Compositio Mathematica **138:** 125–156, 2003.

Symplectic Topology of Integrable Hamiltonian Systems, II: Topological Classification

Dedicated to Professor Charles-Michel Marle

NGUYEN TIEN ZUNG
Département de Mathématiques, Université Toulouse 3, 118 Route de Narbonne, 31062 Toulouse, France. e-mail: tienzung@picard.ups-tlse.fr

(Received: 21 May 2001; accepted in final form: 10 June 2002)

Abstract. The main purpose of this paper is to give a topological and symplectic classification of completely integrable Hamiltonian systems in terms of characteristic classes and other local and global invariants.

Mathematics Subject Classifications (2000). 70H06, 53Dxx.

Key words. characteristic class, integrable Hamiltonian system, topological classification.

1. Introduction

In this paper we are concerned with topological aspects of completely integrable Hamiltonian systems, or integrable systems in short. The problem of studying local and global topological properties of integrable systems is a very natural problem, with many possible applications, and it has attracted many mathematicians over the past decades. There are even a few recent books on the subject (see, e.g., [2, 5, 8, 17, 23]).

The purpose of this paper is to give a topological and symplectic classification of integrable systems in terms of characteristic classes and other local and global invariants. Before trying to formulate our theorems, let us recall some of the main results that have been obtained to date in this direction.

One of the most remarkable results is due to Duistermaat [14], who defined the *monodromy* and the *Chern class* of the *regular part* of an integrable system. The monodromy phenomenon has since then been studied by many authors, for both classical and quantum integrable systems (see, e.g., [8, 9, 22, 35, 36]). It is observed in [40] that the so-called *focus-focus singularities* are the main (if not the only practical) source of nontrivial monodromy for 'real-world' integrable systems. The work of Duistermaat was developed further and extended to the case of complete isotropic fibrations by Dazord and Delzant [10], and extended by Boucetta and Molino [6] to include nondegenerate elliptic singularities of integrable systems.

On the other hand, Fomenko and his collaborators (see, e.g., [5, 17]) developed a *Morse theory* for integrable systems, which takes into account *corank*-1 singularities.

Fomenko and his school studied mainly systems with two degrees of freedom, though some results are also valid for higher-dimensional systems. In particular, they obtained a complete topological classification of nondegenerate integrable systems on isoenergy 3-manifolds.

The main drawback of both Duistermaat's and Fomenko's theories is the absence of higher-corank nonelliptic singularities in their picture. It is not surprising, since the topological structure of higher-corank nonelliptic singularities was almost completely unknown until more recently. This drawback is quite significant, because interesting things often happen at singularities, and a lot of global information is contained there.

Lerman and Umanskii [23] were among the first people who attacked the problem of describing the topology of corank-2 nondegenerate singularities of integrable systems. However, their description is a little bit too complicated in our view, and they restricted their attention to the case with only one fixed point in a system with two degrees of freedom.

In a series of papers [39–42], we studied local and semi-local aspects of singularities integrable systems. Our main results include: the existence of a local converging Birkhoff normal form for any analytic integrable system, the existence of local torus actions and partial action-angle coordinates, and the topological decomposition of nondegenerate singularities of higher corank into almost direct products of simplest singularities (elliptic and hyperbolic corank-1 and focus-focus corank-2). In particular, we gave a much simpler description of singularities studied by Lerman and Umanskii.

What we do in this paper is to combine our knowledge of singularities with the ideas of Duistermaat, Fomenko and others in order to study global aspects of 'generic' integrable systems with singularities. In particular, we will develop the notions of monodromy and Chern class to take singularities into account. It was a non-tri vial task, because we don't know of a general recipe to define characteristic classes for singular fibrations: In our case in general there are no 'local sections' or 'trivial systems' to speak of, so the obstruction theory does not work directly. One may try to define some kind of classifying space and universal system, but we have no idea what they might look like at the moment. And one may try to use the sheaf of local automorphisms, but this is a very big non-Abelian sheaf, not easy to deal with. So our first attempts at defining characteristic classes [38] were not very successful. We then realized that a more detailed study of the sheaf of local automorphisms of an integrable system allows us to reduce this non-Abelian structural sheaf to a very nice finite-dimensional Abelian subsheaf, which we will call the *affine monodromy sheaf*. Our main characteristic class, which will be called the *Chern class* and which classifies 'generic' integrable systems topologically, is an element of the second cohomology group of this affine monodromy sheaf. In the regular part of the system, the affine monodromy sheaf is essentially the same as the monodromy defined by Duistermaat, and our Chern class is also essentially the same as the one defined by Duistermaat. In the case of 2-degree-of-freedom systems on isoenergy 3-manifolds studied by Fomenko *et al.*, our affine monodromy contains information about the 'marks' in the so-called 'marked molecules' in Fomenko's classification.

Let us now formulate the main results of this paper.

Let $\mathbf{F} = (F_1, \dots, F_n): (M^{2n}, \omega) \to \mathbb{R}^{z\,n}$ be a smooth moment map of a completely integrable Hamiltonian system with n degrees of freedom. We will always assume $\mathbf{F}$ to be a proper map. In particular, regular connected components of the level sets of $\mathbf{F}$ are Lagrangian tori, according to a classical theorem of Liouville. Denote by O the space of connected components of the level sets of F. We will call O the *base space*, or also the *orbit space*, of the system. We have a projection π from (M^{2n}, ω) to O and a map $\tilde{\mathbf{F}}: O \to \mathbb{R}^n$, such that $\mathbf{F} = \tilde{\mathbf{F}} \circ \pi$. The topology of O is induced from M^{2n}. For 'generic' integrable systems, O is a stratified n-dimensional manifold. We may view $\pi: (M^{2n}, \omega) \to O$ as the projection map of a singular Lagrangian torus fibration. We will call it the *associated singular Lagrangian fibration* of the system, and denote it by $\mathcal{L}$: fibers of $\mathcal{L}$ are connected components of preimages of $\mathbf{F}$. If the original Hamiltonian system is nonresonant, then $\mathcal{L}$ is essentially unique, i.e. it does not depend on the choice of the moment map $\mathbf{F}$.

Since we are dealing with topological aspects of integrable systems, we will be more interested in the singular fibration $\mathcal{L}$ than in the moment map $\mathbf{F}$. In particular, throughout this paper, we will adopt the following definition of integrable systems:

DEFINITION 1.1. A singular Lagrangian torus fibration $\pi: (M^{2n}, \omega, \mathcal{L}) \to O$ is called an *integrable system* (from the geometric point of view) if near each fiber of $\mathcal{L}$ (i.e. preimage of π) it is defined by a set of n commuting (with respect to the Poisson bracket) functionally independent functions on M^{2n}. Two integrable systems are called *topologically equivalent* if there is a fibration-preserving homeomorphism between them, and they are called *symplectically equivalent* if there is a smooth fibration-preserving symplectomorphism between them.

Notice that, in the above definition, we don't require the global existence of a moment map. We just require it to exist in a neighborhood of every singular fiber.

The affine monodromy sheaf is a sheaf over the base space O and is defined as follows (see Subsection 3.3):

DEFINITION 1.2. The sheaf $\mathcal{R}$ over the base space O, which associates to each open subset $U \subset O$ the free Abelian group $R(U)$ of symplectic system-preserving $\mathbb{S}^1$-actions in $(\pi^{-1}(U), \omega, \mathcal{L})$ is called the *affine monodromy sheaf* of the system.

Besides affine monodromy, we will need another notion of monodromy which we call *homological monodromy*, which involves first homology groups of the strata of the fibers of the system and of the fibers themselves, and which will be explained in Subsection 4.1. In Subsection 4.1 we introduce the notion of *rough equivalence* of integrable systems, which may be reformulated as follows:

DEFINITION 1.3. Two integrable systems over the same base spaces are called *roughly topologically equivalent* if they have the same singularities topologically, and the same homological monodromy.

It is evident that if two systems are topologically equivalent, then they are also roughly topologically equivalent, after an appropriate identification of their base spaces. And under some hypotheses made in Section 3 about 'genericity' of systems under consideration, if two systems are roughly topologically equivalent then they will have the same affine monodromy.

In order to define the Chern class of an integrable system, we will have to compare it to a *reference system* which is roughly topologically equivalent to it. (For regular fibrations, there is a natural choice of the reference system, which is the one with a global section, so one does not have to mention it. But in our general case there is no such a-priori choice). The definition of the Chern class involves some cohomological exact sequence and is explained in Subsection 4.2. The Chern class is an element of $H^2(O, \mathcal{R})$, where O is the base space and $\mathcal{R}$ is the affine monodromy sheaf. Our main result is the following (see Subsection 4.5)

THEOREM 1.4. *Two roughly topologically equivalent integrable Hamiltonian systems are topologically equivalent if and only if they have the same Chern class with respect to a common reference system.*

What makes the above theorem effective is that in many cases it is relatively easy to compute the cohomology group $H^2(O, \mathcal{R})$. Though $\mathcal{R}$ is not a locally constant sheaf in general, it is locally constant on the strata of O, and is a 'constructible' free Abelian sheaf nevertheless.

Similarly, we have the following symplectic classification of integrable systems: in Subsection 4.1 we introduce the notion of *rough symplectic equivalence*, which is rough topological equivalence plus a condition of the symplectic nature of the involved local automorphisms. In particular, if two systems are roughly symplectically equivalent then their singularities are symplectically equivalent (see, e.g., [5, 13, 36] for some results on symplectic invariants of singularities of integrable systems). Then we introduce in Subsection 4.2 the *Lagrangian class*, which is a characteristic class which lies in $H^1(O, \mathcal{Z}^1/\mathcal{R})$. Here $\mathcal{Z}^1$ is the sheaf of local closed differential 1-forms on O (see Subsection 3.6), and there is a natural injection from $\mathcal{R}$ to $\mathcal{Z}^1$. We have (see Subsection 4.5):

THEOREM 1.5. *Two roughly symplectically equivalent integrable Hamiltonian systems are symplectically equivalent if and only if they have the same Lagrangian class with respect to a common reference system.*

Theorem 1.5 is in fact much easier to prove than Theorem 1.4, because the sheaf of local system-preserving symplectomorphisms is much smaller than the sheaf of local system-preserving homeomorphisms. On the other hand, it is a highly non-trivial problem to classify higher corank singularities of integrable systems symplectically. For systems without singularities or with only elliptic singularities, Theorem 1.5 coincides with some results obtained earlier by Dazord and Delzant [10] and Boucetta and Molino [6].

A bonus of our classification results is the possibility to construct new integrable systems and underlying symplectic structures from the old ones by means of surgery. We call it *integrable surgery*, and give a few examples of this method (exotic symplectic spaces, toric manifolds, K3, etc.) in Subsection 4.8.

In this introduction, we often use the word 'generic' without explaining what it is. The problem is, though we know intuitively what does a 'generic' integrable system mean, we don't have a clear-cut definition, and we know very little about degenerate higher corank singularities. So in Section 3, we will give a series of 7 'very reasonable' hypotheses, (H1)–(H7), about singularities of integrable systems, and conjecture that all singularities of a 'generic' integrable system, whatever it means, must satisfy these hypotheses. These hypotheses have been verified for nondegenerate singularities. Theorem 1.4 is proved under the assumption that hypotheses (H1)–(H5) are satisfied, and Theorem 1.5 is proved under all 7 hypotheses.

The rest of this paper is organized as follows: In Section 2 we give an exposition of the theory of regular Lagrangian torus fibrations developed by Duistermaat, Dazord and Delzant [10, 14] (see also [6, 25]). Though most of the material of this section is not new, we include it here for the convenience of readers, and to make it easier to see the similarities between the regular case and the case with singularities. Section 3 contains the main preparation work of this paper, where we will study the structure of the base space, local automorphisms of singularities, and write down a series of hypotheses about singularities. In particular, we will define and study the necessary sheaves for our characteristic classes there. Section 4 contains the definition of characteristic classes, the classification theorems, and a discussion of the realization problem and integrable surgery.

Remark. In some of our previous papers, we used the words 'associated singular Lagrangian *foliation*' to call the associated fibration $\mathcal{L}$ of an integrable system, but now we feel that the word *fibration* is a more correct one. Likewise, we will use the word *fiber* instead of *leaf* when we talk about a connected component of a level set of the moment map. This new convention will save us from confusions when dealing with hyperbolic-type singular fibers, i.e. fibers that contain more than one orbit of the Poisson action of the moment map, because each (singular) orbit of the Poisson action is a (singular) leaf of the associated singular foliation in the sense of Stefan-Sussmann.

2. Regular Lagrangian Torus Fibrations

If we throw out all singular fibers from an integrable system, then what remains is a regular Lagrangian torus fibration.

2.1. ACTION-ANGLE COORDINATES

Let $\pi\colon (M^{2n}, \omega, \mathcal{L}) \to O$ be a regular Lagrangian fibration with compact fibers. Then according to Arnold–Liouville theorem, each fiber of this fibration is a Lagrangian torus of the symplectic manifold (M^{2n}, ω), called a *Liouville torus*. Moreover, for each point $x \in O$ there is a neighborhood $D^n = D(x)$ of x in O such that $(\pi^{-1}(D^n), \omega) \to D^n$ can be written as $(D^n \times \mathbb{T}^n, \sum_1^n \mathrm{d}p_i \wedge \mathrm{d}q_i) \to D^n$ via a fibration-preserving symplectomorphism, where (p_i) is a system of coordinates on D^n and $(q_i \bmod 1)$ is a system of periodic coordinates on $\mathbb{T}^n$. The functions p_i and q_i are called *action* and *angle* coordinates, respectively.

