Canad. J. Math. Vol. **65** (4), 2013 pp. 823–842
http://dx.doi.org/10.4153/CJM-2012-045-3

Symbolic Powers Versus Regular Powers of Ideals of General Points in $\mathbb{P}^1 \times \mathbb{P}^1$

Elena Guardo, Brian Harbourne, and Adam Van Tuyl

Abstract. Recent work of Ein–Lazarsfeld–Smith and Hochster–Huneke raised the problem of which symbolic powers of an ideal are contained in a given ordinary power of the ideal. Bocci–Harbourne developed methods to address this problem, which involve asymptotic numerical characters of symbolic powers of the ideals. Most of the work done up to now has been done for ideals defining 0-dimensional subschemes of projective space. Here we focus on certain subschemes given by a union of lines in $\mathbb{P}^3$ that can also be viewed as points in $\mathbb{P}^1 \times \mathbb{P}^1$. We also obtain results on the closely related problem, studied by Hochster and by Li and Swanson, of determining situations for which each symbolic power of an ideal is an ordinary power.

1 Introduction

Refinements of the groundbreaking results of [7, 21] regarding which symbolic powers of ideals are contained in a given ordinary power of the ideal have recently been given in [1–3, 23], with a focus on ideals defining 0-dimensional subschemes of projective space. The methods mainly involve giving numerical criteria, both for containment and for non-containment. These criteria have been extended in [16] to ideals defining smooth subschemes in $\mathbb{P}^N$ and applied to the case of disjoint unions of lines. The most difficult numerical character needed for these results is denoted in these papers by $\gamma(I)$. We pause briefly to recall its definition.

Throughout this paper we work over an algebraically closed field k of arbitrary characteristic. Let $k[\mathbb{P}^N]$ denote the polynomial ring $k[x_0, \dots, x_N]$ with the standard grading (so each variable has degree 1). Given any homogeneous ideal $(0) \neq I \subseteq k[\mathbb{P}^N]$, $\alpha(I)$ denotes the least degree of a nonzero form (*i.e.,* homogeneous element) in I. Then the limit $\lim_{m\to\infty} \alpha(I^{(m)})/m$ is known to exist (see, for example, [1, Lemma 2.3.1]), and is denoted by $\gamma(I)$.

A large amount of work has been done studying $\gamma(I)$ in a range of contexts (including number theory [4, 32, 33], complex analysis [29], algebraic geometry [1, 2, 8, 27], and commutative algebra [21]), with an emphasis on the case that I defines a 0-dimensional subscheme. Our focus here will be on computing $\gamma(I)$ for ideals of lines in $\mathbb{P}^3$. A special case for which $\gamma(I)$ can be computed is when the symbolic powers $I^{(m)}$ and ordinary powers I^m all coincide. This is because if $I^{(m)} = I^m$ for

Received by the editors March 14, 2012.
Published electronically November 13, 2012.
The second author's work on this project was sponsored by the National Security Agency under Grant/Cooperative agreement "Advances on Fat Points and Symbolic Powers," Number H98230-11-1-0139. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notice. The third author acknowledges the support provided by NSERC.
AMS subject classification: **13F20**, 13A15,14C20.
Keywords: symbolic powers, multigraded, points.

all $m \geq 1$, then $\alpha(I^{(m)}) = \alpha(I^m) = m\alpha(I)$, hence $\gamma(I) = \alpha(I)$. Thus we will also be interested in distinguishing when $I^{(m)} = I^m$ for $m \geq 1$ occurs and when it does not. It has been known for a long time that $I^{(m)} = I^m$ holds for all $m \geq 1$ when I is a complete intersection (*i.e.,* defined by a regular sequence; see [34, Lemma 5, Appendix 6]). What is of interest is when I is not a complete intersection. This is also a remarkably difficult problem; partial results have been obtained, for example, in [22, 24].

The reason for our focus on ideals of certain unions of lines in $\mathbb{P}^3$ is that, for the cases we will consider, the questions can be converted into ones involving symbolic powers of ideals of points in $\mathbb{P}^1 \times \mathbb{P}^1$. The ideal I of a point in $\mathbb{P}^1 \times \mathbb{P}^1$ is a bigraded ideal. Since $k[\mathbb{P}^1 \times \mathbb{P}^1] = k[\mathbb{P}^3]$ as rings, we can regard I as defining a subscheme of $\mathbb{P}^3$, the key being that even though I as a bigraded ideal defines a point in $\mathbb{P}^1 \times \mathbb{P}^1$, it defines a line in $\mathbb{P}^3$ when regarded as a singly graded ideal in the usual grading on $k[\mathbb{P}^3]$; see Remark 2.1.1. Thus the ideal of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is simultaneously (but with respect to a different grading) the ideal of a finite set of lines $\mathbb{P}^3$. (As a specific example, the ideal of $s \leq 4$ general points of $\mathbb{P}^1 \times \mathbb{P}^1$ is the ideal of s general lines of $\mathbb{P}^3$; see Remark 2.1.1. For $s > 4$, the ideal of s points of $\mathbb{P}^1 \times \mathbb{P}^1$ is the ideal of s lines in $\mathbb{P}^3$, but the lines are never general, even if the points are.) Moving to $\mathbb{P}^1 \times \mathbb{P}^1$ makes available to us the vast array of work done on products of projective spaces and surfaces in general, and on $\mathbb{P}^1 \times \mathbb{P}^1$ in particular; see, for example [12, 14, 18, 25, 28, 30].

Our main results are Theorems 1.1 and 1.2.

Theorem 1.1 *Let I be the ideal of $s \geq 1$ general points of $\mathbb{P}^1 \times \mathbb{P}^1$.*

- *If $s = 1$, then $\gamma(I) = 1$.*
- *If $s = 2$ or 3, then $\gamma(I) = 2$.*
- *If $s = 4$, then $\gamma(I) = 8/3$.*
- *If $s = 5$, then $\gamma(I) = 3$.*
- *If $s = 6$, then $\gamma(I) = 24/7$.*
- *If $s = 7$, then $\gamma(I) = 56/15$.*
- *If $s = 8$, then $\gamma(I) = 4$.*
- *If $9 \leq s$, then $\sqrt{s} - 1 < \alpha(I)/2 \leq \gamma(I) \leq \sqrt{2s}$ and $4 \leq \gamma(I)$.*

See Section 2 for the proof.

Theorem 1.2 *Let I be the ideal of a set Z of s general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $I^m = I^{(m)}$ for all $m > 0$ if and only if s is $1, 2, 3,$ or 5. Moreover, $I^{(3)} \neq I^3$ if $s = 4$ and $I^{(2)} \neq I^2$ if $s \geq 6$.*

See Section 3 for the proof. We note that the ideal I of s general points of $\mathbb{P}^1 \times \mathbb{P}^1$ is a complete intersection if and only if $s = 1$ (see the paragraph right before Proposition 2.1.2).

Whereas most of our focus in this paper is on sets of points in general position in $\mathbb{P}^1 \times \mathbb{P}^1$, points not in general position can also be of interest; note for example that a reduced scheme consisting of $s > 1$ general points in $\mathbb{P}^1 \times \mathbb{P}^1$ is never arithmetically Cohen–Macaulay. In a recent preprint[17] we studied this problem for finite sets of points that are arithmetically Cohen–Macaulay subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$.

2 Background

2.1 Points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their Ideals

For the convenience of the reader, we begin with a review of multi-graded ideals arising in the context of products of projective space.

The multi-homogeneous coordinate ring $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ is

$$k[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{t,0}, \dots, x_{t,n_t}].$$

It has a multi-grading given by

$$\deg(x_{i,j}) = e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^t,$$

where the 1 is in the i-th position. The ring $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ is a direct sum of its multi-homogeneous components $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\dots,a_t)}$, where $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\dots,a_t)}$ is the k-vector space span of the monomials of multi-degree $(a_1, \dots, a_t)$. An ideal $I \subseteq k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ is multi-homogeneous if it is the direct sum of its multi-homogeneous components (*i.e.,* of $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\dots,a_t)} \cap I$). Note that a multi-homogeneous ideal I can be regarded as a homogeneous ideal in $k[\mathbb{P}^N]$, $N = n_1 + \cdots + n_t + t - 1$, where a monomial of multi-degree $(a_1, \dots, a_t)$ has degree $d = a_1 + \cdots + a_t$ and the homogeneous component of I of degree d is $I_d = \bigoplus_{\sum_i a_i = d} I_{(a_1,\dots,a_t)}$. However, when $t > 1$, a multi-homogeneous ideal I when regarded as being homogeneous never defines a 0-dimensional subscheme of $\mathbb{P}^N$, even if I defines a zero-dimensional subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. For example, the multi-homogeneous ideal I of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ defines a finite set of lines in $\mathbb{P}^3$, which are skew (and thus not a cone) if no two of the points lie on the same horizontal or vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$ (see Remark 2.1.1), and not a complete intersection unless the points comprise a rectangular array in $\mathbb{P}^1 \times \mathbb{P}^1$ (see the paragraph right before Proposition 2.1.2).

