-complemented subspace of X and Y' is μ-isomorphic to a μ'-complemented subspace of Y, then $K[X', Y'] \leq \lambda\lambda'\mu\mu' K[X, Y]$. If all the spaces are p-Banach then one can replace $K[\cdot, \cdot]$ by $K^{(p)}[\cdot, \cdot]$.*

We thus discover an alleyway passing from an individual result to a uniformity result:

Proposition 5.2.16 *Assume that X contains the class $\mathscr{X}$ uniformly complemented and Y contains the class $\mathscr{Y}$ uniformly complemented. If* $\operatorname{Ext}(X, Y) = 0$ *then* $\operatorname{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ *uniformly. If X, Y are p-Banach spaces then one can replace* Ext *by* $\operatorname{Ext}_{p\mathbf{B}}$.

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In particular, since B-convex spaces contain ℓ_2^n uniformly complemented and $\sup_n K^{(1)}[\ell_2^n, \ell_2^n] = \infty$, we have:

Corollary 5.2.17 *If X, Y are infinite-dimensional B-convex Banach spaces then* $\mathrm{Ext}_{\mathbf{B}}(X, Y) \neq 0$.

Moving up the hill backwards, we now want to find a passage from $\mathrm{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ uniformly to $\mathrm{Ext}(X, Y) = 0$ for particular spaces X and Y:

Proposition 5.2.18 *Let X, Y be quasi-Banach spaces that are locally $\mathscr{X}$ and $\mathscr{Y}$, respectively. If* $\mathrm{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ *uniformly and Y is an ultrasummand then* $\mathrm{Ext}(X, Y) = 0$. *If $\mathscr{X}$ and $\mathscr{Y}$ consist of p-Banach spaces then one can replace* Ext *by* $\mathrm{Ext}_{p\mathbf{B}}$.

Proof It is easy to guess that the idea behind the proof is to show that every extension of X by Y splits locally and then use the hypothesis on the target space to guarantee global splitting. Assume without loss of generality that X and Y are p-normed spaces and let $\Phi\colon X \longrightarrow Y$ be a quasilinear map with $Q(\Phi) \leq 1$. Use Lemma 3.9.3 to get a family of quasilinear maps $\Phi_F : F \longrightarrow Y$, indexed by $F \in \mathscr{F}(X)$, such that $\Phi_F[F]$ spans a finite-dimensional subspace of Y and

$$\sup_{F \in \mathscr{F}(X)} \big(Q(\Phi_F), \|\Phi|_F - \Phi_F\|\big) < \infty.$$

Assume Y is λ-locally $\mathscr{Y}$. For each $F \in \mathscr{F}(X)$, the set $\Phi_F[F]$ lies inside a finite-dimensional subspace of Y that is λ-isomorphic to a certain $G \in \mathscr{Y}$. Since $K[\mathscr{F}(X), \mathscr{Y}] < \infty$, one can select, for each F, a linear map $L_F : F \longrightarrow Y$ such that $M = \sup_F \|\Phi_F - L_F\| < \infty$; i.e.,

$$\|\Phi_F(x) - L_F(x)\| \leq M\|x\| \tag{5.2}$$

for all $x \in F$. The remainder of the proof is the dullest thing we can do with those ingredients. Pick $\mathfrak{U}$ an ultrafilter refining the order filter of $\mathscr{F}(X)$ and $P\colon Y_{\mathfrak{U}} \longrightarrow Y$ a projection through the canonical embedding $\delta\colon Y \longrightarrow Y_{\mathfrak{U}}$. Define the mapping $L\colon X \longrightarrow Y_{\mathfrak{U}}$ given by $L(x) = [L_F(x1_F(x))]$ and then check that L is linear to conclude from (5.2) that $\|\delta \circ \Phi - L\| \leq M$. Finally, the composition PL yields $\|\Phi - PL\| \leq M\|P\|$, and this makes Φ trivial. The proof of the second assertion is analogous using p-linear maps. □

Finally, we build a bridge from one particular pair of spaces to another:

Proposition 5.2.19 *Let $\mathscr{X}$ and $\mathscr{Y}$ be families of quasi-Banach spaces and let X, Y, X', Y' be quasi-Banach spaces such that*

- *X contains $\mathscr{X}$ uniformly complemented and X' is locally $\mathscr{X}$.*

- *Y contains $\mathscr{Y}$ uniformly complemented and Y' is an ultrasummand that is locally $\mathscr{Y}$.*

Then $\operatorname{Ext}(X, Y) = 0$ *implies* $\operatorname{Ext}(X', Y) = 0$. *If all the spaces in consideration are p-Banach then we can replace* Ext *with* $\operatorname{Ext}_{p\mathbf{B}}$.

Proof Just go step by step: from $\operatorname{Ext}(X, Y) = 0$ to $\operatorname{Ext}(\mathscr{X}, \mathscr{Y}) = 0$ uniformly via Proposition 5.2.16, and from there to $\operatorname{Ext}(X', Y') = 0$ crossing through Proposition 5.2.18. □

Twisting $\mathscr{L}_p$-Spaces, $1 \leq p \leq \infty$

We now study the existence of non-trivial twisted sums of Banach spaces of type $\mathscr{L}_p$ for equal or different values of p. For obvious reasons, all spaces in this section will be considered infinite-dimensional without further notice. Since all $\mathscr{L}_p$-spaces have the same local structure (for a fixed p), it is a good opportunity to check how far this local approach can go. Observe that the local theory has nothing else to say about $\operatorname{Ext}_{\mathbf{B}}(\cdot, \mathscr{L}_\infty)$ or $\operatorname{Ext}_{\mathbf{B}}(\mathscr{L}_1, \cdot)$, thus those cases must be treated on an individual basis. A perhaps surprising assertion we will have opportunity to assess is that most of the results presented here are, at the end of the day, formal consequences of the fact $\operatorname{Ext}_{\mathbf{B}}(\ell_2, \ell_2) \neq 0$. To avoid annoying repetitions, we will adopt the following (illogical, but not absurd) convention: if $\mathscr{X}$ and $\mathscr{Y}$ are families of Banach spaces, then $\operatorname{Ext}_{\mathbf{B}}(\mathscr{X}, \mathscr{Y}) \neq 0$ means that $\operatorname{Ext}_{\mathbf{B}}(X, Y) \neq 0$ for every $X \in \mathscr{X}$ and every $Y \in \mathscr{Y}$.

Proposition 5.2.20 $\operatorname{Ext}_{\mathbf{B}}(\mathscr{L}_p, \mathscr{L}_q) \neq 0$ *for* $1 \leq p, q \leq \infty$ *unless* $p = 1$ *or* $q = \infty$.

Proof That $\operatorname{Ext}_{\mathbf{B}}(\mathscr{L}_p, \mathscr{L}_q) \neq 0$ for $1 < p, q < \infty$ is contained in Corollary 5.2.17. We pass to $\operatorname{Ext}_{\mathbf{B}}(\mathscr{L}_p, \mathscr{L}_1) \neq 0$ for $1 < p < \infty$, which, as we observed first and will show now, is somehow a consequence of the existence of twisted Hilbert spaces. Fix an isomorphic embedding $u \colon \ell_2 \longrightarrow L_1$ (think of the Rademacher functions, if you prefer to be more specific), pick the Kalton–Peck Z_2 space and form the pushout diagram

$$\begin{CD} 0 @>>> \ell_2 @>>> Z_2 @>>> \ell_2 @>>> 0\\ @. @VuVV @VVV @| \\ 0 @>>> L_1 @>>> \mathrm{PO} @>>> \ell_2 @>>> 0 \end{CD} \tag{5.3}$$

The pushout sequence cannot split because every subspace of $L_1 \times \ell_2$ has cotype 2, while Z_2 does not. Thus, $\operatorname{Ext}_{\mathbf{B}}(\ell_2, L_1) \neq 0$. Set

$$\mathscr{X} = (\ell_2^n)_n, \quad X = \text{any } \mathscr{L}_p\text{-space}, \quad X' = \ell_2,$$
$$\mathscr{Y} = (\ell_1^n)_n, \quad Y = \text{any } \mathscr{L}_1\text{-space}, \quad Y' = L_1$$

and apply Proposition 5.2.19 to get $\operatorname{Ext}_{\mathbf{B}}(\mathscr{L}_p, \mathscr{L}_1) \neq 0$ for $1 < p < \infty$. Since L_1 is an ultrasummand, the dual of a non-trivial sequence $0 \longrightarrow L_1 \longrightarrow \cdot \longrightarrow \ell_2 \longrightarrow 0$ cannot split, and one also has $\operatorname{Ext}_{\mathbf{B}}(L_\infty, \ell_2) \neq 0$ and thus $\operatorname{Ext}_{\mathbf{B}}(\mathscr{L}_\infty, \ell_2) \neq 0$. To settle the remaining case, namely $\operatorname{Ext}_{\mathbf{B}}(\mathscr{L}_\infty, \mathscr{L}_1) \neq 0$, let us first show how to construct a non-trivial extension of L_∞ by L_1. As one might guess, we start once more with the Kalton–Peck Z_2 sequence and an isomorphic embedding $u\colon \ell_2 \longrightarrow L_1$ to which we add now a quotient map $Q\colon L_\infty \longrightarrow \ell_2$ to form the pushout / pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "u"'] & Z_2 \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & \mathrm{PO} \arrow[r, "\overline{\rho}"] & \ell_2 \arrow[r] & 0 \\
0 \arrow[r] & L_1 \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & L_\infty \arrow[r] \arrow[u, "Q"'] & 0
\end{tikzcd}$$