If (u_i, v_i) is another system of action-angle coordinates in $(\pi^{-1}(D^n), \omega, \mathcal{L})$, then we have

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = A \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} + \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = (A^{-1})^T \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} + \begin{pmatrix} g_1(p_i) \\ \vdots \\ g_n(p_i) \end{pmatrix},$$

where A is an element of $\mathrm{GL}(n, \mathbb{Z})$, c_i are constants, and $\sum_1^n g_i \,\mathrm{d}p_i$ is a closed differential 1-form on D^n.

In particular, the two local systems of action coordinates (p_i) and (u_i) on O are related by an integral affine transformation. Thus the base space O admits a unique natural integral affine structure. This integral affine structure provides O with a volume element, which is equal to $\mathrm{d}p_1, \ldots, \mathrm{d}p_n$ in any local system of action coordinates, and the volume of O is equal to the volume of (M^{2n}, ω) (for the standard volume form $\omega^n/n!$). Each Liouville torus also admits a unique natural affine structure of a flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, given by a system of angle coordinates.

Given a regular Lagrangian torus fibration $\pi\colon (M^{2n}, \omega, \mathcal{L}) \to O$, we can ask if there are global action-angle coordinates. That is, can $(M^{2n}, \omega) \to O$ be written in the form

$$\left(O \times \mathbb{T}^n, \sum_1^n \mathrm{d}p_i \wedge \mathrm{d}q_i\right) \to O$$

where $(p_i)\colon O \to \mathbb{R}^n$ is an immersion and $(q_i \bmod 1)$ is a system of periodic coordinates on $\mathbb{T}^n$.

A natural way to solve the above problem is via *obstruction theory*. If $\pi\colon (M^{2n}, \omega) \to O$ admits a global system of action-angle coordinates, then it has the following properties:

(a) $\pi\colon M^{2n} \to O$ is a principal $\mathbb{T}^n$-bundle.
(b) $\pi\colon M^{2n} \to O$ has a global section.
(c) Moreover, it has a global *Lagrangian* section.

Conversely, if the above conditions are satisfied then one can show easily that $\pi\colon (M^{2n}, \omega) \to O$ admits global action-angle coordinates.

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

The obstruction for the condition (a) to be fulfilled will be called the *monodromy*, or also the *affine monodromy*, because it can be determined completely by the affine structure of the base space O. Obstructions to (b) and (c) will be characterized by the so-called *Chern class* and *Lagrangian class*, respectively.

2.2. MONODROMY

Given a Lagrangian torus fibration $\pi\colon (M^{2n}, \omega) \to O$, we will associate to it the $\mathbb{Z}^n$-bundle of first homology groups of the fibers of $\mathcal{L}$, denoted by $E_{\mathbb{Z}} \xrightarrow{H_1(\mathbb{T}^n,\mathbb{Z})} O$. The holonomy of this bundle and is an element of $\hom(\pi_1(O), \mathrm{GL}(n, \mathbb{Z}))$, defined up to conjugacy, and is called the *monodromy* of the torus fibration.

The symplectic form ω gives rise to a natural isomorphism from the vector bundle $E_{\mathbb{R}} \xrightarrow{H_1(\mathbb{T}^n,\mathbb{R})} O$ (of first homology groups over $\mathbb{R}$ of the fibers of $\mathcal{L}$) to the cotangent bundle T^*O, defined as follows: Let $T_x = \pi^{-1}(x)$ be a fiber of $\mathcal{L}$. Then T_x has a unique canonical flat structure, and constant vector fields on T_x can be identified with $H_1(T_x, \mathbb{R})$ via 'rotation numbers'. On the other hand, if X is a constant vector field on T_x, then the covector field $\alpha(X) = -i_X\omega$ is the pull-back of a covector α on O at x, i.e. an element of T^*O.

Notice that $E_{\mathbb{Z}} \xrightarrow{H_1(\mathbb{T}^n,\mathbb{Z})} O$ is a discrete subbundle of $E_{\mathbb{R}} \xrightarrow{H_1(\mathbb{T}^n,\mathbb{R})} O$. Under the aforementioned identification of $E_{\mathbb{R}}$ with T^*O, $E_{\mathbb{Z}}$ maps to a discrete subbundle of T^*O, consisting of 'integral' covectors. We will denoted this subbundle, or the discrete sheaf associated to it, by $\mathcal{R}$. It follows from Arnold–Liouville theorem that local sections of $\mathcal{R}$ are local differential 1-forms on O which can be written as $\sum m_i\, \mathrm{d}p_i$ in some local system of action coordinates (p_i), with $m_i \in \mathbb{Z}$. Thus $\mathcal{R}$ can be completely determined by the integral affine structure of O. We will call $\mathcal{R}$ the *affine monodromy sheaf*. Since $E_{\mathbb{Z}}$ is isomorphic to $\mathcal{R}$, the monodromy of the system is completely determined by the integral affine structure of O, and will also be called the *affine monodromy*.

There is another characterization of $\mathcal{R}$ as follows: Arnold–Liouville theorem implies that each local differential 1-form on O of the type $\sum m_i\, \mathrm{d}p_i$ with $m_i \in \mathbb{Z}$ in a local system of action coordinates (p_i) gives rise to a symplectic vector field $\sum m_i X_{p_i}$ which generates a symplectic $\mathbb{S}^1$-action which preserves the system, and vice versa. Thus $\mathcal{R}$ is isomorphic to the sheaf of local system-preserving symplectic $\mathbb{S}^1$-actions.

First examples of integrable systems with nontrivial monodromy, namely the spherical pendulum and the Lagrange top, were observed by Cushman and others (e.g., [9, 14]). In these examples and all other known examples arising from classical mechanics and physics, the nontriviality of the monodromy is due to the presence of the so-called focus-focus singularities (see, e.g., [40]).

2.3. CHERN AND LAGRANGIAN CLASSES

The Chern class can be defined as the obstruction for the torus fibration $M \to O$ to admit a global section. Let (U_i) be a trivializing open covering of O. Over each U_i

there is a smooth section, denoted by s_i. The difference between two local sections s_i and s_j, over $U_i \cap U_j$, can be written as

$$\mu_{ij} = s_j - s_i \in C^\infty(E_{\mathbb{R}}/E_{\mathbb{Z}})(U_i \cap U_j) \cong C^\infty(T^*O/\mathcal{R})(U_i \cap U_j).$$

Here $C^\infty(\cdot)$ denotes the sheaf of smooth sections, and $E_{\mathbb{R}}$ and $E_{\mathbb{Z}}$ are the first cohomology bundles defined in the previous subsection. It is immediate that (μ_{ij}) is a 1-cocycle, and it defines a Čech first cohomology class, not depending on the choice of sections:

$$\hat{\mu} \in H^1(O, C^\infty(T^*O/\mathcal{R})).$$

Since $C^\infty(T^*O)$ is a fine sheaf, from the short exact sequence

$$0 \to \mathcal{R} \to C^\infty(T^*O) \to C^\infty(T^*O/\mathcal{R})) \to 0$$

we obtain that the coboundary map $\delta\colon H^1(O, C^\infty(T^*O/\mathcal{R})) \to H^2(O, \mathcal{R})$ in the associated long exact sequence is an isomorphism.

The image μ_C of $\hat{\mu}$ in $H^2(O, \mathcal{R})$ under the isomorphism δ is called the *Chern class* [14]. In the case of trivial monodromy, μ_C coincides with the usual Chern class of principal $\mathbb{T}^n$ bundles (cf. [10]).

If one requires local sections s_i to be Lagrangian, then one has that

$$\mu_{ij} \in \mathcal{Z}(T^*O/\mathcal{R})(U_i \cap U_j)$$

where $\mathcal{Z}$ means closed 1-forms, and it will define another cohomology class which we will call the *Lagrangian class*:

$$\mu_L \in H^1(O, \mathcal{Z}(T^*O/\mathcal{R}))$$

There is another short exact sequence

$$0 \to \mathcal{R} \to \mathcal{Z}(T^*O) \to \mathcal{Z}(T^*O/\mathcal{R}) \to 0,$$

which leads to the following long exact sequence

$$\cdots \to H^1(O, \mathcal{R}) \xrightarrow{\hat{d}} H^1(O, \mathcal{Z}(T^*O)) \equiv H^2(O, \mathbb{R}) \to H^1(O, \mathcal{Z}(T^*O/\mathcal{R}))$$
$$\xrightarrow{\Delta} H^2(O, \mathcal{R}) \xrightarrow{\hat{d}} H^2(O, \mathcal{Z}(T^*O_0)) \equiv H^3(O, \mathbb{R}) \to H^2(O, \mathcal{Z}(T^*O/\mathcal{R})) \to \cdots$$

Under the maps Δ and $\hat{d}$ we have $\mu_L \overset{\Delta}{\mapsto} \mu_C \overset{\hat{d}}{\mapsto} 0$.

Thus, if the integral affine manifold O is given, then any element of the first cohomology group $H^1(O, \mathcal{Z}(T^*O/\mathcal{R}))$ will be the Lagrangian class of some Lagrangian torus fibration over O, and the necessary and sufficient condition for an element μ in $H^2(O, \mathcal{R})$ to be the Chern class of some Lagrangian torus fibration is that $\hat{d}(\mu) = 0$. To each element $\mu_C \in H^2(O, \mathcal{R})$ such that $\hat{d}(\mu_C) = 0$, there are $H^2(O, \mathbb{R})/\hat{d}H^1(O, \mathcal{R})$ choices of the element μ_L such that $\Delta(\mu_L) = \mu_C$, and each choice corresponds to a symplectically different Lagrangian torus fibration with the same Chern class μ_C. If $\mu_L = 0$ then the corresponding fibration is symplectically equivalent to $T^*O/\mathcal{R} \longrightarrow O$ (cf. [10]).

In particular, if the base space O is 2-connected: $\pi_1(O) = \pi_2(O) = 0$, then there is no room for the monodromy and the Lagrangian class, so there always exists a global system of action-angle coordinates, as first observed by Nekhoroshev [32].

It is clear from the definition of the Chern class that if two regular Lagrangian torus fibrations $(M_1, \omega_1, \mathcal{L}_1) \to O$ and $(M_2, \omega_2, \mathcal{L}_2) \to O$ over the same base space O admit a diffeomorphism $\phi\colon M_1 \to M_2$ which projects to the identity map on O and preserves the flat structure of each torus, then they have the same Chern class. In fact, the Chern class is a topological invariant, in the sense that if $\phi\colon M_1 \to M_2$ is a homeomorphism which projects to the identity map on O, but which need not be a diffeomorphism and need not preserve the flat structure of the fibers, then the two fibrations still have the same Chern class. In the following sections we will show this fact for the more general case of systems with singularities.

EXAMPLE 2.1. Systems over flat tori and Kodaira–Thurston example. Assume that $O = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the standard flat torus with the integral affine structure induced from $\mathbb{R}^2$. Then $H^3(O, \mathbb{R}) = 0$, and every element $\mu_{\mathrm{DC}} \in H^2(O, \mathcal{R}) = \mathbb{Z}^2$ is realizable as the Chern class of some Lagrangian torus fibration over O. The automorphism group of the base space acts on $H^2(O, \mathcal{R})$, and the quotient space is isomorphic to $\mathbb{Z}_+$ (nonnegative integers). Thus each integrable system with the base space $\mathbb{T}^2$ is characterized topologically by a nonnegative integer m, and its ambient symplectic manifold M^4_m has $H_1(M^4_m, \mathbb{Z}) = \mathbb{Z}^3 \oplus (\mathbb{Z}/m\mathbb{Z})$ as can be computed easily. For each m there are $H^2(O, \mathbb{R})/\hat{d}H^1(O, \mathcal{R}) = \mathbb{R}/\mathbb{Z}$ choices of the symplectic structure on the fibration $M^4_m \to O$, up to symplectic equivalence. Notice that the fibrations $M^4_m \to \mathbb{T}^2$ are topologically the same as a series of elliptic fibrations over an elliptic curve (see, e.g., [3]). In particular, when $m = 1$, M^4_1 is the well-known Kodaira–Thurston example (see, e.g., [28]) of a manifold admitting both a complex and a symplectic structure but not a Kähler structure.

If, for example, $O = \mathbb{R}^k/\Sigma$ where Σ is a lattice of $\mathbb{R}^k$, and $k \geqslant 3$, then not every element μ of $H^2(O, \mathcal{R})$ will satisfy the condition $\hat{d}(\mu) = 0$. If the lattice Σ is irrational, then it may happen that the operator $\hat{d}$ is injective, and the only Lagrangian torus fibration over O is the one which admits a section.

3. Base Space and Sheaves of Local Automorphisms

From now on, $\pi\colon (M, \omega, \mathcal{L}) \to O$ will always denote an integrable system which may admit singularities.

3.1. SINGULARITIES OF INTEGRABLE SYSTEMS

By a singularity of $\pi\colon (M, \omega, \mathcal{L}) \to O$ we will mean the germ of the fibration $\mathcal{L}$ at a singular fiber $N_x = \pi^{-1}(x)$, $x \in O$, and will denote it by $\pi\colon (U(N_x), \omega, \mathcal{L}) \to (U(x))$, where $U(N_x) = \pi^{-1}(U(x))$ is a tubular neighborhood of N_x. Two singularities are called *topologically* (resp. *symplectically*) *equivalent* if their fibration germs are homeomorphic (resp. symplectomorphic).