Let $R = k[\mathbb{P}^1 \times \mathbb{P}^1]$, where we will use the standard multi-grading for R. That is, $R = k[x_0, x_1, y_0, y_1]$, with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. Let $I \subseteq R$ be a multi-homogeneous ideal (because R is bigraded, we sometimes say I is bihomogeneous). Then I has a multi-homogeneous primary decomposition, *i.e.,* a primary decomposition $I = \bigcap_i Q_i$, where each $\sqrt{Q_i}$ is a multi-homogeneous prime ideal, and Q_i is multi-homogeneous and $\sqrt{Q_i}$-primary [34, Theorem 9, p. 153]. We define the m-th *symbolic power* of I to be the ideal $I^{(m)} = \bigcap_j P_{i_j}$, where $I^m = \bigcap_i P_i$ is a multi-homogeneous primary decomposition, and the intersection $\bigcap_j P_{i_j}$ is over all components P_i such that $\sqrt{P_i}$ is contained in an associated prime of I. In particular, we see that $I^{(1)} = I$ and that $I^m \subseteq I^{(m)}$.

Of particular interest to this paper is the case where I is the ideal of a set Z of s distinct reduced points of $\mathbb{P}^1 \times \mathbb{P}^1$, *i.e.,* $Z = \{P_1, \dots, P_s\}$. A point has the form $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$, and its defining ideal $I(P)$ in R is a prime ideal of the form $I(P) = (F, G)$, where $\deg F = (1, 0)$ and $\deg G = (0, 1)$. The ideal $I(Z)$ is then given by $I(Z) = \bigcap_{i=1}^s I(P_i)$. Furthermore, the m-th symbolic power of $I(Z)$ has the form $I(Z)^{(m)} = \bigcap_{i=1}^s I(P_i)^m$. The scheme defined by $I(Z)^{(m)}$ is sometimes referred to as a *fat point scheme* and denoted $mP_1 + \cdots + mP_s$.

https://doi.org/10.4153/CJM-2012-045-3 Published online by Cambridge University Press

Remark 2.1.1 Note that while the gradings on the rings $k[\mathbb{P}^1 \times \mathbb{P}^1]$ and $k[\mathbb{P}^3]$ are different (and hence $k[\mathbb{P}^1 \times \mathbb{P}^1]$ and $k[\mathbb{P}^3]$ are not isomorphic as graded rings), the underlying rings are the same; in particular, $k[\mathbb{P}^1 \times \mathbb{P}^1] = k[x_0, x_1, y_0, y_1] = k[\mathbb{P}^3]$. A given ideal in this common underlying ring can define non-isomorphic subschemes depending on which graded structure we use. For example, the irrelevant ideals (x_0, x_1) and (y_0, y_1) corresponding to the two factors of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ define a pair of skew lines $L_1 \cong \mathbb{P}^1$ and $L_2 \cong \mathbb{P}^1$ in $\mathbb{P}^3$, where $I(L_1) = (y_0, y_1)$ and hence $k[L_1] = k[x_0, x_1]$, and similarly $I(L_2) = (x_0, x_1)$ and $k[L_2] = k[y_0, y_1]$. Thus the point $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ corresponds to a pair of points $P_1 = [a_0 : a_1] \in L_1$ and $P_2 = [b_0 : b_1] \in L_2$, and the ideal $I(P)$ defines the line L_P in $\mathbb{P}^3$ through the points P_1 and P_2. Given distinct points $P, Q \in \mathbb{P}^1 \times \mathbb{P}^1$, the lines L_P and L_Q meet if and only if either $P_1 = Q_1$ or $P_2 = Q_2$, *i.e.,* if and only if P and Q are both on the same horizontal rule or both on the same vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$.

Given any single line $L \subset \mathbb{P}^3$, lines $L_1 \cong \mathbb{P}^1$ and $L_2 \cong \mathbb{P}^1$ in $\mathbb{P}^3$ can be found such that $I(L)$ is the ideal of a single point in $\mathbb{P}^1 \times \mathbb{P}^1$. Likewise, for any two lines $L, L' \subset \mathbb{P}^3$, lines $L_1 \cong \mathbb{P}^1$ and $L_2 \cong \mathbb{P}^1$ in $\mathbb{P}^3$ can be found such that $I(L \cup L')$ is the ideal of two points in $\mathbb{P}^1 \times \mathbb{P}^1$, and if the lines are general so are the points. Consider three general lines L, L', L''. There is a unique smooth quadric Q (isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) containing them. The lines L, L', L'' lie in a single ruling of Q, and we can take L_1 and L_2 to be any two lines in the other ruling. With respect to L_1 and L_2, $I(L \cup L' \cup L'')$ defines 3 general points of $\mathbb{P}^1 \times \mathbb{P}^1$. (Note that the $\mathbb{P}^1 \times \mathbb{P}^1$ defined by L_1 and L_2 is not canonically the quadric Q itself, although Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ abstractly.) Finally, consider four general lines L, L', L'', L'''. Then L, L', and L'' determine Q and lie in a giving ruling on Q, and L''' meets Q in two points. We take L_1 and L_2 to be the lines of the other ruling through these two points. Now with respect to L_1 and L_2, $I(L \cup L' \cup L'' \cup L''')$ defines four general points in $\mathbb{P}^1 \times \mathbb{P}^1$.

One situation for which $I^{(m)} = I^m$ for all m occurs is the case where I is a complete intersection, meaning that I has a set of t generators, where t is the codimension. For example, suppose I is the ideal of a finite set Z of points of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = (\mathbb{P}^1)^t = Y$. Then $\operatorname{codim}_Y(Z) = t$, so I is a complete intersection if it is generated by t elements of I. As noted in [12, Remark 1.3] for $t = 2$ (but which extends naturally to all $t \geq 2$), an ideal I of a finite set of points $Z \subset Y$ is a complete intersection if and only if Z is a rectangular array of points (*i.e.,* $Z = X_1 \times \cdots \times X_t$ for finite sets $X_i \subset \mathbb{P}^1$).

Proposition 2.1.2 *Let $X_1, \dots, X_t \subseteq \mathbb{P}^1$ be finite sets of points, and let I be the ideal of $Z = X_1 \times X_2 \times \cdots \times X_t \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Then $I^m = I^{(m)}$ for all $m \geq 1$.*

Proof Under these hypotheses, $I = I(X_1)R + \cdots + I(X_t)R$ with $R = k[\mathbb{P}^1 \times \cdots \times \mathbb{P}^1]$ and $I(X_i)$ is the defining ideal of X_i in $k[\mathbb{P}^1]$. The ideal I is then a complete intersection. For any complete intersection I, we have $I^m = I^{(m)}$ for all $m \geq 1$ (see [34, Lemma 5, Appendix 6]). ∎

2.2 Hilbert Functions and Points in Multiplicity 1 Generic Position

Let $Z \subseteq \mathbb{P}^N$ be the subscheme defined by a homogeneous ideal I in $k[\mathbb{P}^N]$. We recall that the *Hilbert function* H_Z of Z is defined to be $H_Z(t) = \dim k[\mathbb{P}^N]_t - \dim I_t$, where

https://doi.org/10.4153/CJM-2012-045-3 Published online by Cambridge University Press

for a graded module M, M_t denotes the homogeneous piece of degree t. Similarly, recall that the Hilbert function H_Z of a subscheme $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is defined to be $H_Z(i, j) = \dim k[\mathbb{P}^1 \times \mathbb{P}^1]_{(i,j)} - \dim I(Z)_{(i,j)}$.

Consider a finite set of points $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ (regarded as a reduced subscheme). We will say Z has *generic* Hilbert function if

$$H_Z(i, j) = \min\left\{ \dim R_{(i,j)}, |Z| \right\} = \min\left\{ (i+1)(j+1), |Z| \right\}.$$

It is well known that points with generic Hilbert function are general; *i.e*, for each $s \geq 1$, there is a non-empty open subset of $U_s \subset (\mathbb{P}^1 \times \mathbb{P}^1)^s$ consisting of distinct ordered sets of s points of $\mathbb{P}^1 \times \mathbb{P}^1$ with generic Hilbert function (see, for example, [30]). In particular, subschemes $Z = P_1 + \cdots + P_s$ consisting of s distinct points for which every subset of the points has generic Hilbert function are general.