The key point is to show that the lower sequence does not split, or, equivalently, that Q cannot be lifted to PO. To this end, let us first point out a special feature of this construction: the quotient map $\overline{\rho}$ is strictly singular because given any infinite-dimensional subspace H of ℓ_2, the lower sequence of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "u"'] & \rho^{-1}[H] \arrow[r, "\rho"] \arrow[d] & H \arrow[r] \arrow[d, "\text{inclusion}"] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & \overline{\rho}^{-1}[H] \arrow[r, "\overline{\rho}"] & \ell_2 \arrow[r] & 0
\end{tikzcd} \tag{5.4}$$

cannot split since $\rho^{-1}[H]$ contains an isomorphic copy of Z_2, as explained in Section 10.9, in the paragraph labelled 'The space Z_2 is "self similar"', and thus it cannot be a subspace of $L_1 \times H$, as would be the case were the lower sequence trivial. Returning to the proof, assume that some linear continuous lifting $L\colon L_\infty \longrightarrow \mathrm{PO}$ for Q exists. Let (x_n) be a bounded sequence in L_∞ such that $Q(x_n) = e_n$. Since PO has finite cotype, we infer from [123, Theorem 2.3] that there is $f \in \mathrm{PO}$ and a subsequence $(L(x_m))_m$ such that $(L(x_m) - f)_m$ is weakly-2-summable, or, equivalently, the continuous image of $(e_m)_m$ [153, Proposition 2.2]. This would imply that the quotient map $\mathrm{PO} \longrightarrow \ell_2$ is invertible on the subspace spanned by the sequence $(e_m)_m$, which is impossible because $\overline{\rho}$ is strictly singular. Thus, $\mathrm{Ext}_{\mathbf{B}}(L_\infty, L_1) \neq 0$ and, setting

$$\begin{array}{lll} \mathscr{X} = (\ell_\infty^n)_n & X = \text{any } \mathscr{L}_\infty\text{-space}, & X' = L_\infty, \\ \mathscr{Y} = (\ell_1^n)_n & Y = \text{any } \mathscr{L}_1\text{-space}, & Y' = L_1 \end{array}$$

in Proposition 5.2.19, we get $\mathrm{Ext}_{\mathbf{B}}(\mathscr{L}_\infty, \mathscr{L}_1) \neq 0$. □

Claim The role of Z_2 can be played by any non-trivial twisted Hilbert space.

Proof of the claim Let $0 \longrightarrow \ell_2 \longrightarrow \diamond \longrightarrow \ell_2 \longrightarrow 0$ be a non-trivial extension. Consider any isomorphic embedding $\jmath\colon \ell_2 \longrightarrow L_1$ and any quotient map $\rho\colon C[0,1] \longrightarrow \ell_2$ and form the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "\jmath"'] & \diamond \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L_1 \arrow[r, "\jmath_2"] & \mathrm{PO} \arrow[r, "\rho_2"] & \ell_2 \arrow[r] & 0 \\ 0 \arrow[r] & L_1 \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r, "\rho_3"] \arrow[u] & C[0,1] \arrow[r] \arrow[u, "\rho"'] & 0 \end{tikzcd}$$

No twisted Hilbert space can have cotype 2 (Section 10.9) and $\diamond$ is no exception, so it cannot be a subspace of the cotype 2 space $L_1 \times \ell_2$. This prevents the middle sequence from splitting. Our goal is to show that the bottom sequence does not split. Suppose it does. Then there exists an operator $s_3\colon C[0,1] \longrightarrow \mathrm{PB}$ such that $\rho_3 s_3 = \mathbf{1}_{C[0,1]}$. Since L_1 has cotype 2, the space PO has cotype q for all $q > 2$, as follows from Corollary 3.11.4. Thus, the operator $\underline{\rho} s_3\colon C[0,1] \longrightarrow \mathrm{PO}$ must factor as $\underline{\rho} s_3 = \beta\alpha$ through some L_r-space with $r > \overline{2}$ [153, Theorem 11.14 (b)]. Form the pullback diagram

$$\begin{tikzcd} L_1 \arrow[r, "\jmath_2"] & \mathrm{PO} \\ \mathrm{PB}(\jmath_2,\beta) \arrow[r] \arrow[u] & L_r \arrow[u, "\beta"'] \end{tikzcd}$$

to discover that since $\rho_2\beta\alpha = \rho$, the map $\rho_2\beta$ is surjective and thus $\mathrm{PB}(\jmath_2,\beta) = \ker \rho_2\beta$, which yields the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L_1 \arrow[r, "\jmath_2"] & \mathrm{PO} \arrow[r, "\rho_2"] & \ell_2 \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{PB}(\jmath_2,\beta) \arrow[r] \arrow[u, "u"] & L_r \arrow[r, "\rho_2\beta"] \arrow[u, "\beta"] & \ell_2 \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

Now, since L_r has type 2 and L_1 has cotype 2, Maurey's extension theorem (see the comments after 1.4.10) yields that every operator from a subspace of

L_r to L_1 extends to L_r. Apply this to u to conclude that the upper sequence must split. □

The case $p = \infty, q = 1$ has already been treated accidentally:

Corollary 5.2.21 *If X, Y are Banach spaces, X contains ℓ_∞^n uniformly and Y contains ℓ_1^n uniformly complemented,* $\operatorname{Ext}_{\mathbf{B}}(X, Y) \neq 0$ *and* $\operatorname{Ext}_{\mathbf{B}}(Y^*, X^*) \neq 0$.

The argument showing that $\operatorname{Ext}_{\mathbf{B}}(\ell_2, L_1) \neq 0$ and $\operatorname{Ext}_{\mathbf{B}}(L_\infty, \ell_2) \neq 0$ that appears in the middle of the proof of Proposition 5.2.20 can be localised in different ways, but only one of them requires a proof:

Proposition 5.2.22 *If X contains ℓ_∞^n uniformly then* $\operatorname{Ext}_{\mathbf{B}}(X, \ell_2) \neq 0$. *If X contains ℓ_1^n uniformly complemented then* $\operatorname{Ext}_{\mathbf{B}}(\ell_2, X) \neq 0$. *If X is an infinite-dimensional Banach space of cotype 2, then* $\operatorname{Ext}_{\mathbf{B}}(\ell_2, X) \neq 0$.

Proof We know that every Banach space contains ℓ_2^n almost isometrically thanks to the Dvoretzky–Rogers theorem [153, Theorem 19.2]. For each $n \in \mathbb{N}$, let $r_n \colon \ell_2^n \longrightarrow X$ be a $\frac{1}{n}$-isometry, let $R_n \colon \ell_2 \longrightarrow X$ be the composition with the projection $\pi_n \colon \ell_2 \longrightarrow \ell_2^n$ onto the first n coordinates and let $R \colon \ell_2 \longrightarrow \ell_\infty(X)$ be the embedding $R(x) = (R_n(x))$. If $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$ and $[\cdot] \colon \ell_\infty(X) \longrightarrow X_{\mathcal{U}}$ is the natural quotient map, it is clear that the composition $[\cdot]R$ is still an embedding. Now, let $0 \longrightarrow \ell_2 \longrightarrow \diamond \longrightarrow \ell_2 \longrightarrow 0$ be a non-trivial twisted Hilbert space and form the successive pushout diagrams

$$\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "R"] & \diamond \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ell_\infty(X) \arrow[r] \arrow[d, "{[\cdot]}"] & \mathrm{PO}_1 \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_{\mathcal{U}} \arrow[r] & \mathrm{PO}_2 \arrow[r] & \ell_2 \arrow[r] & 0
\end{tikzcd}$$

If $\operatorname{Ext}_{\mathbf{B}}(\ell_2, X) = 0$ then $\operatorname{Ext}_{\mathbf{B}}(\ell_2, \ell_\infty(X)) = 0$, and thus the lower sequence splits. This makes $\diamond$ a subspace of $X_{\mathcal{U}} \times \ell_2$, which has cotype 2 since $X_{\mathcal{U}}$ has the same cotype as X: a contradiction. □

The dual result, *If X is a Banach space whose dual has cotype 2, then* $\operatorname{Ext}_{\mathbf{B}}(X, \ell_2) \neq 0$, holds by the duality formula: $\operatorname{Ext}_{\mathbf{B}}(X, \ell_2) = \operatorname{Ext}_{\mathbf{B}}(\ell_2, X^*)$.

5.3 The Mysterious Role of the BAP

We arrive at the third milestone of the chapter: approximation properties. The BAP is no doubt useful in Banach space theory because it allows us to split large objects into smaller ones using finite-rank operators. However, the only whiffs we have had so far of any homological sniff about the BAP are the Pełczyński–Lusky sequence (2.7) and the scent Proposition 2.2.19 left (the sequence splits if and only if X has the BAP) that it somehow detects the BAP. Is there any other homological connection in sight? Yes: the structural theorem of Johnson, Rosenthal and Zippin we have mentioned so often and which is usually seen in its negative form (the BAP is not a 3-space property):

Proposition 5.3.1 *Every separable Banach space X admits a representation $0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$ in which both A, B have a FDD. Moreover, if X^* is separable then A and B may be chosen having a shrinking FDD. In the particular case in which X is a subspace of c_0, X admits a representation $0 \longrightarrow c_0(\mathbb{N}, A_n) \longrightarrow X \longrightarrow c_0(\mathbb{N}, B_n) \longrightarrow 0$, where A_n and B_n are finite-dimensional spaces.*

A neat proof for the first part can be found in [334, Theorem 1.g.2], while the second part can be deduced from [334, Theorem 2.d.1]; see the proof in [334, Theorem 2.f.6]. This failure of the 3-space property for the BAP has a bright side: a Banach space X is separably injective if and only if $\operatorname{Ext}_{\mathbf{B}}(S, X) = 0$ for all separable spaces S with the BAP. And, more good news, the negative 3-space result is not that negative, since restricted forms of the 3-space property are still available:

Proposition 5.3.2 *Let $0 \longrightarrow Y \longrightarrow Z \xrightarrow{\rho} X \longrightarrow 0$ be a locally split sequence.*

(a) *If Y and X have the BAP then Z has the BAP.*
(b) *If Z has the BAP then Y must have the BAP.*

Proof Assertion (b) is trivial. To prove (a), let $M < \infty$ be such that the sequence M-splits locally and both Y and X have the M-AP. Given a finite-dimensional subspace F of Z,

- there is a finite-rank operator $\tau \in \mathfrak{L}(X)$ of norm at most M fixing $\rho[F]$;
- there is a local section $s\colon \tau[X] \longrightarrow Z$ of norm at most M;
- if $E = (\mathbf{1}_F - s\rho)[F] \subset Y$, there is a finite-rank operator $\omega \in \mathfrak{L}(Y)$ fixing E having norm at most M;
- there is a finite-rank extension $\varpi\colon Z \longrightarrow Y$ of ω with norm at most M since the sequence splits locally.