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

The *rank* of a singular fiber N of $\mathcal{L}$ is $\operatorname{rank} N = \max_{p \in N} \operatorname{rank} p$, where, by definition,

$$\operatorname{rank} p = \max_{\mathbf{F}} \dim \langle \mathrm{d}F_1(p), \dots, \mathrm{d}F_n(p) \rangle,$$

where the maximum is taken over all possible moment maps $\mathbf{F} = (F_1, \dots, F_n)\colon (M, \omega, \mathcal{L}) \to \mathbb{R}^n$, and $\langle \mathrm{d}F_1(p), \dots, \mathrm{d}F_n(p) \rangle$ denotes the linear span of the covectors $\mathrm{d}F_1(p), \dots, \mathrm{d}F_n(p)$ in $T_p^* M$.

We put $\operatorname{corank} N = n - \operatorname{rank} N$ and $\operatorname{corank} p = n - \operatorname{rank} p$, where $n = 1/2 \dim M$. If $\operatorname{rank} p < n$, then p is called a *singular point* of the system. If $\operatorname{rank} p = 0$ then p is called a *fixed point*. The rank and corank of a singularity $\pi\colon (U(N_x), \omega, \mathcal{L}) \to U(x)$ is, by definition, the rank and corank of N_x.

A fixed point p of the system is called *nondegenerate* if it satisfies the following condition: there is a moment map $(F_1, \dots, F_n)\colon (M, p, \omega, \mathcal{L}) \to (\mathbb{R}^n, 0)$ such that the quadratic parts $F_1^{(2)}, \dots, F_n^{(2)}$ of $F_1, \dots, F_n$ in a symplectic system of coordinates at p will form a Cartan subalgebra of the Lie algebra of quadratic functions on $\mathbb{R}^{2n}$ under the standard Poisson bracket. Recall that the algebra of quadratic functions on $(\mathbb{R}^{2n}, \omega_0)$ is naturally isomorphic to the simple Lie algebra $\mathrm{sp}(2n, \mathbb{R})$, the functions $F_1^{(2)}, \dots, F_n^{(2)}$ Poisson-commute (because $F_1, \dots, F_n$ Poisson-commute) and span an Abelian subalgebra of $\mathrm{sp}(2n, \mathbb{R})$. The nondegeneracy condition means that this subalgebra is of dimension n and consists of semi-simple elements, i.e. it is a Cartan subalgebra. A classical theorem of Williamson [37] (which essentially classifies Cartan subalgebras of $\mathrm{sp}(2n, \mathbb{R})$ up to conjugacy) implies that, for a nondegenerate fixed point p, there is a moment map $(F_1, \dots, F_n)$ whose quadratic part at p can be decomposed into components of 3 types: elliptic $(F_i^{(2)} = p_i^2 + q_i^2)$, hyperbolic $(F_i^{(2)} = p_i q_i)$, and focus-focus $(F_i^{(2)} = p_i q_i + p_{i+1} q_{i+1}, F_{i+1}^{(2)} = p_i q_{i+1} - p_{i+1} q_i)$, in a symplectic system of coordinates (p_i, q_i) (the symplectic form is $\omega = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$). Note that each focus-focus component consists of two functions. If there are k_e elliptic, k_h hyperbolic and k_f focus-focus components $(k_e + k_h + 2k_f = n)$, then we will say that the *Williamson type* of p is (k_e, k_h, k_f) (cf. [39]). The local normal form theorem for nondegenerate fixed points of integrable systems says that an integrable Hamiltonian system near a nondegenerate fixed point is locally symplectically equivalent to a system given by a quadratic moment map on the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$. This normal form theorem has been proved in the analytic case by Rüssmann [33] and Vey [34] (see also [42]). In the smooth case, it has been proved by Eliasson in his PhD thesis (see also [16]), and partially by Colin de Verdier and Vey [7], and Dufour and Molino [12].

A singular point p with $\operatorname{rank} p = r > 0$ is called *nondegenerate* if it becomes a nondegenerate fixed point for a local integrable Hamiltonian system with $n - r$ degrees of freedom after a local Marsden–Weinstein reduction. A singularity $\pi\colon (U(N_x), \omega, \mathcal{L}) \to U(x)$ is called *nondegenerate* if all singular points in N_x are nondegenerate, plus a natural additional condition which is called 'topological stability' in [39].

Remark. In some references the above additional condition is not included in the definition of nondegenerate singularities, but we will include in here so that we can use the decomposition theorem mentioned below. In [5], Bolsinov and Fomenko suggested another name for this additional condition, which looks better and more to the point than our original name 'topological stability', but unfortunately I could not translate this name into English.

In [39] we studied nondegenerate singularities of integrable Hamiltonian systems, where we showed, among other things, that they are topologically equivalent to almost direct products of simplest singularities. More precisely, if $(U(N_x), \mathcal{L})$ is a nondegenerate singularity, then it is homeomorphic to

$$\{(U(T^r), \mathcal{L}_r) \times (P^2(N_1), \mathcal{L}_1) \times \cdots \times (P^2(N_{k_e+k_h}), \mathcal{L}_{k_e+k_h}) \times \\ \times (P^4(N'_1), \mathcal{L}'_1) \times \cdots \times (P^4(N'_{k_f}), \mathcal{L}'_{k_f})\}/\Gamma$$

where $(U(T^r), \mathcal{L}_r)$ denotes a regular system with r degrees of freedom near a regular torus, $(P^2(N_i), \mathcal{L}_i)$ for $1 \leqslant i \leqslant k_e + k_h$ denotes a corank-1 nondegenerate elliptic or hyperbolic singularity of an system with 1 degree of freedom, $(P^4(N'_i), \mathcal{L}'_i)$ for $1 \leqslant i \leqslant k_f$ denotes a focus-focus singularity of a system with 2 degrees of freedom, Γ is a finite group that acts on the above product freely and component-wise. Moreover, it acts trivially on all possible elliptic components of the product (there are k_e such components if $k_e > 0$). Nondegenerate corank-1 singularities are called 'atoms' in the works of Fomenko and his collaborators (see, e.g., [5, 17]). Focus-focus corank-2 singularities are classified topologically in [40].

To our knowledge, most singularities of integrable Hamiltonian systems are nondegenerate. But starting with 2 degrees of freedom, there are also degenerate singularities. The situation is similar to that of smooth functions: most singularities of smooth functions are nondegenerate, but there are degenerate singularities whose miniversal deformation is of finite dimension k and which can appear in a generic way in k-dimensional families of functions (see, e.g., [41]).

We will make a series of hypotheses about singularities of 'generic' integrable Hamiltonian systems. All nondegenerate singularities will satisfy these hypotheses. We believe that 'generic' degenerate singularities will also satisfy these hypotheses. The first hypothesis is:

(H1) Each fiber N of $\mathcal{L}$ is a disjoint union of a finite number of submanifolds N_i such that if $p \in N_i$ then $\operatorname{rank} p = \dim N_i$.

Recall that if $p \in N_x \subset M$ has rank r, then there is a moment map $\mathbf{F} = (F_1, \ldots, F_n) \colon (M, \omega) \to \mathbb{R}^n$ such that the rank of $\mathbf{F}$ at p is r, and the orbit of the Poisson $\mathbb{R}^n$-action generated by $\mathbf{F}$ which contains p is an immersed r-dimensional submanifold contained in N_x. Hypothesis (H1) says that each such orbit is a stratum of N_x in a natural sense. In particular, $\dim N_x \leqslant n$ for any fiber N_x of $\mathcal{L}$. If, for example, $(U(N_x), \omega, \mathcal{L})$ is a nondegenerate singularity of Williamson type (k_e, k_h, k_f)

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

with $k_e > 0$ (where k_e is the number of elliptic components), then $\dim N_x = n - k_e < n$.

3.2. STRATIFICATION OF THE BASE SPACE

The base space $\mathcal{O}$ of an integrable system $\pi\colon (M, \omega, \mathcal{L}) \to \mathcal{O}$ has a natural topology induced from M. We will always assume $\mathcal{O}$ to be separated and paracompact (it follows from the existence of a proper moment map). If $\mathcal{L}$ is regular, then $\mathcal{O}$ is a manifold. If $\mathcal{L}$ has only nondegenerate elliptic singularities then $\mathcal{O}$ is a manifold with corners (see, e.g., [6]). In the general case, $\mathcal{O}$ is not a manifold, but we can try to give it a natural stratification.

In this paper, a separated paracompact topological space Q will be called a *stratified manifold* of dimension n if Q_i can be written as a disjoint union of topological manifolds Q_i, called strata of Q, in such a way that:

(a) $\max \dim Q_i = n$
(b) For each i, the boundary of the stratum Q_i is a union of strata of dimension smaller than $\dim Q_i$.
(c) If $\dim Q_i = k$ then for each $x \in Q_i$ there is a neighborhood $U(x)$ of x in V which is homeomorphic to the direct product of a k-dimensional disk D^k with a cone over a stratified $(n-k-1)$-dimensional manifold with a finite number of strata. Such a neighborhood will be called a *standard*, or *star-shaped*, neighborhood of x.

Given a singular point x of the base space $\mathcal{O}$, we will denote by S_x the connected component of the set of all points $y \in \mathcal{O}$ such that the singularity at y (i.e. at the singular fiber $N_y = \pi^{-1}(y)$ of the system) is topologically equivalent to the singularity at x. If x is regular then S_x is a connected component of the regular part of $\mathcal{O}$. Our next hypothesis is:

(H2) The base space $\mathcal{O}$ is a stratified manifold for which each S_x defined above is a stratum. If $U(x) \in \mathcal{O}$ is a star-shaped neighborhood of a point $x \in \mathcal{O}$, and $\phi\colon U(x) \to D^k \times V(x)$ is a homeomorphism, where D^k is a k-dimensional disk and $V(x) \in \mathcal{O}$ is a local stratified submanifold transversal to S_x at x (so $V(x)$ is homeomorphic to a cone over a $(n-k-1)$-dimensional stratified manifold), then there is a homeomorphism $\Phi\colon \pi^{-1}(U(x)) \to D^k \times \pi^{-1}(V(x))$ which makes the following diagram commutative:

$$\begin{array}{ccc} \pi^{-1}(U(x)) & \xrightarrow{\Phi} & D^k \times \pi^{-1}(V(x)) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle(\mathrm{id},\,\pi)} \\ U(x) & \xrightarrow{\phi} & D^k \times V(x) \end{array}$$

It is clear that Hypothesis (H2) has a local character: it is satisfied if it is satisfied locally near every singular point. Hypothesis (H2) justifies the use of a tubular neighborhood of a singular fiber to denote a singularity. In fact, it follows from this hypothesis that if $U_1(x)$ and $U_2(x)$ are two small star-shaped neighborhood of a point x in O, then the singular fibration $(\pi^{-1}(U_1(x)), \mathcal{L})$ is topologically equivalent to the singular fibration $(\pi^{-1}(U_2(x)), \mathcal{L})$.

Let $O = \bigsqcup S_x$ be the stratification of the base space O as above. Then we can replace S_i by *thickened strata* as follows: Order the strata of O by dimension: $S_1, S_2, S_3, \ldots$, with $\dim S_1 \leqslant \dim S_2 \leqslant \dim S_3 \leqslant \cdots$ Denote by $\bar{S}_1$ a small closed tubular neighborhood of S_1 in O, $\bar{S}_2$ a sufficiently small closed tubular neighborhood of $(O \setminus \bar{S}_1) \cap S_2$ in $(O \setminus \bar{S}_1)$, $\bar{S}_3$ a sufficiently small closed tubular neighborhood of $(O \setminus (\bar{S}_1 \cup \bar{S}_2)) \cap S_3$ in $(O \setminus (\bar{S}_1 \cup \bar{S}_2))$, etc. Then we have $O = \bigsqcup \bar{S}_i$. This decomposition of O into the disjoint union of $\bar{S}_i$ is unique up to homeomorphisms, and is called a *thickened stratification* of O. The sets $\bar{S}_i$ are called *thickened strata* of O.

Similarly, if $U(x)$ is a star-shaped neighborhood of a point $x \in O$, then we also have a thickened stratification of $U(x)$, $U(x) = \bigsqcup \bar{U}_i$, which consists of a finite number of thickened strata $\bar{U}_i$ and is unique up to homeomorphisms. In an appropriate thickened stratification of O, $O = \bigsqcup \bar{S}_i$, we can put $\bar{U}_i = U(x) \cap \bar{S}_i$ (and then throw out those $\bar{U}_i = U(x) \cap \bar{S}_i$ which are empty).

3.3. LOCAL $\mathbb{S}^1$-ACTIONS AND AFFINE MONODROMY

Let U be an open subset of the base space O. We will denote by $R(U)$ the set of symplectic $\mathbb{S}^1$-actions on $(\pi^{-1}(U), \omega)$ which preserve the system (i.e. preserve every fiber of $\mathcal{L}$). We have the following obvious lemma:

LEMMA 3.1. *$R(U)$ is a free Abelian group of rank less or equal to n. If U_1 is an open subset of U then there is a natural injection from $R(U)$ to $R(U_1)$. If α is a system-preserving symplectic $\mathbb{S}^1$-action on $\pi^{-1}(U_1)$ and β is a system-preserving symplectic $\mathbb{S}^1$-action on $\pi^{-1}(U_2)$ such that their restrictions to $\pi^{-1}(U_1 \cap U_2)$ are the same, then there is a system-preserving symplectic $\mathbb{S}^1$-action on $\pi^{-1}(U_1 \cup U_2)$ which restricts to α and β on $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ respectively.*

Thus, the groups $R(U)$, $U \subset O$, form a free Abelian sheaf over the base space O. We will denote this sheaf by $\mathcal{R}$, and call it the *affine monodromy sheaf*, in analogy with the case of regular Lagrangian torus fibrations.