We will say that a set of s distinct points $P_1, \ldots, P_s$ are *multiplicity* 1 *generic* or are in *multiplicity* 1 *generic position* if for every subscheme $Z = m_1P_1 + \cdots + m_sP_s$ with $0 \leq m_i \leq 1$, Z has generic Hilbert function. Thus being multiplicity 1 generic holds for general points. Note that points $P_1, \ldots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ being generic is not the same as being multiplicity 1 generic. To explain, let $\mathbb{K} \subseteq k$ be a subfield. Then there is a natural inclusion $\mathbb{P}^1_{\mathbb{K}} \subseteq \mathbb{P}^1_k$, and we say that $P_1, \ldots, P_s \in \mathbb{P}^1_k \times \mathbb{P}^1_k = (\mathbb{P}^1 \times \mathbb{P}^1)_k$ are *generic* if $P_i \in (\mathbb{P}^1 \times \mathbb{P}^1)_{k_i} \setminus (\mathbb{P}^1 \times \mathbb{P}^1)_{k_{i-1}}$ for each i, where $k_0 \subsetneq k_1 \subsetneq \cdots \subsetneq k_s = k$ is a tower of algebraically closed fields such that k_0 is the algebraic closure $\overline{k'}$ of the prime field k' of k. Thus for example, if $C \subset \mathbb{P}^2$ is an irreducible reduced cubic with a double point, and if we pick points $p_1, \ldots, p_8 \in C$ such that no three are collinear and no six lie on a conic but such that p_1 is the double point, then the points are multiplicity 1 generic but not generic. On the other hand, s generic points are multiplicity 1 generic.

Example 2.2.1 Any single point of $\mathbb{P}^1 \times \mathbb{P}^1$ is in multiplicity 1 generic position. Two points of $\mathbb{P}^1 \times \mathbb{P}^1$ are in multiplicity 1 generic position if and only if they are not both on the same horizontal or vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$. As a consequence, if $s \geq 3$ points are in multiplicity 1 generic position, then no two of them lie on the same horizontal or vertical rule. For $s = 3$, the converse is also true (since any such three points are equivalent under an isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$), but for $s \geq 4$ points the condition that no two lie on the same horizontal or vertical rule is not sufficient to ensure that the points are in multiplicity 1 generic position. (This is because given three points in multiplicity 1 generic position, there is, up to multiplication by scalars, a unique form of degree $(1, 1)$ that vanishes on the three points. In order for four points to be in multiplicity 1 generic position, the fourth point cannot be in the zero-locus of the $(1, 1)$-form associated with the other three points.)

2.3 Divisors on Blow Ups and a Connection to $\mathbb{P}^2$

Given a finite set of distinct points $P_1, \ldots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$, let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by blowing up the points P_i. Let $\mathrm{Cl}(X)$ be the divisor class group of X. Let H and V be the pullback to X of general members of the rulings on $\mathbb{P}^1 \times \mathbb{P}^1$ (horizontal and vertical, respectively), and for each point P_i let

E_i be the exceptional divisor of the blow up of P_i. Every divisor is linearly equivalent to a unique divisor of the form $aH + bV - m_1E_1 - \cdots - m_sE_s$. Because of this, we can regard $\mathrm{Cl}(X)$ as the free abelian group on the set $\{H, V, E_1, \ldots, E_s\}$. This basis is called an *exceptional configuration*. In particular, when we have a divisor of the form $aH + bV - m_1E_1 - \cdots - m_sE_s$, we will leave it to context whether we really mean a divisor or its linear equivalence class in $\mathrm{Cl}(X)$. We also recall that the intersection form on $\mathrm{Cl}(X)$ is determined by $H \cdot E_i = V \cdot E_i = H^2 = V^2 = E_i \cdot E_j = 0$ for all $i \neq j$, and $-H \cdot V = E_i^2 = -1$ for $i > 0$.

Given a divisor F on X, it will be convenient to write $h^i(X, F)$ in place of $h^i(X, \mathcal{O}_X(F))$, and we will refer to a divisor class as being *effective* if it is the class of an effective divisor. We also sometimes merely say that a divisor is effective when we mean only that it is linearly equivalent to an effective divisor. (When we mean that a divisor is actually effective and not just linearly equivalent to an effective divisor, we will say the divisor is *strictly effective*.) We denote the subsemigroup of classes of effective divisors by $\mathrm{EFF}(X) \subseteq \mathrm{Cl}(X)$. We recall that a divisor or divisor class D is *nef* if $D \cdot C \geq 0$ for every effective divisor C, and we denote the subsemigroup of classes of nef divisors by $\mathrm{NEF}(X) \subseteq \mathrm{Cl}(X)$.

Problems involving fat points $Z = \sum_i m_iP_i$ with support at distinct points $P_i \in \mathbb{P}^1 \times \mathbb{P}^1$ can be translated into problems involving divisors on X. Given $I = I(Z)$ and (i, j), then, as a vector space, $I(Z)_{(i,j)}$ can be identified with $H^0(X, iH + jV - \sum_i m_iE_i)$, which itself can be regarded as a vector subspace of the space of sections $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i, j))$. Thus given (i, j), it is convenient to define the divisor $F(Z, (i, j)) = iH + jV - \sum_i m_iE_i$, in which case we have, under the identifications above,

$$I(Z) = \bigoplus_{i,j} I(Z)_{(i,j)} = \bigoplus_{i,j} H^0\big(X, F\big(Z, (i, j)\big)\big).$$

Remark 2.3.1 It can be useful to reinterpret problems involving points of $\mathbb{P}^1 \times \mathbb{P}^1$ as problems involving points of $\mathbb{P}^2$. Let Y be a finite set of points $p_1, \ldots, p_s$ of $\mathbb{P}^2$. Let Z be the image of Y under the birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ given by blowing up two points $p_{s+1}, p_{s+2} \in \mathbb{P}^2$ such that none of the points p_i, $i < s+1$ is on the line A through p_{s+1} and p_{s+2} and blowing down the proper transform E of A. The divisors $L, E_1, \ldots, E_{s+2}$, where L is a line and E_i is the exceptional curve obtained by blowing up the point p_i, give a basis of the divisor class group $\mathrm{Cl}(X)$ for the surface X obtained by blowing up the points p_i, also called an exceptional configuration. The birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ described above induces a birational morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1$ given by contracting $E_1, \ldots, E_s, L - E_{s+1} - E_{s+2}$. We also have an exceptional configuration on X coming from blowing up points $P_0, P_1, \ldots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ to obtain X; this basis is given by $H = L - E_{s+1}, V = L - E_{s+2}, E_1, \ldots, E_s, E = L - E_{s+1} - E_{s+2}$, where H and V give the rulings on $\mathbb{P}^1 \times \mathbb{P}^1$. We can identify P_i with p_i for $i = 1, \ldots, s$; P_0 is the point obtained by contracting the proper transform of the line through p_{s+1} and p_{s+2}. Thus

$$H^0\big(X, aH+bV-m(E_1+\cdots+E_s)\big) = H^0\big(X, (a+b)L-m(E_1+\cdots+E_s)-aE_{s+1}-bE_{s+2}\big).$$

If I is the ideal of the fat points $mP_1 + \cdots + mP_s$, we note that $\alpha(I^{(m)})$ is then the least t such that $t = a + b$ and $h^0(X, (a + b)L - m(E_1 + \cdots + E_s) - aE_{s+1} - bE_{s+2}) > 0$.

Alternatively, suppose $P_1, \dots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ are such that no two of the points P_i lie on the same horizontal or vertical rule. Let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by blowing up the points P_i. Then there is also a birational morphism $X \to \mathbb{P}^2$. If $H, V, E_1, \dots, E_s$ is the exceptional configuration for $X \to \mathbb{P}^1 \times \mathbb{P}^1$, the exceptional configuration for $X \to \mathbb{P}^2$ can be taken to be $L = H + V - E_s$, $E'_1 = E_1, \dots, E'_{s-1} = E_{s-1}$, $E'_s = H - E_s$, and $E'_{s+1} = V - E_s$.