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Thus, the operator $T = \varpi(\mathbf{1}_X - s\tau\rho) + s\tau\rho$ has finite rank, has controlled norm and fixes F since for $f \in F$, we have $Tf = \varpi(f - s\rho f) + s\rho f = f - s\rho f + s\rho f = f$. □

There are counterexamples for the two remaining cases. To get examples (c) in which both Z, X have the BAP but Y does not, we use Szankowski's remark [448] that the classical Enflo–Davies counterexample for the AP provides a subspace H of c_0 without the AP yielding an exact sequence:

$$0 \longrightarrow H \longrightarrow c_0 \longrightarrow c_0(\mathbb{N}, \ell_2^n) \longrightarrow 0 \tag{5.5}$$

(by the way, this sequence cannot split locally). An example (d) in which Y and Z have the BAP but X does not is the Pełczyński–Lusky sequence for a separable space X without the BAP: the space $c(\mathbb{N}, X_n)$ has the BAP, and since the sequence splits locally, so does $c_0(\mathbb{N}, X_n)$. Another example of this type can be obtained by recalling Lindenstrauss' 'outgrowth' [328] of James [218] according to which every separable space X can be written as $X = Y^{**}/Y$ where Y^{**} (hence Y) has the BAP. The sequence $0 \longrightarrow Y \longrightarrow Y^{**} \longrightarrow X \longrightarrow 0$ splits locally, but X may fail the BAP. The next two sections present two rather surprising results in this context: an example like (c) cannot exist when Z is an $\mathscr{L}_1$-space and an example like (d) cannot exist when Z is an $\mathscr{L}_\infty$-space. If the reader is still sceptical of the homological content of the BAP, we find their lack of faith disturbing: the BAP force is just beginning to manifest.

Projective Presentations and the BAP

Projective presentations are the archetype of exact sequences in which the middle space is an $\mathscr{L}_1$-space. Thus, the paradigmatic result in this context is Lusky's theorem [348] that whenever X has the BAP, the kernel of any of its projective presentations $0 \longrightarrow \kappa(X) \longrightarrow \ell_1(I) \longrightarrow X \longrightarrow 0$ must have the BAP as well. Lusky's proof is technically demanding; we will (somehow) vault such difficulties using the $\mathrm{co}^{(p)}$ spaces appearing in Section 3.10.

Lemma 5.3.3 *If a p-Banach space X has the λ-AP, $\mathrm{co}^{(p)}(X)$ has the 3λ-AP.*

Proof Fix a Hamel basis $\mathscr{H}$ for X and let $\mho\colon X \longrightarrow \mathrm{co}^{(p)}(X)$ be the (version of) the universal p-linear map of Theorem 3.10.2 vanishing on $\mathscr{H}$. Let F be a finite-dimensional subspace of $\mathrm{co}^{(p)}(X)$. We can assume without loss of generality that $F = [\mho(x_1), \dots, \mho(x_m)]$, where $x_j \in X$ for $1 \le j \le m$. Let $\mathscr{H}_0$ be a finite subset of $\mathscr{H}$ whose linear span $[\mathscr{H}_0]$ contains $[x_1, \dots, x_m]$. Now, let $T \in \mathfrak{L}(X)$ be a finite-rank operator fixing $[\mathscr{H}_0]$ and with $\|T\| \le \lambda^+$. Since $Q^{(p)}(\mho) = 1$, Lemma 3.9.1 allows us to obtain a small perturbation $\mho'\colon X \longrightarrow \mathrm{co}^{(p)}(X)$ satisfying:

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- $\mho'(x_j) = \mho(x_j)$ for $1 \le j \le m$,
- $\mho'(b) = \mho(b) = 0$ for $b \in \mathscr{H}$,
- $\|\mho' - \mho\| \le 1 + \varepsilon$,
- $Q^{(p)}(\mho') \le 3 + \varepsilon$,
- $\mho(T[X])$ spans a finite-dimensional subspace of $\mathrm{co}^{(p)}(X)$.

Let $(\mho' \circ T)_{\mathscr{H}} \colon X \longrightarrow \mathrm{co}^{(p)}(X)$ be the version of $\mho' \circ T$ that vanishes on $\mathscr{H}$. The universal property of $\mho$ yields an operator $\phi\colon\ \mathrm{co}^{(p)}(X) \longrightarrow \mathrm{co}^{(p)}(X)$ such that $\phi \circ \mho = (\mho' \circ T)_{\mathscr{H}}$. Let us check that ϕ has the required properties:

- $\|\phi\| = Q^{(p)}((\mho' \circ T)_{\mathscr{H}}) = Q^{(p)}(\mho' \circ T) \le Q^{(p)}(\mho')\|T\| \le (3 + \varepsilon)\lambda^+$.
- ϕ has finite rank: the image of $\mho' \circ T$, and therefore that of $(\mho' \circ T)_{\mathscr{H}}$, spans a finite-dimensional subspace of $\mathrm{co}^{(p)}(X)$. Hence $\{\phi(\mho(x)) : x \in X\} \subset [(\mho' \circ T)_{\mathscr{H}}]$, and since $\mathrm{co}^{(p)}(X)$ is the closure of the space spanned by the points of the form $\mho(x)$ and ϕ is continuous, we also get that $\phi[\mathrm{co}^{(p)}(X)] \subset [(\mho' \circ T)_{\mathscr{H}}]$.
- ϕ fixes F. Indeed, since $x_j = \sum_{b \in \mathscr{H}_0} \lambda_b b$ for $1 \le j \le m$, one has

$$\begin{aligned}\phi(\mho(x_j)) &= (\mho' \circ T)_{\mathscr{H}}(x_j)\\ &= \mho'(Tx_j) - \sum_{b \in \mathscr{H}_0} \lambda_b \mho'(Tb)\\ &= \mho'(x_j) - \sum_{b \in \mathscr{H}_0} \lambda_b \mho'(b)\\ &= \mho(x_j).\end{aligned}$$ □

The stage is set for the proof of the main result. The rest is just throwing balls to the homological wall:

Proposition 5.3.4 *Let $0 < p \le 1$. If X is a p-Banach space with the BAP then $\kappa_p(X)$ has the BAP.*

Proof The universal property of $\mathrm{co}^{(p)}(X)$ yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{co}^{(p)}(X) & \longrightarrow & \diamond & \longrightarrow & X & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \kappa_p(X) & \longrightarrow & \ell_p(I) & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

whose diagonal pushout sequence $0 \longrightarrow \mathrm{co}^{(p)}(X) \longrightarrow \diamond \times \kappa_p(X) \longrightarrow \ell_p(I) \longrightarrow 0$ splits. Thus $\mathrm{co}^{(p)}(X) \times \ell_p(I) \simeq \diamond \times \kappa_p(X)$ and, by Lemma 5.3.3, $\kappa_p(X)$ must have the BAP. □

Corollary 5.3.5 *Let $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ be an exact sequence in which Z is a discrete $\mathscr{L}_p$-space, $0 < p \le 1$. If X has BAP then also Y has BAP.*

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Proof Proceed as before. The diagonal sequence $0 \longrightarrow \mathrm{co}^{(p)}(X) \longrightarrow \diamond{\times}Y \longrightarrow Z \longrightarrow 0$ splits locally. Thus, Proposition 5.3.2 (a) yields that $\diamond{\times}Y$ has BAP. $\square$

Focusing on $p = 1$, given a Banach space X, we get that either all kernels of all quotient maps $\mathscr{L}_1 \longrightarrow X$ enjoy the BAP or none of them does (the result has a straightforward version for discrete $\mathscr{L}_p$-spaces which we just skip):

Proposition 5.3.6 *Given exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & \mathscr{L}_1 \arrow[r] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r] & \mathscr{L}_1' \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

the space Y has BAP if and only if Y' has BAP.

Proof The two sequences passing through PB in the commutative diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & Y \arrow[r] & \mathscr{L}_1 \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & \mathscr{L}_1' \arrow[r] \arrow[u] & 0 \\
 & & Y' \arrow[r, equal] \arrow[u] & Y' \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

split locally. If Y has the BAP then so does PB since $\mathscr{L}_1$-spaces have the BAP. Therefore, the same is true for its locally complemented subspace Y'. $\square$

The sequence $0 \longrightarrow \ell_1(\mathbb{N}, \ell_2^n) \longrightarrow \ell_1 \longrightarrow H^* \longrightarrow 0$, dual of (5.5), shows that ℓ_1 contains a subspace E isomorphic to $\ell_1(\mathbb{N}, \ell_2^n)$ with the BAP such that ℓ_1/E does not have the BAP. Moreover, since every infinite-dimensional $\mathscr{L}_1$-space contains a complemented copy of ℓ_1, taking $\mathscr{L}_1 = \ell_1 \times A$, the sequence $0 \longrightarrow \ell_1(\mathbb{N}, \ell_2^n) \times A \longrightarrow \mathscr{L}_1 \longrightarrow H^* \longrightarrow 0$ shows that every infinite-dimensional $\mathscr{L}_1$-space contains a subspace E' with the BAP such that $\mathscr{L}_1/E'$ does not have the BAP.