If $(g_1, \ldots, g_m)$ is a basis of $R(U) \cong \mathbb{Z}^m$, then these $\mathbb{S}^1$-actions $g_1, \ldots, g_m$ com mute, and together they generate a system-preserving symplectic $\mathbb{T}^m$-action on $(\pi^{-1}(U), \omega, \mathcal{L})$ which is free almost everywhere. Conversely if there is a system-pre serving symplectic action of a torus $\mathbb{T}^m$ on $(\pi^{-1}(U), \omega, \mathcal{L})$ which is locally free somewhere, then the composition of this $\mathbb{T}^m$-action with homomorphisms from $\mathbb{S}^1$ to $\mathbb{T}^m$ gives rise to a subgroup of $R(U)$ which is isomorphic to $\mathbb{Z}^m$. The classical Arnold–Liouville theorem is essentially equivalent to the fact that, if U is a disk in the regular part of O, then $R(U)$ is isomorphic to $\mathbb{Z}^n$. In [39] we have shown that if $\pi\colon (\pi^{-1}(U(x)), \omega, \mathcal{L}) \to U(x)$ is a nondegenerate singularity of rank r and Williamson

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

type (k_e, k_h, k_f) $(r + k_e + k_h + 2k_f = n)$, then $R(U(x))$ is isomorphic to $\mathbb{Z}^{r+k_e+k_f}$. In [41] we have shown that if $\pi\colon (\pi^{-1}(U(x)), \omega, \mathcal{L}) \to U(x)$ is a degenerate singularity of corank 1, then under some mild conditions $R(U(x))$ will be isomorphic to $\mathbb{Z}^{n-1}$. Another simple result which can be proved by the methods of [39, 41] is the following:

LEMMA 3.2. *If* $\pi\colon (\pi^{-1}(U(x)), \omega, \mathcal{L}) \to U(x)$ *is a singularity of rank* r *such that* $\dim \pi^{-1}(x) = n = 1/2 \dim \pi^{-1}(U(x))$, *then there is a locally-free system-preserving symplectic action of* $\mathbb{T}^r$ *on* $(\pi^{-1}(U(x)), \omega, \mathcal{L})$. *In particular, the rank of* $R(U(x))$ *is greater or equal to* r.

Sketch of the Proof. Denote by P an n-dimensional stratum of $\pi^{-1}(x)$. It follows from the facts that P is an orbit of the Poisson action of a moment map $(F-1, \ldots, F_n)\colon (\pi^{-1}(U(x)), \omega, \mathcal{L}) \to \mathbb{R}^n$ and rank $\pi^{-1}(x) = r$, that P is of the type $\mathbb{T}^k \times \mathbb{R}^{n-k}$ with $k \geqslant r$. For each element γ of the fundamental group of P, there are numbers $a_1, \ldots, a_n \in \mathbb{R}$ such that the vector field $\sum a_i X_{F_i}$ is periodic of period 1 on P and its orbits on P are homotopic to γ. There is a unique way to extend a_i into smooth functions on $\pi^{-1}(U(x))$ which are constant on each fiber of $\mathcal{L}$, such that the vector field $\sum a_i X_{F_i}$ is periodic on $\pi^{-1}(U(x))$. Arnold–Liouville theorem (for the regular part of $\pi^{-1}(U(x))$) assures that the vector field $\sum a_i X_{F_i}$ is symplectic. It follows that there is a system-preserving symplectic action of $\mathbb{T}^k$ on $(\pi^{-1}(U(x)), \omega, \mathcal{L})$ which is free almost everywhere. We can choose a subgroup $\mathbb{T}^r \subset \mathbb{T}^k$ such that the action of $\mathbb{T}^r$ will be locally-free everywhere in $\pi^{-1}(U(x))$. □

Let $g\colon \mathbb{S}^1 \times \pi^{-1}(U) \longrightarrow \pi^{-1}(U)$ be a system-preserving symplectic $\mathbb{S}^1$-action, where U is an open subset of the base space O. Then for each stratum P of a fiber in $\pi^{-1}(U)$, g preserves P and the orbits of g on P defines an element γ_P in the fundamental group of P. The association $P \mapsto \gamma_P \in \pi_1(P)$ is continuous in the following sense: each continuous curve $\psi\colon [0,1] \to \pi^{-1}(U)$ can be extended to a continuous family of loops $\phi\colon [0,1] \times S^1 \to \pi^{-1}(U)$ ($\phi\,|_{[0,1]\times\{e\}} = \psi$ where e is a fixed element of S^1) such that each loop $\phi\,|_{\{t\}\times S^1}$ lies on some stratum $P(t)$ of some fiber of $\pi^{-1}(U)$ and is homotopic to $\gamma_{P(t)}$ on $P(t)$. It is evident that the set of continuous association $\{P \mapsto \gamma_P \in \pi_1(P)\}$ is an Abelian group, which we will denote by $\Pi(U)$, and there is a unique natural injective homomorphism from the group $R(U)$ to $\Pi(U)$. Our next hypothesis is:

(H3) If $\pi\colon (\pi^{-1}(U(x)), \omega, \mathcal{L}) \to U(x)$ is a singularity of rank r, then $R(U(x))$ is isomorphic to $\Pi(U(x))$, and there is a locally-free system-preserving symplectic $\mathbb{T}^r$-action on $(\pi^{-1}(U(x)), \omega, \mathcal{L})$. In particular, rank $R(U(x)) =$ rank $\Pi(U(x)) \geqslant$ rank $\pi^{-1}(x)$.

Since $\Pi(U(x))$ is clearly a topological invariant of $(\pi^{-1}(U(x)), \mathcal{L})$, Hypothesis (H3) implies in particular that the affine monodromy sheaf $\mathcal{R}$ is a topological invariant of the system. Moreover, $R(U)$ is naturally isomorphic to $\Pi(U)$ for each open subset U of O.

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

3.4. AFFINE STRUCTURE OF THE BASE SPACE

A local function f on the base space O is called an *integral affine function* if its pull-back $f \circ \pi$ is a local smooth function on M such that the Hamiltonian vector field of $f \circ \pi$ is periodic of period $1/k$ for some natural number k. In other words, local integral affine functions are functions that generate local system-preserving $\mathbb{S}^1$-actions. Denote the sheaf of local integral affine functions on O by $\mathcal{I}$. Then we have the following natural short exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{I} \longrightarrow \mathcal{R} \longrightarrow 0.$$

We may view the sheaf $\mathcal{I}$ as the integral affine structure of O. A local function on O is called *affine* if it can be written as a linear combination of integral affine functions.

As a side note, we remark that the fact that O has a integral affine structure imposes some conditions on the topology of O. For example, for two-dimensional base spaces we have the following analog of a Theorem of Milnor [30] concerning affine structures on 2-surfaces:

Let O be the base space of an integrable system with 2 degrees of freedom, whose singularities are all nondegenerate. A subset C of O will be called a *topological 2-stratum* of O if C is the union of a 2-dimensional stratum of O (with respect to the stratification given in Subsection 3.2) with all possible focus-focus singular points lying on its boundary.

PROPOSITION 3.3. *Let C be a topological 2-stratum of the base O of an integrable Hamiltonian system on a compact four-dimensional symplectic manifold M^4 (maybe with boundary). Assume that the system contains only nondegenerate singularities, and the image of the boundary of M^4 under the projection to O does not intersect with the closure of C (if M^4 is closed then this condition is satisfied automatically). Then C is homeomorphic to either an annulus, a Mobius band, a Klein bottle, a torus, a disk, a projective space, or a sphere (in case of sphere or projective space, C must contain focus-focus points).*

The proof of the above proposition is elementary and is left to the reader.

3.5. LOCAL HOMEOMORPHISMS

For each open subset $U \subset O$, we will denote by $A_t(U)$ the group of homeomorphisms ϕ from $(\pi^{-1}(U), \mathcal{L})$ onto itself which satisfy the following topological properties: ϕ preserves each stratum of each fiber of $(\pi^{-1}(U), \mathcal{L})$ and induces the identity map on the fundamental group of each stratum, and ϕ induces the identity map on the first homology group (with integral coefficients) of each fiber of $(\pi^{-1}(U), \mathcal{L})$. The association $U \mapsto A_t(U)$ is a non-Abelian sheaf over the base space O, which will be denoted by $\mathcal{A}_t$ and called the *sheaf of admissible local topological automorphisms* of the system. (Here t stands for 'topological').

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

If $g: [0, 1] \to A_t(U)$ is a continuous loop in $A_t(U)$, then its orbits on the strata of the fibers of $(\pi^{-1}(U), \mathcal{L})$ define an element of $\Pi(U)$ $(= R(U)$ by Hypothesis (H3)). Moreover, it follows from the definition of $A_t(U)$ that every element of $A_t(U)$ induces the identity map on $\Pi(U)$. Thus there is a natural central extension $\hat{A}'_t(U)$ of $A_t(U)$ by $R(U)$: $0 \to R(U) \to \hat{A}'_t(U) \to A_t(U) \to 0$. Topologically (with respect to the open-compact topology), $\hat{A}'_t(U)$ is an $R(U)$-covering of $A_t(U)$, such that if g is a loop in $A_t(U)$ with $g(0) = g(1) = \text{id}$ and which corresponds to an element γ of $\Pi(U) = R(U)$, then it can be lifted to a curve $\hat{g}$ in $\hat{A}'_t(U)$ such that $\hat{g}(0)$ is the unity element in $\hat{A}'_t(U)$ but $\hat{g}(1)$ is the image of γ in $\hat{A}'_t(U)$ via the injection $R(U) \to \hat{A}'_t(U)$. The association $U \mapsto \hat{A}'_t(U)$ is a presheaf, which can be made in to a sheaf $\hat{\mathcal{A}}_t$ whose stalk at each point $x \in O$ is $\lim_{U(x)\to x} \hat{A}_t(U(x))$. Clearly, $\hat{\mathcal{A}}_t$ is an extension of $\mathcal{A}_t$ by $\mathcal{R}$: $0 \to \mathcal{R} \to \hat{\mathcal{A}}_t \to \mathcal{A}_t \to 0$.

We will define a subgroup $A_c(U)$ of $A_t(U)$, consisting of the so-called *compatible* (with respect to the local $\mathbb{S}^1$-actions) homeomorphisms of $(\pi^{-1}(U), \mathcal{L})$. For this we will fix an appropriate thickened stratification $O = \bigsqcup \bar{S}_i$ of the base space O (recall that such a thickened stratification is unique topologically), and assume that U is compatible with this thickened stratification, i.e. the intersection of $\bar{S}_i$ with U gives rise to a thickened stratification of U. Let y be a point in U. Then there is a point x in U such that y and x belong to the same thickened stratum, and that the stratum that contains x has the same index as the thickened stratum that contains x. Let $V(x)$ be a star-shaped neighborhood of x which contains y and which lies in the thickened stratum that contains x. Then there is a natural system-preserving action of $\mathbb{T}^r$ on $\pi^{-1}(V(x))$, where $r = \text{rank } R(V(x))$, which is unique up to automorphisms of $\mathbb{T}^r$. In particular, there is a natural action of the torus $\mathbb{T}^r$ on $\pi^{-1}(y)$ which is unique up to automorphisms of $\mathbb{T}^r$ and which does not depend on the choice of x. We will denote this action on $\pi^{-1}(y)$ (up to automorphisms) by $T(y)$. We will say that an element ϕ of $A_t(U)$ is a *compatible homeomorphism* if it commutes with the torus action $T(y)$ on $\pi^{-1}(y)$ for each $y \in U$. Evidently, the set of compatible homeomorphisms is a subgroup of $A_t(U)$, which we will denote by $A_c(U)$. The association $U \mapsto A_c(U)$ gives rise to a non-Abelian subsheaf of the sheaf $\mathcal{A}_t$ over O, which we will denote by $\mathcal{A}_c$. Of course, the definition of $\mathcal{A}_c$ depends on the choice of a thickened stratification of O, but since such a stratification is unique up to homeomorphisms, the sheaf $\mathcal{A}_c$ is also unique up to isomorphisms.

Similarly, if $\pi: (\pi^{-1}(U(x)), \omega, \mathcal{L}) \to U(x)$ and $\pi: (\pi^{-1}(V(y)), \omega, \mathcal{L}) \to V(y)$ are two topologically equivalent singularities, then we can talk about compatible homeomorphisms from $(\pi^{-1}(U(x)), \mathcal{L})$ to $(\pi^{-1}(V(y)), \mathcal{L})$ (with respect to a given thickened stratification of $U(x)$ and $V(y)$). Our next hypothesis is that such compatible homeomorphisms always exist.

(H4) If two singularities $(\pi^{-1}(U(x)), \omega, \mathcal{L})$ and $(\pi^{-1}(V(y)), \omega, \mathcal{L})$ are topologically equivalent, then for any fibration-preserving homeomorphism ψ from

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

$(\pi^{-1}(U(x)), \mathcal{L})$ to $(\pi^{-1}(V(y)), \mathcal{L})$ which maps a given thickened stratification of $U(x)$ to a given thickened stratification of $V(y)$, then there is a compatible (with respect to the local $\mathbb{S}^1$-actions and given thickened stratifications) homeomorphism ϕ from $(\pi^{-1}(U(x)), \mathcal{L})$ to $(\pi^{-1}(V(y)), \mathcal{L})$ such that $\phi^{-1} \circ \psi$ belongs to $A_t(U(x))$.

Hypothesis (H4) can be verified directly for nondegenerate singularities. Similarly to the case of $A_t(U)$, the group $A_c(U)$ also has a natural central extension by $R(U)$, which we will denote by $\hat{A}'_c(U)$. In terms of sheaves, $\mathcal{A}_c$ has an extension by $\mathcal{R}$, which we will denote by $\hat{\mathcal{A}}_c$, and we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{R} & \to & \hat{\mathcal{A}}_c & \to & \mathcal{A}_c & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{R} & \to & \hat{\mathcal{A}}_t & \to & \mathcal{A}_t & \to & 0 \end{array}$$

Our next hypothesis is:

(H5) If $\pi: (\pi^{-1}(U(x)), \omega, \mathcal{L}) \to U(x)$ is a singularity then the group $\hat{A}'_c(U(x))$ is contractible. Equivalently, $A_c(U(x))$ is homotopic to $\mathbb{T}^k$ where $k =$ rank R(U(x)).