Lemma 2.3.2 *Let $P_1, \dots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ be distinct points and let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by blowing these points up. Then a divisor $C \subset X$ is a prime divisor with $C^2 < 0$ if and only if $C^2 = C \cdot K_X = -1$ if either $s \leq 8$ and the points are generic or $s \leq 7$ and the points are general. For $s \leq 7$ general points, then in terms of the exceptional configuration for $X \to \mathbb{P}^1 \times \mathbb{P}^1$ the classes of these curves C are (up to permutations of the E_i and swapping H and V) precisely the following:*

$$
\begin{aligned}
&E_1,\\
&H - E_1,\\
&H + V - E_1 - E_2 - E_3,\\
&2H + V - E_1 - \cdots - E_5,\\
&2H + 2V - 2E_1 - E_2 - \cdots - E_6,\\
&3H + V - E_1 - \cdots - E_7,\\
&3H + 2V - 2E_1 - 2E_2 - E_3 - \cdots - E_7,\\
&3H + 3V - 2E_1 - \cdots - 2E_4 - E_5 - E_6 - E_7,\\
&4H + 3V - 2E_1 - \cdots - 2E_6 - E_7,\\
&4H + 4V - 3E_1 - 2E_2 - \cdots - 2E_7.
\end{aligned}
$$

Proof Since $s \leq 8$ and the points are either general or generic, we can regard $X \to \mathbb{P}^2$ as being the blow up of $s+1 \leq 9$ points $p_1, \dots, p_{s+1}$ in $\mathbb{P}^2$, and that there is a smooth cubic curve $D \subset \mathbb{P}^2$ passing through these points. Thus up to linear equivalence we have $D = -K_X = 3L - E'_1 - \cdots - E'_s$ with respect to the exceptional configuration $L, E'_1, \dots, E'_{s+1}$ of the morphism $X \to \mathbb{P}^2$. Since D is irreducible with $D^2 \geq 0$, D is nef, so for any prime divisor C we have $D \cdot C \geq 0$. By the adjunction formula $C^2 - C \cdot D = 2p_C - 2$, we see that $C^2 \geq -2$, with $C \cdot D = 1$ if $C^2 = -1$ and $C \cdot D = 0$ if $C^2 = -2$.

There are only finitely many possible classes of reduced irreducible curves C with $C \cdot D = 0$ when $s \leq 7$ (see [10, Proposition 4.1]). For each of these classes, C is not effective if the points p_i are general, so in fact no such C is effective if $s \leq 7$ and the points p_i are general. (For example, $(L - E'_1 - E'_2 - E'_3) \cdot D = 0$; if $L - E'_1 - E'_2 - E'_3$ is the class of a strictly effective divisor C, then the points p_1, p_2, p_3 are collinear and hence not general.) For $s = 8$ there are infinitely many possible such classes, so it is not enough to assume that the points are general, but if the points are generic, then there are no prime divisors $C \neq D$ with $C \cdot D = 0$ (since $C \cdot D = 0$ implies the coordinates of the points satisfy an algebraic relation coming from the group law on D). Thus the only prime divisors C with $C^2 < 0$ are those that satisfy $C^2 = C \cdot K_X = -1$. Conversely, if C is a divisor with $C^2 = C \cdot K_X = -1$, then by Serre duality

$h^2(X,C) = h^0(X,-D-C)$ but $h^0(X,-D-C) = 0$, since $D\cdot(-D-C) < 0$. Now by Riemann–Roch for surfaces we have $h^0(X,C) - h^1(X,C) = 1 + (C^2 + D\cdot C)/2 = 1$, so C is effective. Up to linear equivalence, if F is a prime divisor with $F \cdot D = 0$, then $F = D$ (otherwise, as above, we would get an algebraic condition on the points p_i), and so $D^2 = 0$ (hence $s = 8$). Now if C is not a prime divisor, then from $D \cdot C = 1$ it follows that $C = G + rD$ with $r > 0$ and $D^2 = 0$, where G is the unique component of C with $D \cdot G = 1$. But then $G^2 = (C - rD)^2 = -1 - 2r < -1$, contrary to what is proved above.

Finally, suppose $s \le 7$. Let C be a prime divisor on X with $C^2 = C \cdot K_X = -1$. Let Y be the surface obtained by blowing up an arbitrary point $P_{s+1} \in \mathbb{P}^1 \times \mathbb{P}^1$. Then, denoting the pullback of C to Y also by C, we have $(C - E_{s+1}) \cdot K_Y = 0$ and $(C - E_{s+1})^2 = -2$. It is not hard to check that the subgroup $K_Y^{\perp}$ of classes orthogonal to K_Y is, for $s < 7$, negative definite, and, if $s = 7$, negative semi-definite, with the only classes F having $F \cdot K_Y = F^2 = 0$ being the multiples of K_Y. Thus for $s < 7$ it follows by negative definiteness that there are only finitely many classes C with $(C - E_{s+1}) \cdot K_Y = 0$ and $(C - E_{s+1})^2 = -2$, and it is not hard to find them all. For $s = 7$, the quotient $K_Y^{\perp}/\langle K_Y\rangle$ is negative definite, so modulo K_Y there are only finitely many classes C with $(C - E_{s+1}) \cdot K_Y = 0$ and $(C - E_{s+1})^2 = -2$. But C must satisfy $C \cdot K_Y = -1$ and $C^2 = -1$, so there is at most one such representative in each coset of $K_Y^{\perp}/\langle K_Y\rangle$. Again it is not hard to find all C. ■

Note that a prime divisor C with $C^2 = C \cdot K_X = -1$ is called an *exceptional curve*. Exceptional curves are smooth rational curves.

Lemma 2.3.3 *Let $P_1, \dots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ be distinct points, let $I \subset k[\mathbb{P}^1 \times \mathbb{P}^1]$ be the ideal generated by all bi-homogeneous forms that vanish at all of the points P_i. Let X be the blow up of these s points of $\mathbb{P}^1 \times \mathbb{P}^1$, with exceptional configuration $H, V, E_1, \dots, E_s$. If for some λ and m we have an effective divisor $C = \lambda(H+V) - m(E_1 + \cdots + E_s)$, then $\gamma(I) \le \frac{2\lambda}{m}$. If moreover for some t and r we have a nef divisor $D = t(H+V) - r(E_1 + \cdots + E_s)$ with $C \cdot D = 0$, then*

$$\gamma(I) = \frac{2\lambda}{m} = \frac{sr}{t}.$$

Proof If C is effective, so is lC and thus $\alpha(I^{(lm)}) \le 2\lambda l$ for all $l \ge 1$, and therefore

$$\frac{\alpha(I^{(lm)})}{lm} \le \frac{2\lambda l}{lm} = \frac{2\lambda}{m}.$$

Now assume that D is nef. From $C \cdot D = 0$ we get

$$\frac{2\lambda}{m} = \frac{sr}{t}.$$

Now, given $\alpha(I^{(j)})$, we can find $a \ge 0$ and $b \ge 0$ with $\alpha(I^{(j)}) = a + b$ such that $(I^{(j)})_{(a,b)} \ne 0$. Moreover, $C' = aH + bV - j(E_1 + \cdots + E_s)$ is effective, so $C' \cdot D = t(a+b) - jrs \ge 0$; hence $\frac{\alpha(I^{(j)})}{j} \ge \frac{rs}{t}$, and therefore

$$\frac{rs}{t} \le \frac{\alpha(I^{(lm)})}{lm} \le \frac{2\lambda l}{lm} = \frac{rs}{t}.$$

Taking the limit as $l \to \infty$ gives the conclusion. ■

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1 Let X be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the s points with exceptional configuration $H, V, E_1, \dots, E_s$.

The case $s = 1$ follows from Proposition 2.1.2, since in this case $\alpha(I) = 1$, so consider $s = 2$. Then $C = D = H + V - E_1 - E_2$ is effective (since $C = (H - E_1) + (V - E_2)$ is a sum of effective divisors) and nef (since $D = (H - E_1) + (V - E_2)$ is a sum of prime divisors, each of which D meets non-negatively). Since $C \cdot D = 0$, we have $\gamma(I) = 2$ by Lemma 2.3.3.

Consider $s = 3$. Then $C = H + V - E_1 - E_2 - E_3$ is effective (being exceptional, by Lemma 2.3.2), and $D = 3H + 3V - 2(E_1 + E_2 + E_3) = H + V + 2C$ is nef with $C \cdot D = 0$ so $\gamma(I) = 2$.

Consider $s = 4$. Then $C = 4(H + V) - 3(E_1 + E_2 + E_3 + E_4) = C_1 + C_2 + C_3 + C_4$ is effective (being the sum of the four exceptional curves C_i, where $C_i = (H + V - E_1 - E_2 - E_3 - E_4) + E_i$) and $D = 3H + 3V - 2(E_1 + E_2 + E_3 + E_4) = 2C_4 + (H - E_4) + (V - E_4)$ is nef with $C \cdot D = 0$ so $\gamma(I) = 8/3$.