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Injective Presentations and the BAP

We now present the corresponding dual results. The key point is the following technical lemma, for which we provide two (two?) proofs.

Lemma 5.3.7 *Let* $0 \longrightarrow Y \longrightarrow X \xrightarrow{\rho} F \longrightarrow 0$ *be an isometrically exact sequence of Banach spaces in which* F *is finite-dimensional and* Y *has the* μ*-AP. Given a finite-dimensional subspace* G *of* X*, there is a finite-rank operator* $T_G \in \mathfrak{L}(X)$ *with norm at most* $3\mu^+$ *fixing* G *and such that* $T_G[Y] \subset Y$*.*

Proof Fix $\varepsilon > 0$. Enlarging G if necessary, we may assume it almost norms $Y^\perp = F^*$: for every $x^* \in Y^\perp$, we have $\|x^*\| \leq (1+\varepsilon)\sup\{|x^*(g)| : g \in G, \|g\| \leq 1\}$. This implies that for each $x \in X$ there is $g \in G$ with $\|g\| \leq (1+\varepsilon)\|x\|$ such that $\rho(x) = \rho(g)$. Let us consider the norm one sum map $\oplus\colon Y \oplus G \longrightarrow X$. Since for each normalised $x \in X$ we can find $g \in G$ such that $\rho(g) = \rho(x)$ and $\|g\| \leq 1^+$, it follows that $(x - g, g) \in Y \oplus G$ has norm at most 3^+ and $\oplus(x - g, g) = x$. Hence $\oplus$ induces an isomorphism $\sigma\colon (Y \oplus G)/\ker\oplus \longrightarrow X$ satisfying $\|\sigma\| \leq 1$ and $\|\sigma^{-1}\| \leq 3^+$. Let $T \in \mathfrak{L}(Y)$ be a finite-rank operator with norm at most μ^+ that fixes $G \cap Y$. The operator $T \times \mathbf{1}_G\colon Y \oplus G \longrightarrow Y \oplus G$ has finite rank and the same norm as T. Moreover, the restriction of $T \times \mathbf{1}_G$ to $\ker\oplus$ is the identity: if $y + g = 0$ then $y \in Y \cap G$, so $T(y) = y$ and $(T \times \mathbf{1}_G)(y, g) = (y, g)$. Thus, $T \times \mathbf{1}_G$ induces the finite-rank operator $T'\colon (Y \oplus G)/\ker\oplus \longrightarrow (Y \oplus G)/\ker\oplus$ given by $T'((y, g) + \ker\oplus) = (Ty, g) + \ker\oplus$ and with the same norm. Hence $T_G = \sigma T' \sigma^{-1}$ is a finite-rank operator on X with norm at most $3\mu^+$, and T_G fixes G: if $g \in G$, we have

$$T_G(g) = \sigma(T'(\sigma^{-1}(g)) = \sigma(T'((0, g) + \ker\oplus)) = \sigma((0, T(g)) + \ker\oplus)) = g. \qquad \square$$

What has happened here? Well, the argument in the preceding proof should elbow us aware of the pushout construction: indeed, draw Lemma 3.9.1 as follows.

5.3.8 *If* $0 \longrightarrow Y \longrightarrow X \xrightarrow{\rho} F \longrightarrow 0$ *is an isometrically exact sequence of Banach spaces with* F *finite-dimensional then for every* $\varepsilon > 0$ *there is a finite-dimensional subspace* $G \subset X$ *and a commutative diagram*

$$\begin{tikzcd}
& 0 \arrow[r] & Y \cap G \arrow[r] \arrow[d] \arrow[ddl] & G \arrow[r] \arrow[d] \arrow[ddl] & F \arrow[r] \arrow[d, equal] \arrow[ddl, equal] & 0 \\
& 0 \arrow[r] & Y \arrow[r] \arrow[dl, equal] & \mathrm{PO} \arrow[r] \arrow[dl, "u"'] & F \arrow[r] \arrow[dl, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & X \arrow[r] & F \arrow[r] & 0 &
\end{tikzcd}$$

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in which u is an isomorphism with $\|u\| \leq 1$ *and* $\|u^{-1}\| \leq 3+\varepsilon$ *and all unlabelled arrows are canonical inclusions.*

To prove it, and keeping the same notation as before, observe that if $T \in \mathfrak{L}(Y)$ is a finite-rank operator fixing $Y \cap G$ with $\|T\| \leq \mu^+$ then the universal property of the pushout yields a commutative the diagram

$$\begin{tikzcd} Y \cap G \arrow[r] \arrow[d] & G \arrow[d] \arrow[ddr, bend left, "\text{inclusion}"] & \\ Y \arrow[r] \arrow[drr, bend right, "T"'] & \mathrm{PO} \arrow[dr, "T'"] & \\ & & X \end{tikzcd}$$

in which $T' \colon \mathrm{PO} \longrightarrow X$ is a finite-rank operator with $\|T'\| \leq \mu^+$. Set $T_G = T'u^{-1}$ to get the desired operator and estimate.

We are ready to deliver the promised result:

Proposition 5.3.9 *Let Y be a Banach space with the μ-AP and let* $Y \longrightarrow \mathscr{L}_{\infty,\lambda}$ *be an isometric embedding. Then* $\mathscr{L}_{\infty,\lambda}/Y$ *has the* $3\mu\lambda$*-AP.*

Proof What we will show is that, given any finite-dimensional subspace $F \subset \mathscr{L}_{\infty,\lambda}$, there is a finite-rank operator $T \in \mathfrak{L}(\mathscr{L}_{\infty,\lambda})$ fixing F such that $T[Y] \subset Y$ and $\|T\| \leq 3\mu\lambda^+$. This already means that $\mathscr{L}_{\infty,\lambda}/Y$ has the $3\mu\lambda$-AP. To that end, fix F and apply Lemma 5.3.7 to the sequence

$$0 \longrightarrow Y \longrightarrow Y+F \longrightarrow (Y+F)/Y \longrightarrow 0$$

to get a finite-rank operator $\tau \in \mathfrak{L}(Y+F)$ fixing F, leaving Y invariant and having norm at most $3\mu^+$. The finite-dimensional subspace $\tau[Y+F]$ must be contained in a subspace λ^+-isomorphic to some ℓ_∞^m, and therefore there is an extension $T \in \mathfrak{L}(\mathscr{L}_{\infty,\lambda})$ of rank at most m and norm at most $3\lambda\mu^+$ that fixes F and leaves Y invariant. □

Proposition 5.3.9 can be completed with:

Lemma 5.3.10 *Given an exact sequence* $0 \longrightarrow \mathscr{L}_\infty \longrightarrow Z \longrightarrow X \longrightarrow 0$ *of Banach spaces in which Z has the BAP, also X has the BAP.*

Proof Form a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{L}_\infty \arrow[r] & \ell_\infty(I) \arrow[r] & \ell_\infty(I)/\mathscr{L}_\infty \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{L}_\infty \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

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The diagonal pullback sequence $0 \longrightarrow Z \longrightarrow \ell_\infty(I) \times X \longrightarrow \ell_\infty(I)/\mathscr{L}_\infty \longrightarrow 0$ splits locally since it is a pushout of the upper row, which splits locally. And the space $\ell_\infty(I)/\mathscr{L}_\infty$ is an $\mathscr{L}_\infty$-space, as is any quotient of two $\mathscr{L}_\infty$-spaces, hence it has the BAP. Thus, if Z has the BAP then the middle space $\ell_\infty(I) \times X$ has the BAP, and this implies that also X has the BAP. □

Independently of whether Y has the BAP, all the quotients $\mathscr{L}_\infty/Y$ have or fail to have the BAP simultaneously:

Proposition 5.3.11 *Given exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & \mathscr{L}_\infty \arrow[r] & X \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r] & \mathscr{L}'_\infty \arrow[r] & X' \arrow[r] & 0 \end{tikzcd}$$

the space X has the BAP if and only if X' has the BAP.

Proof Consider the pushout diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & Y \arrow[r] \arrow[d] & \mathscr{L}_\infty \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{L}'_\infty \arrow[r] \arrow[d] & \mathrm{PO} \arrow[r] \arrow[d] & X \arrow[r] & 0 \\ & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

The two sequences passing through PO split locally. If X' has the BAP then PO must have the BAP, and thus Lemma 5.3.10 applies to the horizontal sequence, allowing us to conclude that also X has the BAP. □

The space ℓ_p contains a subspace without the BAP for every $p \in [1, \infty)$ different from 2 (see [334, Theorem 2.d.6] and [335, Theorem 1.g.4]) and therefore ℓ_p has a quotient E without the BAP. Since ℓ_p is a quotient of $C[0, 1]$ for $p \in [2, \infty)$, the space E is a quotient too. Thus, $C[0, 1]$ contains a subspace Y such that $C[0, 1]/Y = E$ lacks the BAP. This Y cannot have the BAP and no quotient $\mathscr{L}_\infty/Y$ can have the BAP either. Propositions 5.3.4 and 5.3.9 can be forced to cover the UAP case:

Corollary 5.3.12 *If X is a Banach space with the UAP, the kernel of any surjection $\mathscr{L}_1 \longrightarrow X$ and the cokernel of any embedding $X \longrightarrow \mathscr{L}_\infty$ have the UAP.*

Proof Assume X has the UAP so that, for every ultrafilter $\mathfrak{U}$, the ultrapower $X_\mathfrak{U}$ has the BAP. Let $0 \longrightarrow K \longrightarrow \mathscr{L}_1 \longrightarrow X \longrightarrow 0$ be any exact sequence. The ultrapower sequence $0 \longrightarrow K_\mathfrak{U} \longrightarrow (\mathscr{L}_1)_\mathfrak{U} \longrightarrow X_\mathfrak{U} \longrightarrow 0$ is again exact and since ultrapowers of $\mathscr{L}_1$-spaces are $\mathscr{L}_1$-spaces we can apply Proposition 5.3.4 to conclude that $K_\mathfrak{U}$ has the BAP and, therefore, K has the UAP. The dual version is analogous. □

Trivial Twisting and the BAP

The BAP has been lurking behind the vanishing of $\operatorname{Ext}(X, Y)$ spaces at least since Corollary 4.5.12. It is time to reveal its role. To start with, surprising as it may seem, in the presence of the BAP, the existence of non-trivial elements in $\operatorname{Ext}(X, Y)$ can always be detected by a careful observation of the trivial ones.

