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Addendum to "Limit Sets of Typical Homeomorphisms"

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Abstract. Given an integer $n \geq 3$, a metrizable compact topological n-manifold X with boundary, and a finite positive Borel measure μ on X, we prove that for the typical homeomorphism $f\colon X \to X$, it is true that for μ-almost every point x in X the restriction of f (respectively of f^{-1}) to the omega limit set $\omega(f,x)$ (respectively to the alpha limit set $\alpha(f,x)$) is topologically conjugate to the universal odometer.

This note is an addendum to my previous work [3] on limit sets of typical homeomorphisms. The goal is to point out that, by combining the methods used in [3] with a result from [4], the following theorem complementary to [3, Theorem 1.1] can be derived.

Theorem 1 *Let X be a metrizable compact topological manifold with (or without) boundary. Assume X has dimension $n \geq 2$; if the boundary of X is nonempty, then assume $n \geq 3$. Let μ be a finite positive Borel measure on X. For the typical $f \in H(X)$, there exists a residual set G_f such that $\mu(G_f) = \mu(X)$ and the following property holds for every point x in G_f:*

(e) *The restriction of f (respectively of f^{-1}) to the omega limit set $\omega(f,x)$ (respectively to the alpha limit set $\alpha(f,x)$) is topologically conjugate to the universal odometer.*

In the above statement as well as in all that follows we are adopting the notations and terminology used in [3].

Let us recall that the omega limit set $\omega(f,x)$ (respectively the alpha limit set $\alpha(f,x)$) of f at x is the set of all limit points of the sequence $(f^j(x))_{j\geq 0}$ (respectively of the sequence $(f^{-j}(x))_{j\geq 0}$). Let us also recall the notion of *universal odometer*. Given $\alpha \in (\mathbb{N}\backslash\{1\})^{\mathbb{N}}$, where $\mathbb{N}$ denotes the set of all positive integers, we consider the product space

$$\Delta_\alpha = \prod_{i=1}^{\infty} \mathbb{Z}_{\alpha(i)},$$

where $\mathbb{Z}_k = \{0, \dots, k-1\}$ is endowed with the discrete topology. Note that Δ_α is a Cantor space, *i.e.,* a totally disconnected compact metrizable space without isolated points. We consider the operation of addition on Δ_α defined in the following way: if $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ are in Δ_α, then $(x_1, x_2, \dots) + (y_1, y_2, \dots) = (z_1, z_2, \dots)$, where $z_1 = x_1 + y_1 \bmod \alpha(1)$ and, in general, z_i is defined recursively as

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$z_i = x_i + y_i + \epsilon_{i-1} \bmod \alpha(i)$, where $\epsilon_{i-1} = 0$ if $x_{i-1} + y_{i-1} + \epsilon_{i-2} < \alpha(i-1)$ (where $\epsilon_0 = 0$) and $\epsilon_{i-1} = 1$ otherwise. If we let f_α be the "+1" map, that is,

$$f_\alpha(x_1, x_2, \dots) = (x_1, x_2, \dots) + (1, 0, 0, \dots),$$

then $(\Delta_\alpha, f_\alpha)$ is a dynamical system known as a solenoid, adding machine, or odometer. Let M_α be the function from the set of all prime numbers into $\{0, 1, 2, \dots, \infty\}$ given by

$$M_\alpha(p) = \sum_{i=1}^{\infty} n(i),$$

where $n(i)$ is the largest integer such that $p^{n(i)}$ divides $\alpha(i)$. When $M_\alpha(p) = \infty$ for every prime number p, f_α is said to be a universal odometer.

The following beautiful characterization of odometers up to topological conjugacy was obtained by Buescu and Stewart in [5] by using an ergodic theoretic approach. A proof based on a topological approach was obtained later by Block and Keesling in [4].

Let $\alpha, \beta \in (\mathbb{N}\backslash\{1\})^{\mathbb{N}}$. Then f_α and f_β are topologically conjugate if and only if $M_\alpha = M_\beta$.

It follows from this result that any two universal odometers are topologically conjugate to each other.