Consider $s = 5$. Then

$$C = 3(H+V) - 2(E_1 + \cdots + E_5) = (2H + V - E_1 - \cdots - E_5) + (H + 2V - E_1 - \cdots - E_5)$$

is effective (being the sum of two exceptional curves), and $D = 10(H + V) - 6(E_1 + \cdots + E_5) = D_1 + \cdots + D_5$ is nef, where $D_i = 2H + 2V - (E_1 + \cdots + E_5) - E_i$ (and where we see D is nef since each D_i is a sum of two exceptionals, each of which D meets non-negatively; for example, $D_1 = (H + V - E_1 - E_2 - E_3) + (H + V - E_1 - E_4 - E_5)$). Since $C \cdot D = 0$, we have $\gamma(I) = 3$.

Consider $s = 6$. Then $C = 12(H+V) - 7(E_1 + \cdots + E_6) = C_1 + \cdots + C_6$ is effective (since each $C_i = 2(H+V) - (E_1 + \cdots + E_6) - E_i$ is exceptional), and $D = 7(H+V) - 4(E_1 + \cdots + E_6) = (4H + 3V - 2(E_1 + \cdots + E_6)) + (3H + 4V - 2(E_1 + \cdots + E_6))$ is nef (since $4H + 3V - 2(E_1 + \cdots + E_6) = (2H + V - (E_1 + \cdots + E_5)) + (2(H+V) - (E_1 + \cdots + E_5) - 2E_6)$ is a sum of two exceptional curves, and likewise for $(3H + 4V - 2(E_1 + \cdots + E_6))$, each of which D meets non-negatively). Since $C \cdot D = 0$, we have $\gamma(I) = 24/7$.

Consider $s = 7$. Then $C = 28(H + V) - 15(E_1 + \cdots + E_7) = C_1 + \cdots + C_7$ is effective (since each $C_i = 4(H + V) - 2(E_1 + \cdots + E_7) - E_i$ is exceptional), and $D = 15(H + V) - 8(E_1 + \cdots + E_7)$ is nef (since $4D = 2C + (3H + V - (E_1 + \cdots + E_7)) + (H + 3V - (E_1 + \cdots + E_7))$ is a sum of exceptionals, each of which D meets non-negatively). Since $C \cdot D = 0$, we have $\gamma(I) = 56/15$.

Consider $s = 8$. In this case $C = D = 2(H + V) - (E_1 + \cdots + E_8) = -K_X$ is effective, since 8 points impose at most 8 conditions on the 9-dimensional space of forms of degree $(2, 2)$. Since the blow up X of $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 general points is a blow up of $\mathbb{P}^2$ at 9 general points, and since there is an irreducible cubic through 9 general points of $\mathbb{P}^2$, we see that $-K_X$ is nef. Since $C \cdot D = 0$, we have $\gamma(I) = 4$.

Now assume $s \geq 9$. Since $2(H + V) - (E_1 + \cdots + E_8)$ is nef from the preceding case, it follows whenever $aH + bV - m(E_1 + \cdots + E_s)$ is effective that $2(a + b) \geq 8m$ and hence that $\alpha(I^{(m)}) \geq 4m$, so $\gamma(I) \geq 4$. Next we obtain bounds depending on s. Let $C = d(H + V) - m(E_1 + \cdots + E_s)$. If $C^2 > 0$, then tC is effective for $t \gg 0$, so by

Lemma 2.3.3 we have $\gamma(I) \leq 2d/m$. It follows that $\gamma(I) \leq \sqrt{2s}$. It is easy to compute $\alpha(I)$ for any given s. In fact, since the points are general, they impose independent conditions on forms of every bi-degree (i, j); *i.e.*, there are forms of bi-degree (i, j) vanishing at the s points if and only if $(i+1)(j+1) > s$. But for a given degree $t = i + j$, the maximum value of $(i+1)(j+1)$ occurs when $i = j$, and so there are no forms in I of total degree t if $(t/2+1)^2 \leq s$. But $(t/2+1)^2 \leq s$ is equivalent to $t \leq 2(\sqrt{s}-1)$. Thus $\alpha(I) > 2(\sqrt{s}-1)$, hence we get $\sqrt{s}-1 < \alpha(I)/2 \leq \gamma(I)$ from the bound given in [16, Section 2]. ■

3 Additional Results for General Points of $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we consider the problem of whether $I^m = I^{(m)}$ for all m when I is the ideal of s general points of $\mathbb{P}^1 \times \mathbb{P}^1$. For $s = 1, 2, 3, 5$, we verify $I^m = I^{(m)}$ for all m. For $s \geq 6$, we prove that $I^2 \neq I^{(2)}$. For $s = 4$, computer calculations suggest that $I^2 = I^{(2)}$; we show that $I^3 \neq I^{(3)}$.

3.1 Equality of $I^{(m)}$ and I^m

We first consider the case of a set of two points $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ in multiplicity 1 generic position. For this case, the problem reduces to a question of monomial ideals.

Theorem 3.1.1 *Let $I = I(Z)$, where $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ consists of two points in multiplicity 1 generic position. Then $I^{(m)} = I^m$ for all $m \geq 1$.*

Proof Let $Z = P_1 + P_2$. We can assume, after a change of coordinates, that $I(P_1) = (x_0, y_0)$ and $I(P_2) = (x_1, y_1)$. We then apply [16, Lemma 4.1] for the conclusion. ■

We now consider three points in multiplicity 1 generic position.

Theorem 3.1.2 *Let $I = I(Z)$, where $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ consists of three points in multiplicity 1 generic position. Then $I^{(m)} = I^m$ for all $m \geq 1$.*

Proof For specificity, say that the three points are $P_i = P_{i1} \times P_{i2}$, $i = 1, 2, 3$, for points $P_{ij} \in \mathbb{P}^1$ and that $k[\mathbb{P}^1 \times \mathbb{P}^1] = k[a, b, c, d] = k[a, b] \otimes_k k[c, d] = k[\mathbb{P}^1] \otimes k[\mathbb{P}^1]$. Up to change of coordinates, we may as well assume that $P_{11} = P_{12} = [0:1]$, $P_{21} = P_{22} = [1:1]$, and $P_{31} = P_{32} = [1:0]$.

Since the points are multiplicity 1 generic, we know $\dim I_{(1,1)} = 1$, so there is (up to scalar multiples) a unique form F of degree $(1, 1)$ in I. We will show that $I^{(m)} \subseteq I^{(m-1)}I + FI^{(m-1)}$ for each $m \geq 2$. Formally, we can write the right-hand side as $I^{(m-1)}(I+F)$. Iterating $m-1$ times gives $I^{(m)} \subseteq I(I+F)^{m-1} = I^m + FI^{m-1} + \cdots + F^{m-1}I$. Since $F \in I$, we see that $F^iI^{m-i} \subseteq I^m$, hence $I^{(m)} \subseteq I^m$. But $I^m \subseteq I^{(m)}$, so we have $I^{(m)} = I^m$.

We now show $I^{(m)} \subseteq I^{(m-1)}I + FI^{(m-1)}$. This is clear if $m = 1$, so assume that $m \geq 2$. We will consider $(I^{(m)})_{(i,j)}$ for various cases. If $(I^{(m)})_{(i,j)} = 0$, then clearly $I^{(m)} \subseteq I^{(m-1)}I + FI^{(m-1)}$, so we may assume that $(I^{(m)})_{(i,j)} \neq 0$.

If $i + j < 3m$, then apply Bézout's theorem: for any element $G \in (I^{(m)})_{(i,j)}$ the sum of the intersection multiplicities of F with G over all points $P \in \mathbb{P}^1 \times \mathbb{P}^1$ is at least $3m$, since G vanishes at each point P_i with order at least m while F vanishes with order 1,

https://doi.org/10.4153/CJM-2012-045-3 Published online by Cambridge University Press

so summing over the three points gives at least $3m$. But G has degree (i, j) and F has degree $(1, 1)$, so at most $i + j$ common zeros are possible unless F divides G. Since $i + j < 3m$, we see that F divides G, say $G = FH$. Then H has degree $(i - 1, j - 1)$ and vanishes at least $m - 1$ times at each of the three points (since G vanishes at least m times and F vanishes once at each point). Thus $H \in (I^{(m-1)})_{(i-1,j-1)}$, so $(I^{(m)})_{(i,j)} \subseteq F(I^{(m-1)})_{(i-1,j-1)} \subset I^{(m-1)}I + FI^{(m-1)}$.

Hereafter assume that $i + j \geq 3m$. If $j = 0$, then $(I^{(m)})_{(i,j)}$ is the space of polynomials in a and b of degree $(i, 0)$ divisible by $a^m b^m (a - b)^m$. Thus $(I^{(m)})_{(i,j)} = (I_{(3,0)})^m I_{(i-3m,0)}$, hence $(I^{(m)})_{(i,j)} \subseteq I^m \subseteq I^{(m-1)}I$. Similarly, if $i = 0$, swapping c and d for a and b, we again have $(I^{(m)})_{(i,j)} \subseteq I^m \subseteq I^{(m-1)}I$.