Theorem 5.3.13 *Let X be a quasi-Banach space and Y be a μ-ultrasummand such that $K_0[X, Y] < \infty$.*

(a) *If X has the λ-AP then* $\operatorname{Ext}(X, Y) = 0$.
(b) *If X is a $\mathscr{K}$-space and Y has the λ-AP then* $\operatorname{Ext}(X, Y) = 0$.

In both cases, $K[X, Y] \leq \lambda^+ \mu K_0[X, Y]$.

Proof By the Aoki–Rolewicz theorem, we may assume that Y is a p-Banach space. To prove (a), let $\mathfrak{U}$ be any ultrafilter refining the order filter on $\mathscr{F}(X)$, and let us consider the corresponding ultrapower $Y_\mathfrak{U}$ and a bounded projection $P\colon Y_\mathfrak{U} \longrightarrow Y$ along the diagonal embedding $\delta\colon Y \longrightarrow Y_\mathfrak{U}$. Since X has the λ-AP, for each $E \in \mathscr{F}(X)$, there is $T_E \in \mathfrak{F}(X)$ fixing E with $\|T_E\| \leq \lambda^+$. Now, let $\Phi\colon X \longrightarrow Y$ be quasilinear, with $Q(\Phi) \leq 1$. We define the map $\phi\colon X \longrightarrow \ell_\infty(\mathscr{F}(X), Y)$ given by $\phi(x) = (\Phi(T_E x))_E$. Observe that $\phi(x)(E) = \Phi(x)$ when $x \in E$, whence it follows that $[\cdot] \circ \phi = \delta \circ \Phi$. In other drawings, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\phi"'] & Y \arrow[d, "\delta"] \\ \ell_\infty(\mathscr{F}(X), Y) \arrow[r, "{[\cdot]}"] & Y_\mathfrak{U} \end{tikzcd}$$

and consequently, $\Phi = P\,[\cdot] \circ \phi$. Since quasilinear maps are bounded on finite-dimensional spaces, $\Phi \circ T_E\colon X \longrightarrow Y$ is bounded. Thus, for each $E \in \mathscr{F}(X)$, there is $\ell_E \in \mathfrak{L}(X, Y)$ such that

$$\|\ell_E - \Phi\circ T_E\| \le K_0[X,Y]\, Q(\Phi\circ T_E) \le K_0[X,Y]\, Q(\Phi)\,\|T_E\| \le K_0[X,Y]\,\lambda^+.$$

This allows us to define a map $\phi'\colon X \longrightarrow \ell_\infty(\mathscr{F}(X),Y)$ by $\phi'(x)(E) = \ell_E(x1_E(x))$. Of course, we have

$$\|\phi(x)-\phi'(x)\|_\infty = \sup_{x\in E}\|\Phi(T_E(x)) - \ell_E(x)\|_Y \le K_0[X,Y]\lambda^+\|x\|.$$

Hence, if one sets $L = [\cdot]\circ\phi'$ then $\|[\cdot]\circ\phi - L\| \le K_0[X,Y]\lambda^+$. The point is that L is actually linear: it is obviously homogeneous, and moreover, given $x, y\in X$, the set $\{E\in\mathscr{F}(X)\colon x,y\in E\}$ belongs to $\mathfrak{U}$, and thus, if $x,y\in E$, we have

$$\phi'(x+y)(E) = \ell_E(x+y) = \ell_E(x)+\ell_E(y) = \phi'(x)(E)+\phi'(y)(E),$$

which means that $L(x+y) = L(x)+L(y)$. The linear map $PL\colon X\longrightarrow Y$ satisfies

$$\|\Phi - PL\| = \|P[\cdot]\circ\phi - PL\| \le \|P\|\,K_0[X,Y]\,\lambda^+.$$

Thus, $K[X,Y]\le\lambda^+\|P\|K_0[X,Y]$, and every quasilinear map $X\longrightarrow Y$ is trivial.

The proof for (b) is analogous. This time, the index set is $\mathscr{F}(Y)$. For each $E\in\mathscr{F}(Y)$, we pick $T_E\in\mathfrak{F}(Y)$ such that $T_E(y) = y$ for $y\in E$, with $\|T_E\|\le\lambda^+$. Now, let $\Phi\colon X\longrightarrow Y$ be a quasilinear map with $Q(\Phi)\le 1$. For each $E\in\mathscr{F}(Y)$, consider the composition $T_E\circ\Phi\colon X\longrightarrow T_E[Y]$. Since X is a $\mathscr{K}$-space, there is a linear map $\ell_E\colon X\longrightarrow T_E[Y]$ at finite distance from $T_E\circ\Phi$. The problem is that we have no bound for that distance. To overcome this difficulty, just consider $T_E\circ\Phi - \ell_E$ as a bounded homogeneous map $X\longrightarrow Y$. Since $Q(T_E\circ\Phi-\ell_E) = Q(T_E\circ\Phi)\le\lambda^+$, there is a linear map $\ell'_E\colon X\longrightarrow Y$ such that

$$\|(T_E\circ\Phi-\ell_E)-\ell'_E\| = \|T_E\circ\Phi - (\ell_E+\ell'_E)\| \le \lambda^+K_0[X,Y].$$

We define $\phi'\colon X\longrightarrow\ell_\infty(\mathscr{F}(Y),Y)$ by $\phi'(x)(E) = \ell_E(x)+\ell'_E(x)$. Observe that in the worst case, i.e. when $\Phi(x)\in E$, we have $T_E(\Phi(x)) = \Phi(x)$, and therefore $\|\phi'(x)(E)-\Phi(x)\|\le K_0[X,Y]\lambda^+\|x\|$; hence

$$\sup_{E\in\mathscr{F}(Y)}\|\phi'(x)(E)\| \le \Delta_Y\left(\|\Phi(x)\| + K_0[X,Y]\lambda^+\|x\|\right).$$

The rest goes as before. Let $\mathfrak{U}$ be an ultrafilter refining the Fréchet filter on $\mathscr{F}(Y)$, and form the composition $L = [\cdot]\circ\phi'\colon X\longrightarrow\ell_\infty(\mathscr{F}(Y),Y)\longrightarrow Y_{\mathfrak{U}}$, which is linear: it is obviously homogeneous and if $x,y\in X$, then as long as E contains $\Phi(x),\Phi(y)$ and $\Phi(x+y)$, we have $\phi'(x+y)(E) = \phi'(x)(E)+\phi'(y)(E)$, which yields $[\phi'(x+y)(E)] = [\phi'(x)(E)]+[\phi'(y)(E)]$. Pick a bounded projection $P\colon Y_{\mathfrak{U}}\longrightarrow Y$ along the diagonal embedding δ to obtain a linear map $PL\colon X\longrightarrow Y$, which is at finite distance from Φ since $\|\delta\circ\Phi - [\cdot]\circ\phi'\|\le K_0[X,Y]\lambda^+$, and thus $\|\Phi - PL\|\le\|P\|\,K_0[X,Y]\,\lambda^+$. □

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A Banach space version of Theorem 5.3.13 for 1-linear maps is also true, although the requirement of being a $\mathscr{K}$-space can be omitted since 1-linear maps taking values on finite-dimensional spaces are automatically trivial:

Corollary 5.3.14 *If X and Y are Banach spaces, Y an ultrasummand and either X or Y have the BAP and $K_0^{(1)}[X, Y] < \infty$ then $K^{(1)}[X, Y] < \infty$.*

Now let us consider the question of whether $\operatorname{Ext}(X, Y) = 0$ implies the vanishing of any of the spaces $\operatorname{Ext}(X^{**}, Y), \operatorname{Ext}(X, Y^{**})$, $\operatorname{Ext}(X_{\mathcal{U}}, Y)$ or $\operatorname{Ext}(X, Y_{\mathcal{U}})$. The issue was slightly touched in Section 5.2 since X, X^{**} and $X_{\mathcal{U}}$ are perhaps the most natural examples of spaces with the same local structure. The novelty here is the use of the BAP to factorise quasilinear maps through finite-dimensional spaces. Thus, it is about time for the BAP to pounce.

Twisted Sums and Biduals

Does $\operatorname{Ext}(X, Y) = 0$ imply $\operatorname{Ext}(X^{**}, Y) = 0$ or $\operatorname{Ext}(X, Y^{**}) = 0$?

Theorem 5.3.15 *If X is a Banach space whose bidual has the BAP and Y is a quasi-Banach ultrasummand such that* $\operatorname{Ext}(X, Y) = 0$*, then* $\operatorname{Ext}(X^{**}, Y) = 0$. *If Y is a Banach space, we can replace* Ext *by* $\operatorname{Ext}_{\mathbf{B}}$.