In order to prove Theorem 1, we shall need the following result from [4]:

Let $\alpha \in (\mathbb{N}\backslash\{1\})^{\mathbb{N}}$ and $r_m = \alpha(1)\alpha(2)\cdots\alpha(m)$ for each m. Let $f\colon Y \to Y$ be a continuous map of a compact topological space Y. Then f is topologically conjugate to f_α if and only if the following conditions are satisfied:

(1) *for each $m \in \mathbb{N}$, there is a cover $\mathcal{P}_m$ of Y consisting of r_m nonempty pairwise disjoint clopen sets which are cyclically permuted by f;*
(2) *for each $m \in \mathbb{N}$, $\mathcal{P}_{m+1}$ refines $\mathcal{P}_m$;*
(3) *if $W_1 \supset W_2 \supset W_3 \supset \cdots$ is a nested sequence with $W_m \in \mathcal{P}_m$ for each m, then $\bigcap_{m=1}^{\infty} W_m$ consists of a single point.*

We now explain how to prove Theorem 1. First, it is enough to prove the theorem only for omega limit sets, for then we can apply the result to f^{-1} and intersect the two residual sets. Second, as was shown in the proof of [3, Theorem 1.1], it is enough to prove the theorem with "G_δ set" in place of "residual set". Moreover, it is enough to consider the cases "$\mu(b(X)) = 0$" and "$b(X) \neq \varnothing$ and $\mu(i(X)) = 0$" separately.

CASE I: $\mu(b(X)) = 0$.

For each $k \in \mathbb{N}$, let $\mathcal{O}_k$ be the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{W}_X$-trees $T_1, \dots, T_s$ for f such that:

(i) $\theta(T_i) < 1/k$ for all $1 \leq i \leq s$;
(ii) $\mu\big(X - (Q(T_1) \cup \cdots \cup Q(T_s))\big) < 1/k$;

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(iii) For each $1 \leq i \leq s$, the special branch of T_i has the form

$$S_i = A_{i,1}^{(1)} < A_{i,2}^{(1)} < \cdots < A_{i,d_i}^{(1)} < A_{i,1}^{(2)} < A_{i,2}^{(2)} < \cdots < A_{i,d_i}^{(2)} < \cdots \\ \cdots < A_{i,1}^{(k!)} < A_{i,2}^{(k!)} < \cdots < A_{i,d_i}^{(k!)} = R_i.$$

Clearly, each $\mathcal{O}_k$ is open in $H(X)$. Let $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then for each $k \in \mathbb{N}$, there are pairwise disjoint $\mathcal{W}_X$-trees $T_{k,1}, \ldots, T_{k,s_k}$ for f so that properties (i)–(iii) hold with $T_{k,1}, \ldots, T_{k,s_k}$ in place of $T_1, \ldots, T_s$. By replacing each $T_{k,i}$ by a thickening, if necessary, we may assume the following strengthening of property (ii):

(ii') $\mu\big(X - (\mathrm{Int}Q(T_{k,1}) \cup \cdots \cup \mathrm{Int}Q(T_{k,s_k}))\big) < 1/k$.

Put $Q_k = Q(T_{k,1}) \cup \cdots \cup Q(T_{k,s_k})$ ($k \in \mathbb{N}$) and

$$G = \bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} \mathrm{Int}Q_k.$$

Then G is a G_δ set and $\mu(G) = \mu(X)$ because of property (ii'). By property (iii), the special branch of each $T_{k,i}$ has the form

$$S_{k,i} = A_{k,i,1}^{(1)} < \cdots < A_{k,i,d_{k,i}}^{(1)} < \cdots < A_{k,i,1}^{(k!)} < \cdots < A_{k,i,d_{k,i}}^{(k!)} = R_{k,i}.$$

Fix $x \in G$. There exists an increasing sequence $(k_m)_{m \in \mathbb{N}}$ of positive integers such that $x \in Q_{k_m}$ for all $m \in \mathbb{N}$. Hence, for each $m \in \mathbb{N}$, there exists $1 \leq i_m \leq s_{k_m}$ such that $x \in Q(T_{k_m,i_m})$. By [3, Proposition 2.1(B)],

$$\omega(f,x) \subset \bigcup_{t=1}^{k_m!} \bigcup_{j=1}^{d_{k_m,i_m}} A_{k_m,i_m,j}^{(t)} \qquad (m \in \mathbb{N})$$

and $\omega(f,x) \cap A_{k_m,i_m,j}^{(t)} \neq \varnothing$ whenever $m \in \mathbb{N}$, $1 \leq t \leq k_m!$ and $1 \leq j \leq d_{k_m,i_m}$. For each $m \in \mathbb{N}$, let

$$\mathcal{P}_m = \{\omega(f,x) \cap A_{k_m,i_m,j}^{(t)}; 1 \leq t \leq k_m! \text{ and } 1 \leq j \leq d_{k_m,i_m}\}.$$