Now assume $i > 0$ and $j > 0$, in addition to $i+j \geq 3m$. The cases $i \geq j$ and $j \geq i$ are symmetric, so assume $i \geq j$. We work on the surface X obtained by blowing up the points P_i. We have the birational morphism $\pi \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ with exceptional configuration H, V, E_1, E_2, E_3, with respect to which we can identify $(I^{(m)})_{(i,j)}$ with $H^0(X, iH + jV - (m - 1)E)$, where $E = E_1 + E_2 + E_3$.

If $1 \leq j < m$, then we can write $iH + jV - mE = (i - 3m + j)H + j(2H + V - E) + (m - j)(3H - E)$. Note that $3H - E = (H - E_1) + (H - E_2) + (H - E_3)$ is a sum of three disjoint exceptional curves, disjoint also from $(i - 3m + j)H$ and $j(2H + V - E)$. Thus $(i - 3m + j)H + j(2H + V - E)$ is the nef part (with $|(i - 3m + j)H + j(2H + V - E)|$ non-empty and fixed component free) and $(m - j)(3H - E)$ is the negative (and fixed) part of a Zariski decomposition of $iH + jV - mE$. The unique element of $|3H - E|$ corresponds to an element $Q \in I_{(3,0)}$, and since $m - j > 0$ and $|3H - E|$ is the fixed part of $|iH + jV - mE|$, Q is a factor of every element of $(I^{(m)})_{(i,j)}$. Since Q vanishes with order 1 at each point P_1, P_2, P_3, we have $(I^{(m)})_{(i,j)} = Q(I^{(m-1)})_{(i-3,j)} \subset I^{(m-1)}I$, as we wanted to show.

So now we may assume that $i \geq j \geq m > 1$ and $i + j \geq 3m$. We will show that under multiplication we have a surjection $\mu \colon (I^{(m-1)})_{(i-2,j-1)} \otimes_k (I)_{(2,1)} \to (I^{(m)})_{(i,j)}$ and hence $(I^{(m)})_{(i,j)} \subset I^{(m-1)}I$. But surjectivity of μ is equivalent to surjectivity of the corresponding map

$$\lambda \colon H^0\big(X, (i-2)H + (j-1)V - (m-1)E\big) \otimes H^0(X, 2H + V - E) \to H^0(X, iH + jV - mE).$$

Under our assumptions, we have $(i - m) + (j - m) \geq m$ and $i - m \geq j - m \geq 0$, so we can pick integers $0 \leq s \leq r \leq i - m$ and $s \leq j - m$ such that $r + s = m$. Thus

$$iH + jV - mE = r(2H + V - E) + s(H + 2V - E) + (i - m - r)H + (j - m - s)V,$$

and moreover $r \geq 1$ (since $r \geq m/2 > 0$). Note also that $|2H + V - E|$ is non-empty and fixed component free (since we can write $2H + V - E$ as a sum of three exceptional curves $(H - E_u) + (H - E_v) + (V - E_w)$ in three different ways using various permutations of $\{u, v, w\} = \{1, 2, 3\}$, showing that none of the curves occurring as summands is a fixed component), and likewise for $H + 2V - E$. Since $|2H + V - E|$, $|H + 2V - E|$, $|H|$ and $|V|$ are non-empty and fixed component free, $2H + V - E$, $H + 2V - E$, H and V are nef. Since $r \geq 1$ and $m \geq 2$,

$$\big|(i - 2)H + (j - 1)V - (m - 1)E\big| = \big|(r - 1)(2H + V - E) + s(H + 2V - E) + (i - m - r)H + (j - m - s)V\big|$$

https://doi.org/10.4153/CJM-2012-045-3 Published online by Cambridge University Press

is also non-empty and fixed component free, so $(i-2)H+(j-1)V-(m-1)E$ is nef.

As discussed in Remark 2.3.1, we have a birational morphism $p\colon X \to \mathbb{P}^2$ with exceptional configuration $L' = H + V - E_3$, $E_1' = E_1$, $E_2' = E_2$, $E_3' = H - E_3$, and $E_4' = V - E_3$, so $H = L' - E_4'$, $V = L' - E_3'$, $E_1 = E_1'$, $E_2 = E_2'$, and $E_3 = L' - E_3' - E_4'$. Let $p_1, \dots, p_4 \in \mathbb{P}^2$ be the points such that $E_l' = p^{-1}(p_l)$. Because the points P_1, P_2, P_3 are multiplicity 1 generic, no three of the points p_l are collinear. Thus the proper transform E_{uv}' of the line through the points p_u and p_v for $u \neq v$ is an exceptional curve. By contracting E_{14}', E_{24}', E_{12}' and E_3' we get another birational morphism $X \to \mathbb{P}^2$, obtained by blowing up four distinct general points p_u'', this one having exceptional configuration $L'' = 2L' - E_1' - E_2' - E_4'$, $E_1'' = E_{14}'$, $E_2'' = E_{24}'$, $E_3'' = E_{34}'$, and $E_4'' = E_3'$. Note that $2H + V - E = 2L' - E_1' - E_2' - E_4' = L''$.

Thus λ can be written as $\lambda\colon H^0(X, G) \otimes H^0(X, L'') \to H^0(X, L'' + G)$, where $G = (i-2)H + (j-1)V - (m-1)E$ is nef. Since X is the blow up of four points p_u'' and therefore $|2L'' - E_1'' - E_2'' - E_3'' - E_4''| \neq \varnothing$, it follows by [2, Proposition 2.4] that λ is surjective, as claimed. ■

Remark 3.1.3 Li and Swanson [24, Theorem 3.6] have given a criterion under which a radical ideal I in a reduced Noetherian domain has the property that $I^{(m)} = I^m$ for all $m \geq 1$. It is possible that the criterion applies for ideals of any sets of two, three or five multiplicity 1 generic points of $\mathbb{P}^1 \times \mathbb{P}^1$ in any characteristic, but it seems difficult to verify. However, for a specific choice of ground field and a specific choice of points one can use *Macaulay2* to check the criterion. I. Swanson, for example, shared with us such a *Macaulay2* script, which shows over $\mathbb{Q}$ that the ideal I of a reduced set of three points in multiplicity 1 generic position in $\mathbb{P}^1 \times \mathbb{P}^1$ does satisfy the conditions of [24, Theorem 3.6], whence $I^{(m)} = I^m$ for all $m \geq 1$.

Let I be the ideal of five multiplicity 1 generic points $P_1, \dots, P_5 \in \mathbb{P}^1 \times \mathbb{P}^1$. We will show that $I^{(m)} = I^m$ for all $m \geq 1$. The basic argument is the same as we used for the proof of Theorem 3.1.2, but it is now more complicated.

Theorem 3.1.4 *Let $I = I(Z)$ with $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be five multiplicity* 1 *generic points. Then $I^{(m)} = I^m$ for all $m \geq 1$.*

Proof We will show that $(I^{(m)})_{(i,j)} \subset I^{(m-1)}I$ for all i and j, and hence that $I^{(m)} \subseteq I^m$. Since we know that $I^m \subseteq I^{(m)}$, this shows equality. By symmetry, we may assume $i \geq j$. We also know that $I_{(5,0)}$ is 1-dimensional, whose single basis element is the form $G = H_1 \cdots H_5$, where H_s is a form of bi-degree $(1,0)$ defining the horizontal rule through the point P_s. Any form $F \in (I^{(m)})_{(i,j)}$ restricts for each s to a form of degree j on H_s, but with order of vanishing at least m. If $j < m$, then F must vanish on the entire horizontal rule through each P_s, and hence each H_s divides F, so G divides F. That is, if $j < m$, then $(I^{(m)})_{(i,j)} = G(I^{(m-1)})_{(i-5,j)} \subset I^{(m-1)}I$.

We also know that $I_{(2,1)}$ is 1-dimensional, with basis a form D defining a smooth rational curve C vanishing with order 1 at each point P_s. Likewise, if $i + 2j < 5m$, then any form $F \in (I^{(m)})_{(i,j)}$ vanishes on C, and hence D divides F, so $(I^{(m)})_{(i,j)} = D(I^{(m-1)})_{(i-2,j-1)} \subset I^{(m-1)}I$.

We may now assume that $i \geq j \geq m \geq 2$ and $i + 2j \geq 5m$. This implies that $2i + j \geq i + 2j \geq 5m$, and it also implies that $i + j > 3m$. (To see the latter, given $m \geq 2$, consider the system of inequalities $i \geq j$, $j \geq m$, $i + j \leq 3m$. The solution set is a triangular region in the (i, j)-plane with vertices $(3m/2, 3m/2)$, (m, m), and $(2m, m)$. Since each vertex has $i + 2j < 5m$, we see that $i \geq j \geq m \geq 2$ and $i + 2j \geq 5m$ imply $i + j > 3m$.)