Proof Suppose on the contrary that there is a non-trivial quasilinear map $\Phi\colon X^{**} \longrightarrow Y$ with $Q(\Phi) \leq 1$. The idea is that, even if the restriction of Φ itself to X can be trivial – it can be zero, in fact – one can use a finite-rank operator to 'push' Φ down to get a non-trivial quasilinear map from X to Y. To this end, assume that X^{**} has the λ-AP, pick $M > 0$ and choose a finite-dimensional subspace $E \subset X^{**}$ such that $\operatorname{dist}(\Phi|_E, \mathfrak{L}(E, Y)) > \lambda M$. Pick $\varepsilon > 0$ and select a finite-rank operator $\tau\colon X^{**} \longrightarrow X^{**}$ such that $\|\tau\| \leq \lambda$ and $\|\tau(x^{**}) - x^{**}\| \leq \varepsilon\|x^{**}\|$ for all $x^{**} \in E$. We will see that $\Phi \circ \tau$ is a 'bad' quasilinear map for sufficiently small ε that will depend on $n = \dim E$ and Y. Before going further, let us indicate how the hypothesis that the bidual of X has the BAP is to be used: since $\mathfrak{F}(X^{**}) = X^{***} \otimes X^{**}$, a finite-rank operator on X^{**} of given norm that ε-fixes a finite-dimensional subspace $E \subset X^{**}$ can be chosen in $X^{*} \otimes X^{**}$ by an obvious application of the Goldstine theorem. So, the preceding τ can be chosen to be an operator $\tau\colon X \longrightarrow X^{**}$ such that $\|\tau\| \leq \lambda$ and

$$\|\tau^{**}(x^{**}) - x^{**}\| \leq \varepsilon\|x^{**}\| \tag{5.6}$$

for all $x^{**} \in E$. Now set $F = \tau^{**}[X^{**}] = \tau[X]$ and apply the following lemma:

Lemma 5.3.16 *Let $\tau\colon X \longrightarrow F$ be a linear operator, where X and F*

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are Banach spaces, with F finite-dimensional. Let E be a finite-dimensional subspace of X^{**} and $\varepsilon > 0$. Then there is a subspace $E_0 \subset X$ and a surjective ε-isometry $u\colon E_0 \longrightarrow E$ such that $\tau^{**}(u(x)) = \tau(x)$ for every $x \in E$.

Proof The result follows from the principle of local reflexivity: given E, X as in the statement, G a finite-dimensional subspace of X^* and $\varepsilon > 0$, there is an ε-isometry $v\colon E \longrightarrow X$ such that $x^{**}(g) = g(v(x^{**}))$ for every $x^{**} \in E$ and every $g \in G$. Moreover, v can be chosen such that $v(x) = x$ for every $x \in E \cap X$, but we will not use this fact. Assume that $\tau = \sum_{i=1}^n g_i \otimes f_i$, for $g_i \in X^*$ and $f_i \in F$. Fix as G the subspace spanned by $g_1, \ldots, g_n$, and let $\varepsilon > 0$. By the principle of local reflexivity, we obtain an ε-isometry $v\colon E \longrightarrow X$ such that $\tau^{**}(x^{**}) = \tau(v(x^{**}))$ for $x^{**} \in E$. Set $E_0 = v[E]$ and $u = v^{-1}$ to conclude. □

Back to the proof of the theorem, we have obtained a subspace $E_0 \subset X$ together with an ε-isometry $u\colon E_0 \longrightarrow E$ such that $\tau^{**}(u(x)) = \tau(x)$ for $x \in E_0$. Letting $x^{**} = u(x)$ in (5.6), we obtain $\|\tau(x) - u(x)\| \leq \varepsilon\|u(x)\| \leq \varepsilon(1+\varepsilon)\|x\| \leq 2\varepsilon\|x\|$ for all $x \in E_0$. In particular, $\|\tau|_{E_0}\| \leq 2\varepsilon + \|u\| \leq 1 + 3\varepsilon$. We now need to pause to observe that if E, Z are p-Banach spaces and $\dim E = n$ then $K[E, Z]$ can be bounded by a constant $\varkappa(n, p)$ depending only on n and p. We won't spoil the reader's fun here. We also make a detour to obtain a slightly mystifying lemma in which the role of the constant $\varkappa(n, p)$ is finally unmasked.

Lemma 5.3.17 *Let $\Phi\colon X \longrightarrow Y$ be a quasilinear map acting between p-normed spaces, with $Q(\Phi) \leq 1$, and let F be an n-dimensional p-normed space. Given two linear operators $u, v\colon F \longrightarrow X$, we have*

$$\operatorname{dist}(\Phi\circ u, \mathsf{L}(F, Y)) \leq 3^{1/p-1}\left(\operatorname{dist}(\Phi \circ v, \mathsf{L}(F, Y)) + \|v\| + (1 + \varkappa(n, p))\|v - u\|\right).$$

Proof There is no need to freak out about the factor $3^{1/p-1}$: it only appears because we have to sum three chunks to complete the proof. Pick linear maps $L_1, L_2\colon F \longrightarrow Y$ such that

- $D_1 = \|L_1 - \phi \circ v\| \leq \operatorname{dist}(\phi \circ v, \mathsf{L}(F, Y)) + \varepsilon$,
- $D_2 = \|L_2 - \phi \circ (u - v)\| \leq \operatorname{dist}(\phi \circ (u - v), \mathsf{L}(F, Y)) + \varepsilon \leq \varkappa(n, p))\|v - u\| + \varepsilon$

for small $\varepsilon > 0$. Let us estimate $\|\phi \circ u - (L_1 + L_2)\|$. Pick a normalised $f \in F$:

$$\begin{aligned}
&\|\Phi u f - L_1 f - L_2 f\|^p \\
&= \|\Phi u f - \Phi(u - v) f - \Phi v f + \Phi v f - L_1 f + \Phi(u - v) f - L_2 f\|^p \\
&\leq \|\Phi u f - \Phi(u - v) f - \Phi v f\|^p + \|\Phi v f - L_1 f\|^p + \|\Phi(u - v) f - L_2 f\|^p \\
&\leq (\|u - v\| + \|v\|)^p + D_1^p + D_2^p,
\end{aligned}$$

whence, as required,

$$\begin{aligned}
&\|\Phi \circ u - (L_1 + L_2)\| \\
&\leq 3^{1/p-1}(\|u - v\| + \|v\| + D_1 + D_2) \\
&\leq 3^{1/p-1}(\|u - v\| + \|v\| + \operatorname{dist}(\Phi \circ v, \mathsf{L}(F, Y)) + \varkappa(n, p)\|v - u\| + 2\varepsilon). \qquad \square
\end{aligned}$$

We are ready to complete the proof. On account of Lemma 5.3.17, one has

$$\begin{aligned}
&\operatorname{dist}(\Phi \circ u, \mathsf{L}(E_0, Y)) \\
&\leq 3^{1/p-1}\left(\operatorname{dist}(\Phi \circ \tau|_{E_0}, \mathsf{L}(E_0, Y)) + \|\tau|_{E_0}\| + (1 + \varkappa(n, p))\|u - \tau|_{E_0}\|\right) \\
&\leq 3^{1/p-1}\left(\operatorname{dist}(\Phi \circ \tau|_{E_0}, \mathsf{L}(E_0, Y)) + 1 + 3\varepsilon + 2\varepsilon(1 + \varkappa(n, p))\right) \\
&\leq 3^{1/p-1}\left(\operatorname{dist}(\Phi \circ \tau|_{E_0}, \mathsf{L}(E_0, Y)) + 6\right),
\end{aligned}$$

provided $\varepsilon \leq 1/(1 + \varkappa(n, p))$ – this is therefore the precise value of ε we need to start the proof! On the other hand,

$$\begin{aligned}
\lambda M \leq \operatorname{dist}(\Phi|_E, \mathsf{L}(E, Y)) = \operatorname{dist}(\Phi \circ uu^{-1}, \\
\mathsf{L}(E, Y)) \leq \|u^{-1}\| \operatorname{dist}(\Phi \circ u, \mathsf{L}(E_0, Y)).
\end{aligned}$$

But $\|u^{-1}\| \leq (1 + \varepsilon) \leq 2$, and so $\operatorname{dist}(\Phi \circ u, \mathsf{L}(E_0, Y)) \geq \lambda M/2$, and therefore $\lambda M/2 \leq 3^{1/p-1}\left(\operatorname{dist}(\Phi \circ \tau|_{E_0}, \mathsf{L}(E_0, Y)) + 6\right)$, whence one gets $\operatorname{dist}(\Phi \circ \tau, \mathsf{L}(X, Y)) \geq \operatorname{dist}(\Phi \circ \tau|_{E_0}, \mathsf{L}(E_0, Y)) \geq \lambda M(2 \cdot 3^{1/p-1})^{-1} - 6$, while $Q(\Phi \circ t) \leq Q(\Phi) \cdot \|\tau\| \leq \lambda$, and thus $K_0[X, Y]$ cannot be finite. $\square$

Twisted Sums and Ultrapowers

We tackle the next two cases: when does $\operatorname{Ext}(X, Y) = 0$ imply $\operatorname{Ext}(X_{\mathcal{U}}, Y) = 0$ or $\operatorname{Ext}(X, Y_{\mathcal{U}}) = 0$? Here the UAP, which is the approximation property most suited to work with ultraproducts, can go berserk. Let us begin with a companion for Theorem 5.3.15:

Theorem 5.3.18 *Let X be a Banach space, and let Y be a quasi-Banach ultrasummand such that* $\operatorname{Ext}(X, Y) = 0$. *If X has the UAP then* $\operatorname{Ext}(X_{\mathcal{U}}, Y) = 0$ *for all ultrapowers of X for every countably incomplete ultrafilter $\mathcal{U}$.*