Again, for nondegenerate singularities, Hypothesis (H5) can be verified directly, and is similar to the fact that the space of homeomorphisms from a cube to itself which are identity on the boundary is contractible (with respect to the open-compact topology).

3.6. DIFFERENTIAL FORMS ON THE BASE SPACE

Recall that a differential k-form β on $(M^{2n}, \mathcal{L})$ is called a *basic form* (with respect to the fibration $\mathcal{L}$) if for any vector X on M^{2n} tangent to $\mathcal{L}$ we have $i_X\beta = i_X \, d\beta = 0$. We will denote the space of basic k-forms on M^{2n} (with respect to a given integrable system $\pi: (M^{2n}, \omega, \mathcal{L}) \to O$) by $\Omega^k(O)$ and consider it as the space of differential k-forms on the base space O. (If O is regular then basic forms on M are pull-backs of differential forms on O). In particular, the space of smooth functions on O, denoted by $C^\infty(O)$, is the space of functions $f: O \to \mathbb{R}$ such that $f \circ \pi$ is a smooth function on M^{2n}. For each k, the linear space $\Omega^k(O)$ is a $C^\infty(O)$-module. The differential of a basic form is again a basic form. Thus we have the following DeRham complex of O:

$$0 \to C^\infty(O) \xrightarrow{d} \Omega^1(O) \xrightarrow{d} \Omega^2(O) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(O) \xrightarrow{d} 0$$

The cohomology of this complex, known as *basic cohomology*, will also be called the *DeRham cohomology of* O, and denoted by $H^k_{\mathrm{DR}}(O, \mathbb{R})$. Of particular interest is the second cohomology group $H^2_{DR}(O, \mathbb{R})$: if β is a closed 2-form on O then $\omega + \pi^*\beta$ will be a new symplectic form on M for which $\pi: (M, \omega + \pi^*\beta, \mathcal{L}) \to O$ remains an integrable Hamiltonian system (see Subsection 4.6).

If all singularities over O are nondegenerate, then it is easy to see that C^∞ is a fine sheaf over O, the Poincaré lemma holds for O (i.e. closed differential forms on O are locally exact), and the DeRham cohomology of O is naturally isomorphic to its singular cohomology. Indeed, near each nondegenerate singular point of the system in the symplectic manifold there is a radiant-like vector field which preserves the fibration, and one can use these radiant vector fields to prove the Poincaré lemma for O by the same Thom-Moser path method as used in the book of Abraham and Marsden [1].

When dealing with the symplectic classification of general integrable Hamiltonian systems, we will make the following hypothesis:

(H6) The sheaf C^∞ of local smooth functions on O is a fine sheaf.

Notice that, when O contains degenerate singularities, closed 1-forms on O are still locally exact: if β is a closed 1-form in a star-shaped neighborhood $U(x)$ of O, then since $\pi^*\beta$ vanishes on $\pi^{-1}(x)$ and homotopically $\pi^{-1}(U(x))$ is the same as $\pi^{-1}(x)$, the cohomological class of $\pi^*\beta$ in $H^1(\pi^{-1}(U(x)), \mathbb{R})$ is zero. Hence $\pi^*\beta = \mathrm{d}h$ for some function h on $\pi^{-1}(U(x))$. It is obvious that h is constant on the fibers of the fibration, i.e. h is a basic function. As a consequence, we have the following short exact sequence of Abelian sheaves over O:

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty \longrightarrow \mathcal{Z}^1 \longrightarrow 0,$$

where $\mathcal{Z}^1$ denotes the sheaf of local closed differential 1-forms. Since C^∞ is a fine sheaf by Hypothesis (H6), the cohomology long exact sequence associated to the above short exact sequence gives rise to a natural isomorphism from $H^k(O, \mathcal{Z}^1)$ to $H^{k+1}(O, \mathbb{R})$ ($k \geqslant 1$).

3.7. LOCAL SYMPLECTOMORPHISMS

Let X be a symplectic vector field on a singularity $(\pi^{-1}(U(x)), \omega, \mathcal{L})$, which is tangent to $\mathcal{L}$. Then the time-1 map of X, denoted by g^1_X, is an element of the group $A_t(U(x))$ of admissible homeomorphisms, which preserves the symplectic form. The inverse is also true, at least for nondegenerate singularities:

LEMMA 3.4. *Let $(\pi^{-1}(U(x)), \omega, \mathcal{L})$ be a nondegenerate singularity, and ϕ a symplectomorphism from $(\pi^{-1}(U(x)), \omega, \mathcal{L})$ onto itself which preserves every stratum of every fiber of the system. Then ϕ can be written as the time one map g^1_X of a symplectic vector field X on $(\pi^{-1}(U(x)), \omega)$ which is tangent to $\mathcal{L}$.*

Sketch of the proof. If the singularity contains only elliptic components, then one can write an explicit formula for X in a canonical system of coordinates. If the singularity does not contain any elliptic component, then the proof is similar to that of Lemma 3.2. The general case is a combination of these two cases. □

We will make the above property into a hypothesis, our last one, which will be used in the symplectic classification of integrable Hamiltonian systems:

(H7) Let $(\pi^{-1}(U(x)), \omega, \mathcal{L})$ be a singularity, and ϕ a symplectomorphism from $(\pi^{-1}(U(x)), \omega, \mathcal{L})$ onto itself which preserves every stratum of every fiber of the system. Then ϕ can be written as the time one map g_X^1 of a symplectic vector field X on $(\pi^{-1}(U(x)), \omega)$ which is tangent to $\mathcal{L}$.

For each open subset U of O, we will denote by $A_s(U)$ the set of smooth symplectomorphisms from $(\pi^{-1}(U), \omega)$ onto itself which belong to the group of compatible homeomorphisms $A_t(U)$. It is obvious that $A_s(U)$ is a subgroup of the group of compatible homeomorphisms $A_c(U)$ (for any thickened stratification of O), and the association $U \mapsto A_s(U)$ is a sheaf over O, which we will call *the sheaf of local symplectomorphisms of the system*, and denote by $\mathcal{A}_s$.

There is a natural projection from the sheaf $\mathcal{Z}^1$ of local closed 1-forms on O to $\mathcal{A}_s$: if β is a basic closed 1-form on $(\pi^{-1}(U(x)), \omega, \mathcal{L})$, then the vector field X defined by $i_X\omega = \beta$ is symplectic and its time-1 map defines an element in $A_s(U(x))$. On the other hand, there is a natural injection from the sheaf $\mathcal{R}$ of local symplectic $\mathbb{S}^1$-actions to $\mathcal{Z}^1$: if g is a symplectic $\mathbb{S}^1$-action on $(\pi^{-1}(U(x)), \omega, \mathcal{L})$ and X the symplectic vector field that generates g, then $i_{X\omega}$ is an element of $\mathcal{Z}^1(U(x))$. Hypothesis (H7) implies that the projection from $\mathcal{Z}^1$ to $\mathcal{A}_s$ is surjective, and its kernel is the image of $\mathcal{R}$ in $\mathcal{Z}^1$. In other words, we have the following exact sequence of Abelian sheaves

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{Z}^1/\mathcal{R} \longrightarrow 0,$$

and $\mathcal{A}_s$, is naturally isomorphic to $Z^1/\mathcal{R}$.

4. Characteristic Classes and Classification

4.1. HOMOLOGICAL MONODROMY AND ROUGH EQUIVALENCE

To define the characteristic classes of an integrable system, we will have to compare it to another 'reference' system which is 'roughly equivalent' to it. Here two integrable Hamiltonian systems will be called roughly equivalent if they have the same base space, the same singularities, and the same 'homological monodromy', in the following sense:

DEFINITION 4.1. Two integrable Hamiltonian systems $(M_a, \omega_a, \mathcal{L}_a) \xrightarrow{\pi_a} O_a$ and $(M_b, \omega_b, \mathcal{L}_b) \xrightarrow{\pi_b} O_b$ are called *roughly topologically equivalent* if there is a homeomorphism $\phi\colon O_a \to O_b$, a covering of O_a by open subsets U_i, a homeomorphism

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

$\Phi_i\colon \pi_a^{-1}(U_i) \to \pi_b^{-1}(\phi(U_i))$ for each i, such that $\pi_b \circ \Phi_i = \phi \circ \pi_a|_{\pi_a^{-1}(U_i)}$, and $\Phi_i^{-1}\Phi_j$ induces the identity map on the fundamental groups of the strata of $\pi_a^{-1}(x)$ and the identity map on $H_1(\pi_a^{-1}(x), \mathbb{Z})$ for each point $x \in U_i \cap U_j$. The two systems are called *roughly symplectically equivalent* if, in addition, Φ_i are smooth symplectomorphisms.

It follows from the above definition and the hypotheses made in the previous section that, if two systems are roughly topologically equivalent, then they have the same affine monodromy sheaf $\mathcal{R}$. For regular Lagrangian torus fibrations, the condition of rough topological equivalence is equivalent to the condition of having the same base space up to homeomorphisms and the same affine monodromy.

If two systems are roughly symplectically equivalent, then their base spaces are diffeomorphic and have the same integral affine structure, in the sense that there is a homeomorphism between them which preserves the algebra of differential forms and the sheaf of local integral affine functions. For two roughly topologically equivalent systems to be roughly symplectically equivalent, a necessary and sufficient condition is that their base spaces have the same affine structure and their singularities are not only topologically equivalent but also symplectically equivalent. Symplectic invariants of simplest singularities and systems (with one or one and a half degrees of freedom) have been studied by some authors (see, e.g., [5, 13, 36]).

If two systems are roughly topologically (resp. symplectically) equivalent, then we will also say that they have the same *rough topological* (*resp. symplectic*) *type*. Given an integrable Hamiltonian system $\pi\colon (M, \omega, \mathcal{L}) \to O$, we will denote its rough topological and symplectic types by $\hat{O}_{\text{top}}$ and $\hat{O}_{\text{symp}}$ respectively, and view them as *framed base spaces*, with the framing given by singularities and 'monodromies'. For each rough topological or symplectic type, we will try and choose a 'reference system' with this rough type. For example, if the base space O is regular, then the framing is given by the affine monodromy, and the obvious reference system is the one which admits a global (Lagrangian) section. In the general case, where there is no obvious choice, we'll pick an arbitrary system and call it the reference system for a given rough type.

4.2. DEFINITION OF CHARACTERISTIC CLASSES

Assume that an integrable Hamiltonian system $\pi\colon (M, \omega, \mathcal{L}) \to O$ is roughly topologically equivalent to a reference system $\pi_0\colon (M_0, \omega_0, \mathcal{L}_0) \to O$ over the same base space O. By definition, there is a covering of O by open subsets U_i, a homeomorphism $\Phi_i\colon \pi_0^{-1}(U_i) \to \pi^{-1}(\phi(U_i))$ for each i, such that $\pi \circ \Phi_i = \pi_0|_{\pi_0^{-1}(U_i)}$, and $\mu_{ij} = \Phi_i^{-1} \circ \Phi_j$ belongs to the group $A_t(U_i \cap U_j)$ (defined in the previous section, for the reference system). Its clear that the family $(\mu_{ij} \in A_t(U_i \cap U_j))$ is a Cech 1-cocycle in the non-Abelian sheaf $\mathcal{A}_t$, which will define an element μ in $H^1(\mathcal{A}_t)$. The first cohomology class μ is a topological invariant of the system $\pi\colon (M, \omega, \mathcal{L}) \to O$, and it is trivial if and only if $\pi\colon (M, \omega, \mathcal{L}) \to O$ is topologically equivalent to the reference system $\pi_0\colon (M_0, \omega_0, \mathcal{L}_0) \to O$ by a homeomorphism which projects to the identity map on O.

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

Similarly, if $\pi: (M, \omega, \mathcal{L}) \to O$ is roughly symplectically equivalent to a given reference system $\pi_0: (M_0, \omega_0, \mathcal{L}_0) \to O$, then we will be able to define an element μ_L in $H^1(\mathcal{A}_s)$, where $\mathcal{A}_s$ is the sheaf of local symplectomorphisms of the reference system. Recall that $\mathcal{A}_s \cong \mathcal{Z}^1/\mathcal{R}$ (and is an Abelian sheaf). Thus we can write $\mu_L \in H^1(\mathcal{Z}^1/\mathcal{R})$.

For a given reference system, the short exact sequence $0 \to \mathcal{R} \to \hat{\mathcal{A}}_t \to \mathcal{A}_t \to 0$ gives rise to the following long exact sequence:

$$\cdots \to H^1(\hat{\mathcal{A}}_t) \to H^1(\mathcal{A}_t) \xrightarrow{\delta} H^2(\mathcal{R}) \to H^2(\hat{\mathcal{A}}_t) \to \cdots$$

DEFINITION 4.2. The second cohomology class $\mu_C \in H^2(O, \mathcal{R})$ which is the image of μ under the coboundary map $\delta: H^1(O, \mathcal{A}_t) \to H^2(O, \mathcal{R})$ arising from the short exact sequence $0 \to \mathcal{R} \to \hat{\mathcal{A}}_t \to \mathcal{A}_t \to 0$ is called the *Chern class* of the system $(M, \omega, \mathcal{L})$ (with respect to a given reference system $(M_0, \omega_0, \mathcal{L}_0)$). The first cohomology class $\mu_L \in H^1(O, \mathcal{Z}^1/\mathcal{R})$ is called the *Lagrangian class* of $(M, \omega, \mathcal{L})$.