There is a natural map

$$\mu_{(i,j)}\colon \left(I^{(m-1)}\right)_{(i-3,j-1)} \otimes I_{(3,1)} \to \left(I^{(m)}\right)_{(i,j)}.$$

Since $\operatorname{Im}(\mu_{(i,j)}) = (I^{(m-1)})_{(i-3,j-1)}I_{(3,1)}$, to finish, it is enough to show that $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ whenever $\mu_{(i,j)}$ is not surjective. We can identify $(I^{(m)})_{(i,j)}$ with $H^0(X, A)$, and $I_{(3,1)}$ with $H^0(X, L)$, where $A = iH + jV - mE$, $L = 3H + V - E$, and $E = E_1 + \cdots + E_5$ are divisors on the blow up X of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $P_1, \ldots, P_5$ with respect to the usual exceptional configuration $H, V, E_1, \ldots, E_5$. Surjectivity of $\mu_{(i,j)}$ is equivalent to surjectivity of the map $H^0(X, A - L) \otimes H^0(X, L) \to H^0(X, A)$, which we will also denote by $\mu_{(i,j)}$.

Using Lemma 2.3.2, the inequalities $i \geq j \geq m \geq 2$, $i + 2j \geq 5m$, $2i + j \geq 5m$, and $i + j > 3m$ show that $A \cdot B \geq 0$ for every exceptional curve B on X, and hence A is effective and nef (since for a blow up X of $\mathbb{P}^1 \times \mathbb{P}^1$ at five multiplicity 1 generic points, and thus 6 general points of $\mathbb{P}^2$, using the results of [10] one checks that the only prime divisors of negative self-intersection are the exceptional curves, but any divisor meeting every exceptional curve non-negatively is effective and nef [10, Proposition 4.1]).

Note that the exceptional configuration

$$L, E_1' = H - E_1, E_2' = H - E_2, E_3' = H - E_3, E_4' = H - E_4, E_5' = H - E_5, E_6' = 2H + V - E$$

corresponds to a birational morphism $X \to \mathbb{P}^2$ obtained by blowing up 6 general points of $\mathbb{P}^2$ and that L is the pullback of a line in $\mathbb{P}^2$. By [15], $\mu_{(i,j)}$ always has maximal rank. Determining whether $\mu_{(i,j)}$ is surjective or injective is now purely numerical, and by [9, Theorem 3.4], $\mu_{(i,j)}$ is surjective if $A - L$ is nef, unless either $A - L = 5L - 2E_1' - \cdots - 2E_6' = H + 3V - E$ or $A - L = t(-K_X - E_s')$ for $t > 0$. Note that $-K_X - E_s' = H + 2V - E + E_s$ for $1 \leq s \leq 5$, while $-K_X - E_6' = V$. Since each term E_s of $A - L$ has the same coefficient, $A - L = t(-K_X - E_s')$ is impossible for $s \neq 6$. Thus $\mu_{(i,j)}$ is surjective if $A - L$ is nef, unless either $A - L = H + 3V - E$ or $A - L = tV$ for $t > 0$, *i.e.,* unless either $A = 4H + 4V - 2E$ or $A = 3H + tV - E$ for $t > 1$. But $A = 3H + tV - E$ is not relevant, since we are interested in cases with $m > 1$. For the case $A = 4H + 4V - 2E = -2K_X$, we have surjectivity of $H^0(X, -K_X)^{\otimes 2} \to H^0(X, A)$ by [19, Proposition 3.1(a)]. Thus $(I^{(2)})_{(4,4)} = (I_{(2,2)})^2 \subset I^2$.

So it now suffices to show that $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ whenever $A - L$ is not nef but A is nef and $m \geq 2$. First we must find all such A.

Either by hand or using software such as Normaliz [6], we can find generators for the semigroup of all (i, j, m) such that $i \geq j \geq m \geq 0$ and $i + 2j \geq 5m$. The result is that every such (i, j, m) is a non-negative integer linear combination of $(1, 0, 0)$, $(1, 1, 0)$, $(2, 2, 1)$, $(3, 1, 1)$, $(4, 3, 2)$, and $(5, 5, 3)$. So consider $A = a(1, 0, 0)+$

$b(1,1,0) + c(2,2,1) + d(3,1,1) + e(4,3,2) + f(5,5,3)$, where we use (i,j,m) as shorthand for $iH + jV - mE$.

Note that $A - L$ is nef for any $A = a(1,0,0) + b(1,1,0) + c(2,2,1) + d(3,1,1) + e(4,3,2) + f(5,5,3)$ with $d > 0$, since $(3,1,1) = L$. So we may assume that $d = 0$. However, $f(5H+5V-3E) - L = (t-1)(5H+5V-3E) + 2(H+2V-E)$, where $H+2V-E$ is an exceptional curve by Lemma 2.3.2 with $(5H+5V-3E)\cdot(H+2V-E) = 0$, so $A - L$ is effective but never nef for $A = f(5H+5V-3E)$.

In contrast, $(e(4H+3V-2E) - L)\cdot(H+2V-E) < 0$ for $e = 1$, but for $e > 1$ we have $e(4H+3V-2E) - L = (5H+5V-3E) + (e-2)(4H+3V-2E)$, so, for $e > 0$, $A - L$ is not nef for $A = e(4H+3V-2E)$ if and only if $e = 1$. In particular, if $e > 1$, then $A - L$ is nef for $A = a(1,0,0) + b(1,1,0) + c(2,2,1) + e(4,3,2) + f(5,5,3)$ regardless of the values of a, b, c, and f. However,

$$\big((4H+3V-2E) + f(5H+5V-3E) - L\big)\cdot(H+2V-E) < 0$$

for all $f \geq 0$, $A - L$ is never nef for $A = (4H+3V-2E) + f(5H+5V-3E)$.

Similarly, for $c \geq 0$, $c(2H+2V-E) - L$ is nef if and only if $c > 1$, and $(2H+2V-E) + f(5H+5V-3E) - L$ is never nef, but $(2H+2V-E) + (4H+3V-2E) - L$ is nef. Thus the only cases with $A = c(2,2,1) + d(3,1,1) + e(4,3,2) + f(5,5,3)$ for which $A - L$ is not nef, but $m \geq 2$ are: $A = f(5,5,3)$, $f \geq 1$; $A = (4,3,2) + f(5,5,3)$, $f \geq 0$; and $A = (2,2,1) + f(5,5,3)$, $f \geq 1$.

The only other possible cases are obtained from these by adding on to one of these multiples of either $(1,0,0)$ or $(1,1,0)$. But $(A - L) + (1,0,0)$ for any of these A is nef, so we do not get any additional cases by allowing $a > 0$ or $b > 0$. That is, we must check that $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ only when (i,j,m) is either $(2,2,1) + f(5,5,3)$, $(4,3,2) + f(5,5,3)$, or $f(5,5,3)$.

First, we show that $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ holds for the cases $f(5,5,3)$. Let $F = 5H + 5V - 3E$. The divisor $E_6' = 2H + V - E$ is linearly equivalent to the exceptional curve that is the proper transform C' of the curve above denoted as C. Likewise, $H + 2V - E$ is linearly equivalent to an exceptional curve; denote this exceptional curve by C''. Note that $F = 2C' + (H + 3V - E) = 2C'' + (3H + V - E)$. Thus $(I_{(2,1)})^2 I_{(1,3)} \subseteq (I^{(3)})_{(5,5)}$ and $(I_{(1,2)})^2 I_{(3,1)} \subseteq (I^{(3)})_{(5,5)}$, but

$$\dim I_{(1,2)} = \dim I_{(2,1)} = 1 \quad \text{and} \quad \dim I_{(3,1)} = \dim I_{(1,3)} = 3,$$

while $\dim(((I_{(2,1)})^2 I_{(1,3)}) \cap ((I_{(1,2)})^2 I_{(3,1)})) = 0$ since $F - 2C' - 2C''$ is not linearly equivalent to an effective divisor. Thus

$$\dim\Big(\big((I_{(2,1)})^2 I_{(1,3)}\big) + \big((I_{(1,2)})^2 I_{(3,1)}\big)\Big) = 6 = \dim(I^{(3)})_{(5,5)},$$

hence $(I^{(3)})_{(5,5)} \subset I^3$. Moreover, $F = 5H + 5V - 3E$ is normally generated by [19, Proposition 3.1(a)], which means that $H^0(X,F)^{\otimes n} \to H^0(X,nF)$ is surjective. Thus $(I^{(3f)})_{(5f,5f)} = ((I^{(3)})_{(5,5)})^f$ and hence $(I^{(3f)})_{(5f,5f)} \subset (I^3)^f = I^{3f}$, as we needed to show.