Proof The proof follows that of Theorem 5.3.15, but it is simpler. Suppose there is a countably incomplete ultrafilter $\mathcal{U}$, based on I, such that $\operatorname{Ext}(X_{\mathcal{U}}, Y) \neq 0$, and let $\Phi\colon X_{\mathcal{U}} \longrightarrow Y$ be a quasilinear map with $Q(\Phi) \leq 1$. Fix $M > 0$ and pick a finite-dimensional subspace F of $X_{\mathcal{U}}$ such that $\operatorname{dist}(\Phi|_F, L(F, Y)) > \lambda M$. Let $f^1, \ldots, f^n$ be a (normalised) basis of F. Write $f^k = [(f_i^k)]$, and for each $i \in I$, put $F_i = [f_i^1, \ldots, f_i^n]$. For each i, take $\tau_i \in \mathfrak{L}(X)$ such that $\tau_i|_{F_i} = \mathbf{1}_{F_i}$, with $\|\tau_i\| \leq \lambda$ and $\dim(\tau_i[X]) \leq r(n)$, and set $F_i = \tau_i[X]$ and $G = [G_i]_{\mathcal{U}}$. Obviously, G contains F. Take $f^{n+1}, \ldots, f^m \in F$ such that the enlarged system $f^1, \ldots, f^n, f^{n+1}, \ldots, f^m$ is a basis of G. For $n + 1 \leq k \leq m$, we can write

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$f^k = [(f_i^k)]$, where $f_i^k \in G_i$. Given $i \in I$, we define an operator $u_i\colon G \longrightarrow G_i$, taking $u_i(f^k) = f_i^k$ for $1 \le k \le m$. Now, for every $\varepsilon > 0$, the set $\{i \in I\colon u_i$ is an ε-isometry$\}$ belongs to $\mathfrak{U}$. Fix $s \in I$ such that u_s is a 1-isometry. In particular, $\|u_s\| \le 2$ and $\|u_s^{-1}\| \le 2$. We will prove that the composition $\Phi \circ u_s^{-1}\tau_s$ is a 'bad' quasilinear map. The following commutative diagram, where unlabelled arrows are plain inclusion maps, can help the reader to visualise the relevant information:

$$\begin{tikzcd}
 & F_s \arrow[dl] \arrow[r, "u_s^{-1}"] \arrow[dd] & F \arrow[dd] \arrow[dr] & \\
X \arrow[dr, "\tau_s"] \arrow[ddrrr, bend right, "\Phi\circ u_s^{-1}\tau_s"'] & & & X_{\mathfrak{U}} \arrow[dl, "{[\tau_i]_{\mathfrak{U}}}"] \\
 & G_s \arrow[r, "u_s^{-1}"] & G \arrow[dr, "\Phi|_G"] & \\
 & & & Y
\end{tikzcd}$$

Observe that $Q(\Phi \circ u_s^{-1}\tau_s) \le Q(\Phi)\|u_s^{-1}\|\|\tau_s\| \le 2\lambda$, while

$$\begin{aligned}
\operatorname{dist}(\Phi \circ u_s^{-1}\tau_s, \mathsf{L}(X,Y)) &\ge \operatorname{dist}(\Phi \circ u_s^{-1}\tau_s|_{F_s}, \mathsf{L}(F_s, Y)) \\
&= \operatorname{dist}(\Phi \circ u_s^{-1}|_{F_s}, \mathsf{L}(F_s, Y)) \\
&\ge \frac{1}{\|u_s\|}\operatorname{dist}(\Phi \circ u_s^{-1}u_s, \mathsf{L}(E,Y)) \ge \frac{\lambda M}{2}.
\end{aligned}$$

Since M is arbitrary, we get $K_0[X,Y] = \infty$. □

And so we arrive at:

Theorem 5.3.19 *Let X be a separable Banach space and let Y be a Banach space such that* $\operatorname{Ext}_{\mathbf{B}}(X,Y) = 0$. *If X has the BAP or Y has the UAP then* $\operatorname{Ext}_{\mathbf{B}}(X, Y_{\mathfrak{U}}) = 0$ *for all countably incomplete* $\mathfrak{U}$.

Proof Let $0 \longrightarrow \kappa(X) \longrightarrow \ell_1 \longrightarrow X \longrightarrow 0$ be a projective presentation of X and let $\tau : \kappa(X) \longrightarrow Y_{\mathfrak{U}}$ be an operator. We must show that τ extends to ℓ_1. To that end, suppose Y has the UAP so that $Y_{\mathfrak{U}}$ enjoys the BAP. Then the range of τ is contained in a separable subspace of $Y_{\mathfrak{U}}$ with the BAP and, by Theorem 2.14.5, τ lifts to an operator $t\colon \kappa(X) \longrightarrow \ell_\infty(I,Y)$ that can be written as $t = (t_i)$, with $t_i \in \mathfrak{L}(\kappa(X), Y)$. Since $\operatorname{Ext}_{\mathbf{B}}(X,Y) = 0$, each t_i can be extended to an operator $T_i\colon \ell_1 \longrightarrow Y$ with $\|T_i\| \le C\|t_i\|$. Then $T = (T_i)$ is an operator $T\colon \ell_1 \longrightarrow \ell_\infty(I,Y)$, and the composition

$$\ell_1 \xrightarrow{\ T\ } \ell_\infty(I,Y) \xrightarrow{\ [\cdot]\ } Y_{\mathfrak{U}}$$

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is an extension of τ. If instead we use the hypothesis that X has the BAP then we use Proposition 5.3.4 to get that $\kappa(X)$ has the BAP, and Theorem 2.14.5 once more yields the required lifting of τ. □

Finally, we treat the remaining case: does $\operatorname{Ext}(X,Y) = 0 \Rightarrow \operatorname{Ext}(X,Y^{**}) = 0$? Since all even-order duals of Y are complemented in suitable ultrapowers of Y, and since $\operatorname{Ext}_{\mathbf{B}}(X,Y^{**}) = \operatorname{Ext}_{\mathbf{B}}(Y^*,X^*)$, Theorem 5.3.15 yields:

Corollary 5.3.20 *Let X, Y be Banach spaces, with X separable. Assume that either X has the BAP or Y has the UAP and that* $\operatorname{Ext}_{\mathbf{B}}(X,Y) = 0$. *Then* $\operatorname{Ext}_{\mathbf{B}}(X,Y^{**}) = \operatorname{Ext}_{\mathbf{B}}(Y^*,X^*) = 0$.

This provides a rather unexpected partial answer for what we might call the duality problem: does $\operatorname{Ext}(X,Y) = 0$ imply $\operatorname{Ext}(Y^*,X^*) = 0$? It would be interesting to know if the approximation properties are truly necessary here. The main difficulty for a direct attack is that there are elements in $\operatorname{Ext}_{\mathbf{B}}(Y^*,X^*)$ that are not duals of elements of $\operatorname{Ext}_{\mathbf{B}}(X,Y)$, as it has been shown in Proposition 2.12.3 and will again be proved in Theorem 10.5.12. Separability cannot be removed in Theorem 5.3.19 because infinite-dimensional ultraproducts via countably incomplete ultrafilters are never injective [22, Theorem 4.6]. Thus, there is some Banach space X for which $\operatorname{Ext}_{\mathbf{B}}(X,(\ell_\infty)_{\mathcal{U}}) \neq 0$, despite having $\operatorname{Ext}_{\mathbf{B}}(X,\ell_\infty) = 0$.

5.4 Notes and Remarks

5.4.1 Which Banach Spaces Are $\mathscr{K}$-Spaces?

From Theorem 5.2.1 we immediately get: *X is a $\mathscr{K}$-space if and only if* $K[X,\mathbb{K}] < \infty$. The following variation there makes sense: X is a $\mathscr{K}_0$-space if $K_0[X,\mathbb{K}] < \infty$; namely, there is a constant C such that for every bounded quasilinear functional $\phi\colon X \longrightarrow \mathbb{K}$ there is $x^* \in X^*$ such that $\|\phi - x^*\| \leq CQ(\phi)$; equivalently, $\operatorname{Ext}(X,\mathbb{K})$ is Hausdorff. Perhaps the most interesting problems on *Banach* $\mathscr{K}$-spaces are deciding whether every $\mathscr{K}_0$-space is a $\mathscr{K}$-space (the converse is obvious) and characterising Banach $\mathscr{K}$- and $\mathscr{K}_0$-spaces. Kalton repeatedly conjectured that 'not containing ℓ_1^n uniformly complemented' is the right characterisation of $\mathscr{K}$-spaces; cf. [285, p. 815], [257, p. 11], [279, Remark on p. 44], [269, Problem 4.2]. In any case, anyone daydreaming about proving this conjecture should take into account that it implies that ultrapowers, thus all even duals, of Banach $\mathscr{K}$-spaces are $\mathscr{K}$-spaces too and also that all Banach $\mathscr{K}_0$-spaces are $\mathscr{K}$-spaces. A first step in this direction follows from Theorems 5.3.15 and 5.3.18:

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Corollary *Let X be a Banach $\mathscr{K}$-space. If X^{**} has the BAP then it is a $\mathscr{K}$-space, and if X has the UAP then all ultrapowers of X are $\mathscr{K}$-spaces.*

Not much is known about the nature of Banach $\mathscr{K}$-spaces, and the gap between Kalton's conjecture and the current list of members of the club is indeed oceanic. In particular, we do not know whether the following are or are not $\mathscr{K}$-spaces: Pisier's spaces, i.e. spaces P such that $P \otimes_\pi P = P \otimes_\varepsilon P$ [389]; James' quasireflexive space [216; 217]; the spaces $\mathfrak{K}(\ell_2)$ and $\mathfrak{L}(\ell_2)$; non-commutative L_p spaces built over a von Neumann algebra with no minimal projection and $0 < p < 1$; the p-Gurariy spaces, $0 < p < 1$ in Chapter 6; the Hardy classes H_p for $0 < p < 1$ (see [251, Problem 6]); the spaces of vector-valued functions $\ell_p(E), L_p(E), c_0(E), C(K, E)$ when $p \neq 1$ and E is a $\mathscr{K}$-space, as is the case when $E = \ell_2$ and $p > 1$. Of course, we know no example of a Banach $\mathscr{K}_0$-space whose ultrapowers fail to be $\mathscr{K}_0$-spaces, and the same for $\mathscr{K}$-spaces or for quasi-Banach spaces. On the other hand, if $X_\mathfrak{U}$ is either a $\mathscr{K}_0$-space or a $\mathscr{K}$-space then so is the base space X.