It is clear from the above definition that the Chern class is a topological invariant: if two systems are topologically equivalent, then they will have the same Chern class (with respect to any common reference system) after an appropriate homeomorphism of their respective base spaces. Similarly, the Lagrangian class is obviously a symplectic invariant. Notice that in the definition of the characteristic classes, we need a reference system, therefore the Chern and Lagrangian classes are not 'absolute' classes but rather 'relative' ones. In other words, they live in an affine space rather than a vector space.

4.3. CHERN CLASS VIA COMPATIBLE HOMEOMORPHISMS

It follows from Hypothesis (H4) that, in the definition of the Chern class, we may replace the sheaf $\mathcal{A}_t$ of local admissible homeomorphisms by the sheaf $\mathcal{A}_c$ of local compatible (with respect to a given thickened stratification of the base space) homeomorphisms: we can choose homeomorphisms $\Phi_i: \pi_0^{-1}(U_i) \to \pi^{-1}(\phi(U_i))$ such that $\mu_{ij} = \Phi_i^{-1} \circ \Phi_j \in A_c(U_i \cap U_j)$. Thus the cocycle μ_{ij} defines an element $\bar{\mu} \in H^1(O, \mathcal{A}_c)$ which maps to $\mu \in H^1(O, \mathcal{A}_t)$ under the natural homomorphism $H^*(O, \mathcal{A}_c) \to H^*(O, \mathcal{A}_t)$. Under the long exact sequence

$$\cdots \to H^1(\hat{\mathcal{A}}_c) \to H^1(\mathcal{A}_c) \xrightarrow{\bar{\delta}} H^2(\mathcal{R}) \to H^2(\hat{\mathcal{A}}_c) \to \cdots$$

we have $\bar{\delta}\bar{\mu} = \mu_C$.

If $\bar{\mu} = 0$ then of course the system $(M, \omega, \mathcal{L})$ is topologically equivalent to the reference system $(M_0, \omega_0, \mathcal{L}_0)$, and vice versa. If we can show that $H^1(O, \hat{\mathcal{A}}_c) = 0$, then we will have $\bar{\mu} = 0$ if and only if the Chern class μ_C vanishes.

LEMMA 4.3. *The sheaf $\hat{\mathcal{A}}_c$ is acyclic. In particular, we have $H^1(O, \hat{\mathcal{A}}_c) = 0$.*

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

Proof. The idea is that, due to Hypothesis (H5), the sheaf $\hat{\mathcal{A}}_c$ is similar to a fine sheaf, though it is not Abelian. It suffices to prove the above lemma for an arbitrary thickened stratum in O (more precisely, an arbitrary thickened open subset of a stratum). The acyclicity of $\hat{\mathcal{A}}_c$ over O will then follow from the Meyer–Vietoris exact sequences. Let $\bar{S}$ be a thickened stratum of O, and consider the sheaf $\hat{\mathcal{A}}_c$ over $\bar{S}$. $\bar{S}$ may be considered as a locally trivial fibration over a stratum S, with the fiber being local stratified submanifolds in $\bar{S}$ which are transversal to S. We will consider only open subsets of $\bar{S}$ which are saturated by the fibers of this fibration. In other words, we will consider $\hat{\mathcal{A}}_c$ as a sheaf over the S. Then it will become the sheaf of local continuous sections of a locally trivial fibration over S whose fibers are isomorphic to a contractible topological group, which implies the acyclicity. □

4.4. ADDITIVITY OF CHARACTERISTIC CLASSES

The Chern class is additive in the following sense:

LEMMA 4.4. *If three systems $\pi_i: (M_i, \omega_i, \mathcal{L}_i) \to O$, $i = 1, 2, 3$, over the same base space O, are roughly topologically equivalent, and the Chern class of $(M_i, \omega_i, \mathcal{L}_i)$ with respect to $(M_j, \omega_j, \mathcal{L}_j)$ is $\mu_C^{ij} \in H^2(O, \mathcal{R})$ (the three systems have the same affine monodromy sheaf $\mathcal{R}$), then we have $\mu_C^{12} + \mu_C^{23} + \mu_C^{31} = 0$.*

Proof. The proof is straightforward and uses the fact that $\mathcal{R}$ lies in the center of $\hat{\mathcal{A}}_t$: Denote by (U_i) an appropriate open covering of $O, U_{ij} = U_i \cap U_j$, ϕ_i^{ab}: $\pi_a^{-1}(U_i) \to \pi_b^{-1}(U_i)$ $(a, b = 1, 2, 3)$ the homeomorphisms that define the rough topological equivalences. We can assume that $\phi_i^{ba} = (\phi_i^{ab})^{-1}$ and $\phi_i^{31} \circ \phi_i^{23} \circ \phi_i^{12} = Id$. Put $\phi_{ij}^{ab} = (\phi_i^{ab})^{-1} \circ \phi_j^{ab} \in A_t^a(U_{ij})$, and denote by ψ_{ij}^{ab} a lifting of ϕ_{ij}^{ab} from the group $A_t^a(U_{ij})$ to the group $\hat{A}_t^a(U_{ij})$ such that $\psi_{ji}^{ab} = (\psi_{ij}^{ab})^{-1}$. Then $\mu_{ijk}^{ab} = \psi_{ij}^{ab}\psi_{jk}^{ab}\psi_{ki}^{ab} \in R(U_{ijk})$ is a 2-cocycle that defines the Chern class of the system $(M_b, \omega_b, \mathcal{L}_b)$ with respect to the system $(M_a, \omega_a, \mathcal{L}_a)$. If we can choose the elements ψ_{ij}^{ab} in such a way that $\mu_{ijk}^{12} + \mu_{ijk}^{23} + \mu_{ijk}^{31} = 0$ then we are done. Since $\phi_{ij}^{23} = \phi_i^{12} \circ \phi_{ij}^{13}(\phi_{ij}^{12})^{-1} \circ (\phi_i^{12})^{-1}$, we can choose ψ_{ij}^{ab} so that $\psi_{ij}^{23} = \mathrm{Ad}_{\phi_i^{12}}(\psi_{ij}^{13}(\psi_{ij}^{12})^{-1})$ (the operator $\mathrm{Ad}_{\phi_i^{12}}: \hat{A}_t^1(U_{ij}) \to \hat{A}_t^2(U_{ij})$) is well defined). Then we have:

$$\mu_{ijk}^{23} = \psi_{ij}^{23}\psi_{jk}^{23}\psi_{ki}^{23} = \mathrm{Ad}_{\phi_i^{21}}(\psi_{ij}^{23}\psi_{jk}^{23}\psi_{ki}^{23})$$

because $\mathrm{Ad}_{\phi_i^{21}}$ is identity when restricted to

$$\begin{aligned} R(U_{ijk}) &= (\psi_{ij}^{13}(\psi_{ij}^{12})^{-1}) \circ \mathrm{Ad}_{\psi_{ij}^{12}}(\psi_{jk}^{13}(\psi_{jk}^{12})^{-1}) \circ \mathrm{Ad}_{\psi_{ik}^{12}}(\psi_{ki}^{13}(\psi_{ki}^{12})^{-1}) \\ &= \psi_{ij}^{13}\psi_{jk}^{13}(\psi_{jk}^{12})^{-1}(\psi_{ij}^{12})^{-1}\psi_{ik}^{12}\psi_{ki}^{13} \\ &= (\psi_{ki}^{13}\psi_{ij}^{13}\psi_{jk}^{13})((\psi_{jk}^{12})^{-1}(\psi_{ij}^{12})^{-1}\psi_{ik}^{12}) \\ &= \mu_{ijk}^{13} - \mu_{ijk}^{12}. \end{aligned}$$

□

Similarly, the Lagrangian class is also additive:

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

LEMMA 4.5. *If three systems* $\pi_i: (M_i, \omega_i, \mathcal{L}_i) \to O, i = 1, 2, 3,$ *over the same base space* O*, are roughly symplectically equivalent, and the Lagrangian class of* $(M_i, \omega_i, \mathcal{L}_i)$ *with respect to* $(M_j, \omega_j, \mathcal{L}_j)$ *is* $\mu_L^{ij} \in H^1(O, \mathcal{Z}^1/\mathcal{R})$ (*the three systems have the same sheaf* $\mathcal{Z}^1/\mathcal{R}$)*, then we have* $\mu_L^{12} + \mu_L^{23} + \mu_L^{31} = 0$.

The *proof* is obvious. □

4.5. CLASSIFICATION THEOREMS

We can now present the main theorems. Their formulations are similar to the case of regular Lagrangian torus fibrations.

THEOREM 4.6. *If two roughly topologically equivalent integrable Hamiltonian systems have the same Chern class* (*with respect to a common reference system*)*, then they are topologically equivalent.*

The *proof* of the above theorem follows directly from Lemma 4.3 and Lemma 4.4. □

THEOREM 4.7. *If two roughly symplectically equivalent integrable Hamiltonian systems have the same Lagrangian class* (*with respect to a common reference system*)*, then they are symplectically equivalent.*

The *proof* of the above theorem follows directly from the definition. □

For a given reference system, the following commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{R} & \to & \hat{\mathcal{A}}_t & \to & \mathcal{A}_t & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{R} & \to & \mathcal{Z}^1 & \to & \mathcal{A}_s & \to & 0 \end{array}$$

give rise to the following commutative diagram of associated long exact sequences of cohomologies over the base space O:

$$\begin{array}{ccccccccccc} \cdots & \to & H^1(\hat{\mathcal{A}}_t) & \to & H^1(\mathcal{A}_t) & \xrightarrow{\delta} & H^2(\mathcal{R}) & \to & H^2(\hat{\mathcal{A}}_t) & \to & \cdots \\ & & \uparrow & & \uparrow & & \| & & \uparrow & & \\ \cdots & \xrightarrow{d} & H^1(\mathcal{Z}^1) & \to & H^1(\mathcal{A}_s) & \xrightarrow{\Delta} & H^2(\mathcal{R}) & \xrightarrow{d} & H^2(\mathcal{Z}^1) & \to & \cdots \\ & & \| & & & & & & \| & & \\ & & H^2(O, \mathbb{R}) & & & & & & H^3(O, \mathbb{R}) & & \end{array}$$

In the above diagram, $H^k(O, \mathcal{Z}^1)$ are identified with $H^{k+1}(O, \mathbb{R})$ via the isomorphisms arising from the short exact sequence $0 \to \mathbb{R} \to C^\infty \xrightarrow{d} \mathcal{Z}^1 \to 0$. Remark that, in the above diagram, the operators δ depend on the topological type of the reference system $\pi_0: (M_0, \omega_0, \mathcal{L}_0) \to O$, while the operators $\hat{d}$ and Δ depend on its rough symplectic type.

It follows from the above commutative diagram that if the system $(M, \omega, \mathcal{L})$ is roughly symplectically equivalent to a reference system $(M_0, \omega_0, \mathcal{L}_0)$, and both the

Chern and the Lagrangian classes of $(M, \omega, \mathcal{L})$ are taken with respect to $(M_0, \omega_0, \mathcal{L}_0)$, then under the maps Δ and $\hat{d}$ we have $\mu_L \overset{\Delta}{\to} \mu_C \overset{d}{\to} 0$. It is clear from the construction of characteristic classes that any element in $H^1(O, \mathcal{Z}^1/\mathcal{R})$ is the Lagrangian class of some integrable Hamiltonian system which is roughly symplectically equivalent to $(M_0, \omega_0, \mathcal{L}_0)$. Thus we have the following proposition which is similar to a result of Dazord and Delzant [10] for the regular case:

PROPOSITION 4.8. *An element $\mu_C \in H^2(O, \mathcal{R})$ is the Chern class of some integrable Hamiltonian system $\pi\colon (M, \omega, \mathcal{L}) \to O$ roughly symplectically equivalent to a given reference system $\pi_0\colon (M_0, \omega_0, \mathcal{L}_0) \to O$ over a given base space O if and only if $\hat{d}(\mu_C) = 0$. Under this condition, the space of integrable Hamiltonian systems which are roughly symplectically equivalent to $\pi_0\colon (M_0, \omega_0, \mathcal{L}_0) \to O$ and which have μ_C as the Chern class, considered up to symplectic equivalence, is naturally isomorphic to $H^2(O, \mathbb{R})/\hat{d}(H^1(O, \mathcal{R}))$.*

Notice that the above proposition does not solve the following problem: which elements in $H^2(O, \mathcal{R})$ can be realized as the Chern class of a system which is roughly topologically equivalent to $\pi_0\colon (M_0, \omega_0, \mathcal{L}_0) \to O$? The condition $\hat{d}(\mu_C) = 0$ needs not hold for systems which are roughly topologically equivalent but not roughly symplectically equivalent to the reference system (except for the case $H^3(O, \mathbb{R}) = 0$, when this condition is empty). I don't know which conditions must μ_C satisfy in general.