Then each $\mathcal{P}_m$ is a partition of $\omega(f,x)$ in nonempty clopen subsets that are cyclically permuted by f. Hence, condition (1) of the above-mentioned result from [4] holds (with $Y = \omega(f,x)$ and $r_m = d_{k_m,i_m} k_m!$). By passing to a subsequence, if necessary, we can also assume that condition (2) holds (because of property (i)) and that

$$\frac{k_{m+1}!}{d_{k_m,i_m} k_m!} \text{ is a multiple of } m!,$$

for every $m \in \mathbb{N}$. Moreover, condition (3) also follows from property (i). Thus, $f|_{\omega(f,x)} : \omega(f,x) \to \omega(f,x)$ is topologically conjugate to f_α, where

$$\alpha = \Big(d_{k_1,i_1} k_1!, \frac{d_{k_2,i_2} k_2!}{d_{k_1,i_1} k_1!}, \frac{d_{k_3,i_3} k_3!}{d_{k_2,i_2} k_2!}, \ldots\Big).$$

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Since $m!$ divides the $(m+1)^{\text{th}}$-coordinate of α for every $m \in \mathbb{N}$, it follows that f_α is a universal odometer.

In order to prove that each $\mathcal{O}_k$ is dense in $H(X)$, let us fix $k \in \mathbb{N}$, $h \in H(X)$, and $\epsilon > 0$. In the proof of [2, Theorem 1 (CASE I)], we saw that the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ for f satisfying properties (i) and (ii) is dense in $H(X)$. Therefore, if we choose an integer $k' \geq k$ such that $1/k' < \epsilon/3$, then there are a function $g \in H(X)$ and pairwise disjoint $\mathcal{W}_X$-trees $T'_1, \ldots, T'_s$ for g so that $\widetilde{d}(g, h) < \epsilon/3$ and (i) and (ii) hold with $T'_1, \ldots, T'_s$ in place of $T_1, \ldots, T_s$ and k' in place of k. Let

$$S'_i = A^{(1)}_{i,1} < \cdots < A^{(1)}_{i,d_i} = R'_i$$

be the special branch of T'_i $(1 \leq i \leq s)$. In the proof of [3, Theorem 1.1 (CASE I)], we explained how to duplicate the lengths of the special branches of the trees $T'_1, \ldots, T'_s$ and to make small pertubations on g in order to obtain $u \in H(X)$ and pairwise disjoint $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ for u so that $\widetilde{d}(u, g) < \epsilon/3$ and (i)–(iii) hold with the number 2 instead of $k!$ in property (iii). By using a similar reasoning, we can triplicate, quadruplicate,... the lengths of the special branches of the trees. Hence, by multiplying these lengths by $k!$, we obtain $f \in H(X)$ and pairwise disjoint $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ for f so that $\widetilde{d}(f, g) < \epsilon/3$ and (i)–(iii) hold. So, $f \in \mathcal{O}_k$ and $\widetilde{d}(f, h) < \epsilon$.

CASE II: $b(X) \neq \varnothing$ and $\mu(i(X)) = 0$.

For each $k \in \mathbb{N}$, let $\mathcal{O}_k$ be the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$ for f satisfying properties (i)–(iii) as in CASE I. Then each $\mathcal{O}_k$ is open in $H(X)$ and each $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$ has the desired property. In order to prove that each $\mathcal{O}_k$ is dense in $H(X)$, we fix $k \in \mathbb{N}$, $h \in H(X)$ and $\epsilon > 0$. In the proof of [2, Theorem 1 (CASE II)], we saw that the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$ for f satisfying properties (i) and (ii) is dense in $H(X)$. Now, it is enough to continue by arguing as in CASE I, but we need to consider the collection $\mathcal{Z}_X$ instead of $\mathcal{W}_X$ and the collection $\mathcal{V}_X$ instead of $\mathcal{U}_X$.

By using the same methods we can obtain the following version of Theorem 1 for the space $C(X)$.

Theorem 2 *Let X be a metrizable compact topological n-manifold with (or without) boundary, where $n \geq 1$, and let μ be a finite positive Borel measure on X. For the typical $f \in C(X)$, there exists a residual set G_f such that $\mu(G_f) = \mu(X)$ and the following property holds for every point x in G_f: the restriction of f to the omega limit set $\omega(f, x)$ is topologically conjugate to the universal odometer.*

This theorem complements the main reults in [1].

Let us mention that D'Aniello, Darji, and Steele proved in [6] that for the typical $f \in C(X)$, it is true that for the typical point x in X the restriction of f to the omega limit set $\omega(f, x)$ is topologically conjugate to the universal odometer. Our main goal in the previous theorem was to establish the same conclusion with probability 1, *i.e.,* for μ-almost every point x in X.

It is interesting to call attention to the fact that in the previous theorems, as well as in the main results in [1, 3], it turned out that the same properties hold simultaneously on a residual set and on a set of full measure. As is well known, this is not always the case in dynamics.

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