5.4.2 Twisting a Few Exotic Banach Spaces

There are three methods available for twisting exotic Banach spaces: the local methods developed in this chapter, forming pullbacks / pushouts from other examples and, in the presence of unconditional basis, the quasilinear Kalton–Peck technique as well. In this section, written in a hakuna matata style, we will make all approaches cavort together.

Corollary 5.2.21 implies that if a Banach space X contains ℓ_1^n uniformly complemented then $\operatorname{Ext}_{\mathbf{B}}(X^*, X) \neq 0$. If X contains both ℓ_∞^n and ℓ_1^n uniformly complemented then so does X^*, and thus $\operatorname{Ext}_{\mathbf{B}}(X, X) \neq 0$ and $\operatorname{Ext}_{\mathbf{B}}(X^*, X^*) \neq 0$. Most reflexive spaces X contain uniformly complemented copies of ℓ_p^n for some p, which, reasoning similarly, implies that none of the spaces $\operatorname{Ext}_{\mathbf{B}}(X^*, X)$, $\operatorname{Ext}_{\mathbf{B}}(X, X)$, $\operatorname{Ext}_{\mathbf{B}}(X^*, X^*)$ or $\operatorname{Ext}_{\mathbf{B}}(X, X^*)$ is 0. That B-convex spaces can always be twisted with themselves (Corollary 5.2.17) is perhaps the most staggering result in this line.

Basic information on Schrerier, Baernstein and Tsirelson spaces can be found in [88]. The Schreier space S is likely the *fons et origo* of non-classical spaces and the one that opened the door to Tsirelson-like spaces and these to H.I. spaces. A general construction of Schreier-like spaces can be simply done by fixing a compact family $\mathcal{A} \subset \{0, 1\}^{\mathbb{N}}$ of finite subsets of $\mathbb{N}$ containing the singletons and such that $G \subset F \in \mathcal{A} \implies G \in \mathcal{A}$ and defining the space $\mathsf{S}_\mathcal{A}$ to be the completion of the space of finitely supported sequences with respect to the norm $\|x\|_\mathcal{A} = \sup_{F \in \mathcal{A}} \|1_F x\|_1$. The space $\mathsf{S}_\mathcal{A}$ has a shrinking unconditional

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basis formed by the unit vectors. It is a subspace of $C(\mathcal{A})$ which, since $\mathcal{A}$ is countable, is c_0-saturated as well as, necessarily, $\mathsf{S}_{\mathcal{A}}$. In particular, $\mathsf{S}_{\mathcal{A}}$ always contains ℓ_∞^n uniformly complemented. If one chooses the family of finite subsets of size at most n for $\mathcal{A}$ then $\mathsf{S}_{\mathcal{A}}$ is just a renorming of c_0. Thus, to obtain something interesting, one needs to assume that $\mathcal{A}$ contains arbitrarily large sets, and that forces $\mathsf{S}_{\mathcal{A}}$ to contain ℓ_1^n uniformly complemented. Since both $\mathsf{S}_{\mathcal{A}}$ and $\mathsf{S}_{\mathcal{A}}^*$ have unconditional basis, the Kalton–Peck map provides non-trivial elements of $\operatorname{Ext}(\mathsf{S}_{\mathcal{A}}, \mathsf{S}_{\mathcal{A}})$ and $\operatorname{Ext}(\mathsf{S}_{\mathcal{A}}^*, \mathsf{S}_{\mathcal{A}}^*)$ that must be non-locally convex by the just mentioned presence of complemented copies of ℓ_1^n. However, the local argument displayed at the beginning of this section yields $\operatorname{Ext}_{\mathbf{B}}(\mathsf{S}_{\mathcal{A}}, \mathsf{S}_{\mathcal{A}}) \neq 0$ as well as $\operatorname{Ext}_{\mathbf{B}}(\mathsf{S}_{\mathcal{A}}^*, \mathsf{S}_{\mathcal{A}}^*)$, $\operatorname{Ext}_{\mathbf{B}}(\mathsf{S}_{\mathcal{A}}^*, \mathsf{S}_{\mathcal{A}})$ and $\operatorname{Ext}_{\mathbf{B}}(\mathsf{S}_{\mathcal{A}}, \mathsf{S}_{\mathcal{A}}^*)$. The first example of a non-trivial family was introduced by Schreier [428], the *admissible* sets – those such that $|A| \leq \min A$. There are many more interesting families in sight (e.g. [7; 104; 105]) generating Schreier-like spaces with their own twisted properties. Baernstein spaces come next as reflexive versions of the Schreier space, and they, too, can be twisted. The next cairn in this road is Tsirelson's space T: a reflexive space with unconditional basis without copies of ℓ_p but containing ℓ_1^n uniformly complemented. Therefore, $\operatorname{Ext}_{\mathbf{B}}(\mathsf{T}, \mathsf{T}^*) \neq 0$ and $\operatorname{Ext}(\mathsf{T}, \mathsf{T}) \neq 0$. We do not, however, know whether $\operatorname{Ext}_{\mathbf{B}}(\mathsf{T}, \mathsf{T}) \neq 0$. Other examples could be given, such as asymptotically ℓ_1-spaces which, containing ℓ_1^n uniformly complemented, are twistable. Or the James Tree space JT, which became famous in the 1960s because the quotient $\mathsf{JT}^{**}/\mathsf{JT}$ is a Hilbert space whose dimension is the continuum. Hence JT and its predual JT_* contain ℓ_2^n uniformly complemented. Since JT_*, moreover, contains ℓ_1^n uniformly complemented, and so JT contains ℓ_∞^n uniformly, these spaces turn out to be very twistable.

Sources

Most of the material opening Section 5.1 is taken from [255], where Kalton introduced $\mathscr{L}_p$-spaces for $0 < p < 1$ and classified them as discrete, continuous and hybrid. Definition 5.1.2 is modelled on the notion of locally complemented subspace of Fakhouri [168]. The trick of Proposition 5.1.6 is often called the 'Lindenstrauss compactness argument' and appears in [323, Proof of Theorem 2.1] and [324]. Lindenstrauss works with Banach spaces complemented in the bidual and uses the weak* topology to paste the pieces together. The quasi-Banach version of Proposition 5.1.6 is reminiscent of the classical proof that biduals are complemented subspaces of suitable ultraproducts. That adaptation already appeared in [255] and has been used several times throughout the chapter (cf. Propositions 5.1.6 and 5.2.18, Theorem 5.3.13). The uniform boundedness principle for quasilinear maps is a gem that Kalton obtained in his first paper on twisted sums [251]. The interpretation Kalton gives in that paper, keen to follow the ideas of Enflo, Lindenstrauss and Pisier [167], is not as clean as it seems nowadays since there was not a clear connection between quasilinear maps and extensions by then. The ideas of Section 5.2 were known for a long time by all those who knew them. The exposition follows [67] from where the examples in Proposition 5.2.20 were taken. The general construction in Proposition 5.2.20 is, however, from [73], while the dual version of Proposition 5.2.22 is from the Kalton–Pełczyński paper [284]. The literature contains several glimmering (or less) variations of Proposition 5.3.4. Indeed, Lusky proved in [348; 349] that if X is a separable Banach space with a basis, then the kernel of any quotient map $\rho\colon \ell_1 \longrightarrow X$ has basis. Now, when X has the BAP, $X \times C_0$ has basis [348]. So, let $\rho\colon \ell_1 \longrightarrow X$ and $\rho'\colon \ell_1 \longrightarrow C_0$ be quotient maps. The operator $\rho \times \rho'\colon \ell_1 \times \ell_1 \longrightarrow X \times C_0$ is a quotient map, and thus $\ker(\rho \times \rho') = \ker\rho \times \ker\rho'$ has a basis. Therefore $\ker\rho$ has the BAP. Another forerunner of Proposition 5.3.4 appears in [173]: Figiel, Johnson and Pełczyński proved that if X^* has the BAP, then the space Y in any exact sequence $0 \longrightarrow Y \longrightarrow \mathscr{L}_1 \longrightarrow X \longrightarrow 0$ must have the BAP. In fact, they establish that Y^* has the BAP through Proposition 5.3.9, which corresponds to their [173, Theorem 2.1.b]. The results about BAP in kernels of projective presentations in Section 5.3 come from [115] and those of Section 5.3 from [173]. Section 5.3 is taken from [115]. Theorems 5.3.15 and 5.3.18 appear here for the first time. Theorem 5.3.19 and its corollary appear in [22] (see also [22, Sections 4.4 and 4.5]). The analogue of Proposition 5.3.4 is due to Figiel, Johnson and Pełczyński [173]. The subsequent paper [172] contains versions of Proposition 5.3.9 and Lemma 5.3.10 for the bare approximation property, namely, that if a Banach space with the AP embeds into an $\mathscr{L}_\infty$-space then the quotient has the AP (Corollary 2 in [172]) and also, that when an $\mathscr{L}_\infty$-space embeds into a Banach space with the AP, the quotient space has the AP too (Corollary 1 in [172]). The twisting of Schreier, Tsirelson and James Tree spaces was first performed in [220] using what the authors called ‘co-local structures’.

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