4.6. THE MAGNETIC TERM

In analogy with classical electromagnetism (see, e.g., [21]), by a *magnetic term* we will mean (the pull-back $\pi^*\beta$ of) a closed 2-form β on the base space O of an integrable system $\pi\colon (M, \omega, \mathcal{L}) \to O$. We have:

LEMMA 4.9. *For any closed 2-form β on O, the form $\omega + \pi^*\beta$ is a symplectic form on M, and $\pi\colon (M, \omega + \pi^*\beta, \mathcal{L}) \to O$ is an integrable system.*

Proof. We will show that $\omega + \pi^*\beta$ is nondegenerate everywhere on M, the rest is obvious. Let p be an arbitrary point of M. We will show that the kernel K_p of $\pi^*\beta$ at p is a coisotropic subspace of the tangent space at p with respect to the symplectic form ω. Then for any vector $X \in K_p$ there is a vector $Y \in T_pM$ such that $\omega(X, Y) + \pi^*\beta(X, Y) = \omega(X, Y) \neq 0$, and for any vector $X \in T_pM$ not belonging to K_p there is a vector $Y \in K_p$ such that $\omega(X, Y) + \pi^*\beta(X, Y) = \omega(X, Y) \neq 0$. According to Hypotheses (H1) and (H2), there is a sequence of regular points $p_i \in M$, $i \in \mathbb{N}$, of the fibration $\mathcal{L}$ which tend to p. Denote the tangent space of $\mathcal{L}$ at p_i by K_i. By taking a subsequence of (y_p), we can assume that K_i tends to a subspace $K_0 \subset T_pM$ at p. Since each K_i is Lagrangian, K_0 is also Lagrangian. On the other hand, K_i lies in the kernel of $\pi^*\beta$ at p_i, and it follows from the semi-continuity of the kernel that $K_0 \subset K_p$. Thus K_p is coisotropic. □

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

It is easy to see that if the magnetic term β is locally exact on O, then the two systems $\pi: (M, \omega + \pi^*\beta, \mathcal{L}) \to O$ and $\pi: (M, \omega, \mathcal{L}) \to O$ are roughly symplectically equivalent, and if β is exact then the two systems are symplectically equivalent.

Assume in this paragraph that the Poincaré lemma holds for 2-forms on O, i.e. any closed 2-form on O is exact, and denote by $\mathcal{Z}^2$ the sheaf of local closed 2-forms. Then we have the following short exact sequence: $0 \to \mathcal{Z}^1 \to \Omega^1 \stackrel{d}{\to} \mathcal{Z}^2 \to 0$, where the sheaf Ω^1 of local 1-forms on O is a fine sheaf (because C^∞ is a fine sheaf by Hypothesis (H6)). The associated long exact sequence gives rise to the following isomorphisms: $H^2_{\mathrm{DR}}(O, \mathbb{R}) = H^0(O, \mathcal{Z}^2)/\mathrm{d}H^0(O, \Omega^1) = H^1(O, \mathcal{Z}^1) = H^2(O, \mathbb{R})$. In other words, the second DeRham cohomology group of O is isomorphic to the second singular cohomology group of O. This isomorphism, together with the natural operator from $H^2(O, \mathbb{R}) = H^1(O, \mathcal{Z}^1)$ to $H^1(O, \mathcal{Z}^1/\mathcal{R})$, will send each closed 2-form β on O to an element $\mu_L(\beta) \in H^1(O, \mathcal{Z}^1/\mathcal{R})$ which is nothing else but the Lagrangian class of the system $\pi: (M, \omega + \pi^*\beta, \mathcal{L}) \to O$ with respect to the system $\pi: (M, \omega, \mathcal{L}) \to O$. In particular, if the Poincaré lemma holds for 2-forms on O, then any two systems over O which are roughly symplectically equivalent and have the same Chern class will differ from each other by only a magnetic term.

4.7. REALIZATION PROBLEM AND INTEGRABLE SURGERY

Given a stratified manifold O equipped with an integral affine structure (in the sense of Subsection 3.4), one can ask whether it can be realized as the base space of some integrable Hamiltonian system. If it is the case, we will say that O is *realizable*. Of course, if O is to be realizable, it has to be *locally realizable*: each singular point y in O corresponds to some singularity of some integrable system, that is a singular Lagrangian torus fibration with the base space $U(y)$ where $U(y)$ is a neighborhood of y in O, in such a way that the following compatibility is satisfied: If $U(y_1) \cap U(y_2) \neq \emptyset$ then there is a fibration-preserving symplectomorphism $\Phi_{y_1 y_2}$ between the restrictions of the two fibrations to $U(y_1) \cap U(y_2)$; if $U(y_1) \cap U(y_2) \cap U(y_3) \neq \emptyset$, then the map $\Phi_{y_1 y_2} \circ \Phi_{y_2 y_3} \circ \Phi_{y_3 y_1}$ (on a restricted fibration over $U(y_1) \cap U(y_2) \cap U(y_3)$) is isomorphic to identity. A stratified integral affine manifold O equipped with such singularities will be called a *formal rough symplectic type*, and denoted by $\hat{O}_{\mathrm{symp}}$ as before.

A natural problem arises: given a formal rough symplectic type $\hat{O}_{\mathrm{symp}}$, is there any integrable system roughly symplectically equivalent to it ? A natural way to solve this problem is via *integrable surgery*. In this paper, by an integrable surgery, we mean a surgery of an integrable Hamiltonian system which projects to a surgery on its base space.

For example, given two integrable systems $(M_1, \omega_1, \mathcal{L}_1) \stackrel{\pi_1}{\to} O_1$ and $(M_2, \omega_2, \mathcal{L}_2) \stackrel{\pi_2}{\to} O_2$ over two subsets O_1 and O_2 of a space O, such that they are roughly symplectically equivalent when restricted to the common base space $O_1 \cap O_2$. Does there exist an integrable system over $O_1 \cup O_2$ which is roughly symplectically equivalent to the above systems when restricted to O_1 and O_2 ? The answer to this question may be given in terms of characteristic classes:

PROPOSITION 4.10. *Denote the difference between the Lagrangian classes of the systems* $(M_1, \omega_1, \mathcal{L}_1) \xrightarrow{\pi_1} O_1$ *and* $(M_2, \omega_2, \mathcal{L}_2) \xrightarrow{\pi_2} O_2$ *restricted to* $Q_1 \cap O_2$ *by* $\mu_L \in H^1(O_1 \cap O_2, \mathcal{Z}^1/\mathcal{R})$. *Then there is an integrable system with the base space* $O_1 \cup O_2$ *which is roughly symplectically equivalent to the above two systems when restricted to* O_1 *and* O_2 *if and only if* μ_L *lies in the sum of the images of* $H^1(O_1, \mathcal{Z}^1/\mathcal{R})$ *and* $H^1(O_2, \mathcal{Z}^1/\mathcal{R})$ *in* $H^1(O_1 \cap O_2, \mathcal{Z}^1/\mathcal{R})$ *under the restriction maps. In particular, if* $(M_1, \omega_1, \mathcal{L}_1) \xrightarrow{\pi_1} O_1$ *and* $(M_2, \omega_2, \mathcal{L}_2) \xrightarrow{\pi_2} O_2$ *are topologically equivalent when restricted to* $O_1 \cap O_2$, *then the obstruction to the existence of the required system over* $O_1 \cup O_2$ *lies in the quotient of the group* $H^2(O_1 \cap O_2, \mathbb{R})/\hat{d}H^1(O_1 \cap O_2, \mathcal{R})$ *by the sum of the images of* $H^2(O_1, \mathbb{R})/\hat{d}H^1(O_1, \mathcal{R})$ *and* $H^2(O_2, \mathbb{R})/\hat{d}H^1(O_2, \mathcal{R})$ *under the restriction maps.*

Proof. It is a direct consequence of the results of Subsection 4.5 □

For the case of systems with 2 degrees of freedom we have:

PROPOSITION 4.11. *Any formal rough symplectic type* $\hat{O}_{\text{symp}}$ *with* O *two-dimensional is realizable.*

Proof. If O is 2-dimensional, then we can always choose $O_1 \cap O_2$ in Proposition 4.10 to be a tubular neighborhood of something one-dimensional, so all the obstructions vanish. □

Starting with 3 degrees of freedom, there are formal rough symplectic types which are not realizable, as the following example shows.

EXAMPLE 4.12. A fake base space. Let (S^2, ω) be a symplectic 2-sphere, $f\colon S^2 \to \mathbb{R}$ a Morse function with 2 maximal points of the same value ($= 1$), 2 minimal points of the same value ($= -1$), two saddle points of different values ($= \pm 1/2$), such that f is invariant under an involution of S^2 which preserves the symplectic form. Denote the base space of this integrable system with one degree of freedom by $G = G_+ \cup G_-$, where G_+ (resp., G_-) corresponds to the part of the sphere with $f \geqslant 0$ (resp., $f \leqslant 0$). G is a tree with 5 edges: 2 upper, 2 lower, and one middle. Denote by σ the involution of G which preserves f and the lower edges but interchanges the two upper edges (so σ cannot be lifted to an involution on S^2). Denote by K^2 the Klein bottle with a standard integral affine structure. We have $\pi_1(K^2) = \langle a, b \mid abab^{-1} = 1\rangle$. Denote by $\bar{K}$ the double covering of K^2 corresponding to the subgroup of $\pi_1(K^2)$ which is generated by a^2 and b (so $\bar{K}$ is again a Klein bottle), and denote the involution on $\bar{K}$ corresponding to that double covering also by σ. Put $O = \bar{K} \times_\sigma G = (\bar{K} \times G)/\mathbb{Z}_2$, with the induced integral affine structure from the direct product. We have $O = O_+ \cup O_-$, with $O_- = \bar{K} \times_\sigma G_- = K^2 \times G_-$, and $O_+ = \bar{K} \times_\sigma G_+$ a twisted product. The spaces O_- and O_+ are base spaces of integrable systems induced from the direct product of the subsystems over G_- and G_+ with a system over $\hat{K}$. These two systems are roughly equivalent over $O_0 = O_+ \cap O_- \approx K^2$, but they are not equivalent, hence they cannot be glued together to obtain a system over O. More

https://doi.org/10.1023/A:1026133814607 Published online by Cambridge University Press

precisely, the affine monodromy sheaf over $O_0 \approx K^2$ in O is $\mathcal{R} \cong \mathcal{R}_{K^2} \oplus \mathbb{Z}$ where $\mathcal{R}_{K^2}$ is the affine monodromy of K^2 as an affine manifold itself;

$$H^2(O_0, \mathcal{R}) \cong H^2(K^2, \mathcal{R}_{K^2}) \oplus H^2(K^2, \mathbb{Z}) = H^2(K^2, \mathcal{R}_{K^2}) \oplus \mathbb{Z}_2,$$

and we have a natural map to the second component: $H^2(O_0, \mathcal{R}) \xrightarrow{\rho} \mathbb{Z}_2$. Any system over O_- will have a Chern class which when restricted to O_0 will map to 0 under the map ρ, but any system over O_- will have a Chern class which when restricted to O_0 will map to the nontrivial element of $\mathbb{Z}_2$. hence, those systems can never be glued together to form a system over O. In other words, O is not realizable.

4.8. Some Examples of Integrable Surgery

EXAMPLE 4.13. Exotic symplectic $\mathbb{R}^{2n}s$. Start with the following two integrable systems: The first one is given by the moment map $\mathbf{F} = (F_1, \ldots, F_n) = (\pi x_1^2 + \pi y_1^2, \ldots, \pi x_n^2 + \pi y_n^2)$ on the open ball of radius 1 of $\mathbb{R}^{2n}$ with coordinates x_i, y_i and with the standard metric and symplectic structure (i.e. a harmonic oscillator; here $\pi = 3.14159\ldots$). On the base space O_1 of this system, the functions F_i are also integral affine coordinates of the induced affine structures outside the singularities. Let O_2 be an open n-disk attached to O_1 in such a way that $O_1 \cup O_2$ is diffeomorphic to O_1 *rel*. Singularities of O_1, and $O_1 \cap O_2$ is contractible. Extend the functions $F_1, \ldots, F_n$ from O_1 to O_2 in such a way that $\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_n \neq 0$ everywhere on O_2 and there is a point $y \in O_2$ with $F_1(y) = \cdots = F_n(y) = 0$. O_2 with the integral affine structure given by the functions F_i is the base space of a unique integrable system $(M_2, \omega_2) \to O_2$. This is our second system. By construction, our two systems can be glued in a unique natural way into an integrable system living on a symplectic manifold diffeomorphic to $\mathbb{R}^{2n}$. The preimage of y in this manifold is a Lagrangian torus, and in fact it is an *exact* Lagrangian torus (i.e. for any 1-form α such that $\mathrm{d}\alpha$ is equal to the symplectic form, the restriction of α on this torus is cohomologous to 0). On the other hand, a famous result of Gromov [19] says that in the standard symplectic space there can be no smooth closed exact Lagrangian submanifold. Thus our symplectic space is exotic in the sense that it can not be symplectically embedded to the standard symplectic space of the same dimension. This example is inspired by a different example found by Bates and Peschke [4].

EXAMPLE 4.14. Toric manifolds. Consider a Hamiltonian $\mathbb{T}^n$ action on a closed symplectic $2n$-dimensional manifold (M, ω), which is free somewhere. (M, ω) together with this torus action may be called a *Hamiltonian toric manifold*. The regular (singular) orbits of this $\mathbb{T}^n$ action are Lagrangian (isotropic) tori, and they are fibers of an integrable Hamiltonian system with only elliptic singularities. The base space of this system is integral-affinely equivalent to a polytope in the Euclidean space $\mathbb{R}^n$, whose each vertex has exactly n edges and these edges can be moved to the principal axis of $\mathbb{R}^n$ by an integral affine transformation. (This fact follows easily from the normal form of elliptic singularities given by Eliasson [16] and Dufour and Molino

[12].) A famous theorem of Delzant [11] says that each polytope satisfying the above condition on vertices is the base space of a Hamiltonian toric manifold which is unique up to symplectic equivalence. (These Hamiltonian toric manifolds admit a Kähler structure which make them toric manifolds in the sense of complex algebraic geometry.) The uniqueness in Delzant's theorem is evident from our point of view: Since $\mathcal{R}$ in this case is isomorphic to the constant sheaf $\mathbb{Z}^n$, and the base space is contractible, there is no room for characteristic classes. The existence is also simple: one starts from a Lagrangian section, and reconstructs the system (and the ambient manifold) in a unique way (see [6]).

EXAMPLE 4.15. Twisted Products. We may call a *twisted product* of two integrable systems $(M_1, \omega_1, \mathcal{L}_1) \xrightarrow{\pi_1} O_1$ and $(M_2, \omega_2, \mathcal{L}_2) \xrightarrow{\pi_2} O_2$ an integrable system over $O_1 \times O_2$, which is not topologically equivalent but is roughly symplectically equivalent to the direct product of the two systems, and with the following property: The Marsden-Weinstein reduction of this system to $\{y\} \times O_2$ (resp., $O_1 \times \{y\}$) is symplectically equivalent to $(M_1, \omega_1, \mathcal{L}_1)$ (resp., $(M_2, \omega_2, \mathcal{L}_2)$ for every point $y \in O_1$ (resp., $y \in O_2$). For example, if M_i if M_i are symplectic tori, with the systems given by Morse functions, then $H^2(O_1 \times O_2, \mathcal{R}) = \mathbb{Z}^4$ (here $\mathcal{R}$ is the corresponding affine monodromy sheaf, and the formula can be obtained easily via Meyer-Vietoris sequences), and non-zero elements of this group corresponds to twisted products.

EXAMPLE 4.16. Blow-ups. Blowing-up, one of the main tools in algebraic geometry, is also useful in symplectic geometry (see, e.g., [20, 27, 28]), as well as in the study of singularities of integrable Hamiltonian systems (see, e.g., [12]). To blow up a point in a $2n$-dimensional symplectic manifold, one can cut away a symplectic ball containing this point, and then collapse the sphere which is the boundary of this ball to $\mathbb{C}P^{n-1}$ by collapsing each characteristic curve on this sphere to a point. Since a symplectic ball admits a simple natural integrable system, namely the harmonic oscillator, blowing-up can also be done by integrable surgery: it amounts to cutting out an appropriate simplex which contains a nondegenerate elliptic fixed point from the base space. If instead of an elliptic fixed point, we have a stratum in the base space which is closed and consists of nondegenerate elliptic singular points of constant rank, then cutting the base space by an appropriate 'hyperplane' near this stratum will lead to the symplectic blowing-up along the symplectic submanifold which is the preimage of this stratum.

EXAMPLE 4.17. Dehn surgery. Consider an integrable system with 2 degrees of freedom, $\pi: (M^2, \omega, \mathcal{L}) \to O$, and let $D^2 \in O$ be a closed disk lying in the regular part of O. Cut out the piece $\pi^{-1}(D^2)$ from the system, and then glue it back after some twisting. This operation may be called an *integrable Dehn surgery*, in analogy with the well-known Dehn surgery in low-dimensional topology. It is easy to see that any two roughly symplectically equivalent systems with two degrees of freedom can be transformed to each other by performing Dehn surgeries and then adding a magnetic term to the symplectic form.

EXAMPLE 4.18. Hamiltonian Hopf bifurcation. There is a phenomenon called *Hamiltonian Hopf bifurcation*, which happens in the Lagrange top and many other Hamiltonian systems (see, e.g., [29]): under some parameter change (e.g. the energy), two pairs of purely imaginary eigenvalues (of the reduced linearized system at an equilibrium) tend to each other, coincide, and then jump out of the imaginary axe to form a quadruple of complex eigenvalues which is symmetric with respect to both the real axe and the imaginary axe. One can verify, for example, that a majority of integrable systems arising from the so called *argument shift method* on coadjoint orbits of compact Lie algebras (see, e.g., [31]) exhibit such bifurcations. In terms of integrable systems, this bifurcation amounts to a (generic) degenerate corank 2 singularity which connects an elliptic-elliptic singularity (i.e. a nondegenerate singularity of corank 2 which has 2 elliptic components) to a focus-focus singularity in a 1-parameter family. From the integrable surgery point of view, the passage from an elliptic-elliptic singularity to a focus-focus singularity can be done via a small surgery, without the need of an 1-parameter family. I will omit the explicit operation here.

EXAMPLE 4.19. K3, ruled manifolds, etc. It is easy to construct systems with 2 degrees of freedom for which a topological 2-stratum C of the base space is of any of the allowed cases listed in Proposition 3.3. The most interesting case is S^2. The sphere S^2 admits an integral affine structure with 24 singular points of focus-focus type, which may be constructed as follows: Start with an integral affine triangle ($=$ base space of $\mathbb{C}P^2$ under a Hamiltonian torus action). Cut from this triangle 3 small homothetic triangles, each lying on one edge. Gluing together the pairs of small edges that arise after we cut out the small triangles, we obtain a triangle with an integral affine structure with 3 singular points of focus-focus type. We can glue 8 copies of this new triangle together to obtain a sphere with 24 focus-focus points. Proposition 4.11 shows that this S^2 is the base space of some integrable Hamiltonian system with 24 simple focus-focus singularities. Topologically, it is a torus fibration over S^2 with 24 singular fibers of type I^+ in the sense of Matsumoto, and the ambient space is diffeomorphic to a K3 surface (see [26] and references therein). We can also go the other way around (less explicitly): start with a holomorphic integrable system on a K3 surface (see [24]). Forgetting about the complex structure and taking the real part of the complex symplectic form, we get an integrable system with 2 degrees of freedom whose base space is homeomorphic to S^2. Similarly, we can construct a system whose base space is homeomorphic to $\mathbb{R}P^2$, with 12 focus-focus singular points. The ambient manifold will be diffeomorphic to an Enriquez surface.

Assume now that the base space has no focus-focus singular point and is diffeomorphic to the direct product of a graph or a circle with a closed interval (the affine structure on O needs not be a direct product). The ambient manifold of integrable systems with such an orbit space O are rational and ruled manifolds in the sense of McDuff (see e.g. [27, 28]), which are analogs of complex ruled surfaces (see, e.g., [3]). It can be shown easily that in this case, as in the case of S^2 with 24 focus-focus points, we have $H^2(O, \mathcal{R}) = 0$ (for any realizable affine structure on O). If O is a product of 2 graphs which are not trees, then there will be many topologically non-equivalent integrable systems over O, because $H^2(O, \mathcal{R}) \neq 0$, etc.

Acknowledgements

I would like to thank Jean-Paul Dufour, Pierre Molino, and Lê Hông Vân for conversations on characteristic classes, Heinz Hansmann for explaining to me the Hamiltonian Hopf bifurcation, Kaoru Ono for pointing out to me Matsumoto's work on torus fibrations and the fact that an example of mine of a symplectic 4-manifold is in fact diffeomorphic to a K3 surface. I would like to thank also Michel Boileau, Lubomir Gavrilov, Giuseppe Griffone and Jean-Claude Sikorav for an invitation to Toulouse, for their very warm hospitality and interesting discussions. And I'm especially indebted to Michèle Audin, Anatoly T. Fomenko, and Alberto Verjovsky, without whom this paper would not exist.

References

1. Abraham, R. and Marsden, J. E.: *Foundations of Mechanics*, Benjamin, New York, 1978.
2. Audin, M.: *Spinning Tops*, Cambridge Stud. Adv. Math. 51, Cambridge Univ. Press, 1996.
3. Barth W., Peters, C. and Van de Ven, A.: *Compact Complex Surfaces*, Springer, Berlin, 1984.
4. Bates, L. and Peschke, G.: A remarkable symplectic structure, *J. Differential Geom.* **32** (1990), 533–538.
5. Bolsinov, A. V. and Fomenko, A. T.: *Integrable Hamiltonian Systems, Geometry, Topology, Classification,* Vols. 1 and 2, 1999 (in Russian).
6. Boucetta, M. and Molino, P.: Géométrie globale des systèmes hamiltoniens complètement integrables, *C. R. Acad. Sci. Paris Sér. I Math.* **308** (1989), 421–424.
7. Colin de Verdier, Y. and Vey, J.: Le lemme de Morse isochore, *Topology* **18** (1979), 283–293.
8. Cushman, R. and Bates, L.: *Global Aspects of Classical Integrable Systems*, Birkhäuser, Basel, 1997.
9. Cushman, R. and Knörrer, H.: The Energy-Momentum mapping of the Lagrange top, In: Lecture Notes in Math. 1139, Springer, New York, 1985, pp. 12–24.
10. Dazord, P. and Delzant, T.: Le problème général des variables action-angles, *J. Differential Geom.* **26**(2) (1987), 223–251.
11. Delzant, T.: Hamiltoniens périodiques et image convexe de l'application moment, *Bull. Soc. Math. France*, **116** (1988), 315–339.
12. Dufour, J.-P. and Molino, P.: Compactification d'action de R^n et variables action-angle avec singularités, In: P. Dazord and A. Weinstein (eds.), *Seminaire Sud-Rhodanien de géométric* (*Berkeley 1989*), Math. Sci. Res. Inst. Publ. 20, Cambridge Univ. Press, 1990.
13. Dufour, J.-P., Molino, P. and Toulet, A.: Classification des systemes integrables en dimension 2 et invariants des modeles de Fomenko, *C. R. Acad. Sci.* Paris, 1994.

14. Duistermaat, J. J.: On global action-angle variables, *Comm. Pure Appl. Math.* **33** (1980), 687–706, 151–167.
15. Duistermaat, J. J. and Heckman, G. J.: On the variation in the cohomology of the symplectic form of the reduced phase space and Addendum, *Invent. Math.* **69** (1982), 259–269 and **72** (1983), 153–158.
16. Eliasson, L. H.: Normal form for Hamiltonian systems with poisson commuting integrals – elliptic case, *Comm. Math. Helv.* **65** (1990), 4–35.
17. Fomenko, A. T. (ed.): *Topological Classification of Integrable Systems*, Adv. Soviet Math. 6, Amer. Math. Soc., Providence, RI, 1991.
18. Godement, R.: *Théorie des faisceaux*, 3rd edn, Hermann, Paris, 1973.
19. Gromov, M.: Pseudoholomorphic curves on almost complex manifolds, *Invent. Math.* **82** (1985), 307–347.
20. Guillemin, V. and Sternberg, S.: Birational equivalence in symplectic geometry, *Invent. Math.* **97** (1989), 485–522.
21. Guillemin, V. and Sternberg, S.: *Symplectic Techniques in Physics*, Cambridge Univ. Press, 1984.
22. Guillemin, V. and Uribe, A.: Monodromy in the quantum spherical pendulum, *Comm. Math. Phys.* **122**(4) (1989), 563–574.
23. Lerman, L. and Umanskiy, Ya.: Four-dimensional integrable Hamiltonian systems with simple singular points (Topological aspects), *Transl. Math. Monogr.* 176, Amer. Math. Soc. Providence, 1998.
24. Markushevich, D.: Complex integrable Hamiltonian systems, *Math. USSR Sb.* **59** (1988), 459–469.
25. Marle, C-M.: Variables actions-angles: leur détermination et leurs singularités, In: *La Mécanique analytique de Lagrange et son héritage*, vol. I. Supplemento al numero 124 (1990) *Atti Accad. Sci. Torino*, (1990), 211–243.
26. Matsumoto, Y.: Topology of torus fibrations, Sugaku expositions, **2** (1989), 55–73.
27. McDuff, D.: Lectures on symplectic 4-manifolds, In: M. Audin and J. Lafontaine (eds), *Pseudoholomorphic Curves in Symplectic Geometry*, Progr. in Math. 117, Birkhäuser, Basel, 1994.
28. McDuff, D. and Salamon, D.: *Introduction to Symplectic Geometry*, Oxford Univ. Press, 1995.
29. van der Meer, J. C.: *The Hamiltonian Hopf Bifurcation*, Lecture Notes in Math. 1169, Springer, New York, 1985.
30. Milnor, J.: On the existence of a connection with curvature zero, *Comment. Math. Helv.* **32** (1958), 215–233.
31. Mishchenko, A. S. and Fomenko, A. T.: Euler equations on finite-dimensional Lie groups, *Math. USSR Izv.* **12** (1978), 371–389.
32. Nekhoroshev, N. N.: Action-angle variables and their generalizations, *Trans. Moscow Math. Soc.* **26** (1972), 180–198.
33. Rüssmann, H.: Über das Verhalten analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung, *Math. Ann.* **154** (1964), 285–300.
34. Vey, J.: Sur certaines systèmes dynamiques séparables, *Amer. J. Math.* **100** (1978), 591–614.
35. Vu Ngoc San: Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type, *Comm. Pure Appl. Math.* **53**(2) (2000), 143–217.
36. Vu Ngoc San: On semi-global invariants for focus-focus singularities, *Topology* **42**(2) (2003), 365–380.
37. Williamson, J.: On the algebraic problem concerning the normal forms of linear dynamical systems, *Amer. J. Math.* **58**(1) (1936), 141–163.

38. Nguyen Tien Zung: A topological classification of integrable Hamiltonian systems, *Séminaire Gaston Darboux, Université Montpellier* (II), 1994–1995, pp. 43–54.
39. Nguyen Tien Zung: Symplectic topology of integrable Hamiltonian systems, I: Arnold–Liouville with singularities, *Compositio Math*. **101** (1996), 179–215.
40. Nguyen Tien Zung: A note on focus-focus singularities, *Differential Geom. Appl*. **7** (1997), 123–130.
41. Nguyen Tien Zung: A note on degenerate corank-1 singularities of integrable Hamiltonian systems, *Comment. Math. Helv*. **75** (2000), 271–283.
42. Nguyen Tien Zung: Convergence versus integrability in Birkhoff normal forms, Preprint math.DS/0104279.