Now consider $(I^{(2)})_{(4,3)}$. We have $I_{(1,2)}I_{(3,1)} \subseteq (I^{(2)})_{(4,3)}$ and $I_{(2,1)}I_{(2,2)} \subseteq (I^{(2)})_{(4,3)}$, but $\dim I_{(1,2)}I_{(3,1)} = 3$, $\dim I_{(2,1)}I_{(2,2)} = \dim I_{(2,2)} = 4$, and

$$\dim\big(\big(I_{(1,2)}I_{(3,1)}\big) \cap \big(I_{(2,1)}I_{(2,2)}\big)\big) = \dim H^0(X,H) = 2,$$

https://doi.org/10.4153/CJM-2012-045-3 Published online by Cambridge University Press

so $\dim((I_{(1,2)}I_{(3,1)})+(I_{(2,1)}I_{(2,2)})) = 4+3-2 = 5 = \dim(I^{(2)})_{(4,3)}$, hence $(I^{(2)})_{(4,3)} = ((I_{(1,2)}I_{(3,1)})+(I_{(2,1)}I_{(2,2)})) \subset I^2$, as we needed to show.

Note that $4H+3V-2E = 3L-E_1'-\cdots-E_5'$. Since the points are general, $|3L-E_1'-\cdots-E_5'|$ and hence $|4H+3V-2E|$ contains the class of a smooth elliptic curve, Q. Let $F = 5H+5V-3E$. Tensoring $0 \to \mathcal{O}_X(-Q) \to \mathcal{O}_X \to \mathcal{O}_Q \to 0$ by $\mathcal{O}_X(Q+fF)$ and taking global sections gives

$$0 \to H^0(X, fF) \to H^0(X, Q+fF) \to H^0\big(Q, \mathcal{O}_Q(Q+fF)\big) \to 0.$$

Tensoring by $H^0(X,F) = \Gamma_X(F)$ and applying the natural multiplication maps gives the following commutative diagram (see [26], or [11, Lemma 2.3.1]):

$$\begin{array}{ccccccccc} 0 & \to & H^0(X,fF)\otimes\Gamma_X(F) & \to & H^0(X,Q+fF)\otimes\Gamma_X(F) & \to & H^0(Q,\mathcal{O}_Q(Q+fF))\otimes\Gamma_X(F) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^0(X,(f+1)F) & \to & H^0(X,Q+(f+1)F) & \to & H^0(Q,\mathcal{O}_Q(Q+(f+1)F)) & \to & 0 \end{array}$$

Since $F-Q$ is linearly equivalent to an exceptional curve and hence $h^1(X,F-Q) = 0$, the sequence $0 \to \mathcal{O}_X(F-Q) \to \mathcal{O}_X(F) \to \mathcal{O}_Q(F) \to 0$ is exact on global sections. Thus the map $H^0(Q,\mathcal{O}_Q(Q+fF))\otimes\Gamma_X(F) \to H^0(Q,\mathcal{O}_Q(Q+(f+1)F))$ has the same image as $H^0(Q,\mathcal{O}_Q(Q+fF))\otimes\Gamma_Q(F) \to H^0(Q,\mathcal{O}_Q(Q+(f+1)F))$, and the latter is surjective by [26, Theorem 6] (or see [20, Proposition II.5(c)]). We saw above that F is normally generated, and hence that the map $H^0(X,fF)\otimes\Gamma_X(F) \to H^0(X,(f+1)F)$ is surjective. Now apply the snake lemma to the above diagram to conclude that $H^0(X,Q+fF)\otimes\Gamma_X(F) \to H^0(X,Q+(f+1)F)$ is surjective. By induction, we have surjectivity for all $f \geq 0$ and hence

$$(I^{(2+3f)})_{(4+5f,3+5f)} = (I^{(2)})_{(4,3)}\big((I^{(3)})_{(5,5)}\big)^f \subset I^2 I^{3f} = I^{2+3f}.$$

Finally we consider the case of $(2,2,1)+f(5,5,3)$. The proof here is the same as for $(4,3,2)+f(5,5,3)$, except now Q is a smooth elliptic curve linearly equivalent to $-K_X = 3L-E_1'-\cdots-E_6'$, and $F-Q$ is linearly equivalent to the sum $C'+C''$ of two disjoint exceptional curves, so as before we have $h^1(X,F-Q) = 0$. Thus $(I^{(1+3f)})_{(2+5f,2+5f)} = (I)_{(2,2)}((I^{(3)})_{(5,5)})^f \subset I^{1+3f}$. ■

3.2 Non-equality of $I^{(m)}$ and I^m

While computer calculations suggest that $I^{(2)} = I^2$ for the ideal I of four multiplicity 1 generic points in $\mathbb{P}^1\times\mathbb{P}^1$, it is not hard to see that $I^{(3)} \neq I^3$. This is because $\alpha(I) = 3$, so $\alpha(I^3) = 9$, but there is a unique curve of bi-degree $(1,1)$ through any three of the four points (corresponding to the divisors $H+V-E_1-E_2-E_3-E_4+E_i$ in Lemma 2.3.2), hence the sum of these four curves corresponds to a non-trivial form in $(I^{(3)})_{(4,4)}$. Thus $\alpha(I^{(3)}) \leq 8$, so $I^{(3)} \not\subseteq I^3$.

In fact, the case of four multiplicity 1 generic points is part of a much larger family, namely a set Z of s points in multiplicity 1 generic position when $s = t^2$ for some integer $t \geq 2$. For this family, we can, in a similar way, verify failures of containments of certain symbolic powers of the ideal $I(Z)$ of the points in various ordinary powers of the ideal.

Theorem 3.2.1 *Let $I = I(Z)$, where $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of $s = t^2$ points in multiplicity 1 generic position with $t \geq 2$. Then for all integers $n \geq 1$,*

$$I^{((s-1)(2t-1)n)} \not\subseteq I^{2s(t-1)n+1}.$$

Proof We begin by showing that the symbolic power $I^{((s-1)(2t-1)n)}$ has a nonzero element of bidegree $((t-1)s(2t-1)n, (t-1)s(2t-1)n)$. For each point $P_i \in Z$, let $Y_i = Z \setminus \{P_i\}$. Then Y_i is a set of $s-1$ points in multiplicity 1 generic position for each $i = 1, \ldots, s$, and hence

$$\dim\big(I(Y_i)_{(t-1,t-1)}\big) = \max\big\{\, t^2 - |Y_i|, 0\big\} = \max\big\{\, s - (s-1), 0\big\} = 1.$$

Thus, for each $i = 1, \ldots, s$, there is a form F_i (unique up to scalar multiplication) that vanishes at all of the points of Y_i. Moreover, F_i does not vanish at P_i. Indeed, if $F_i(P_i) = 0$, then $F_i \in I(Z)_{(t-1,t-1)}$, but $I(Z)_{(t-1,t-1)} = 0$, since $\dim(I_{(t-1,t-1)}) = \max\{t^2 - |Z|, 0\} = 0$.

Set $F = \prod_{i=1}^{s} F_i$. The form F has degree $((t-1)s, (t-1)s)$ and passes through all the points of Z with multiplicity at least $s-1$, so $F \in I^{(s-1)}$. Thus

$$F^{(2t-1)n} \in (I^{(s-1)})^{(2t-1)n} \subseteq I^{((s-1)(2t-1)n)}$$

and

$$\deg F^{(2t-1)n} = \big((t-1)s(2t-1)n, (t-1)s(2t-1)n\big)$$

for each $n \geq 1$.

To show $I^{((s-1)(2t-1)n)} \not\subseteq I^{2s(t-1)n+1}$, it is now enough to check that

$$(I^{2s(t-1)n+1})_{((t-1)s(2t-1)n,(t-1)s(2t-1)n)} = 0.$$

Because the points of Z are in multiplicity 1 generic position, then for $i+j = 2(t-1)$, $i, j \geq 0$, we have $(i+1)(j+1) \leq t^2 = |Z|$, so $\dim(I_{(i,j)}) = 0$. Thus, viewing I as a singly graded ideal, we have $\alpha(I) \geq 2t-1$, hence

$$\alpha\big(I^{2s(t-1)n+1}\big) \geq \big(2s(t-1)n+1\big)(2t-1) > 2s(t-1)n(2t-1)$$

and so $(I^{2s(t-1)n+1})_{(s(t-1)n(2t-1),s(t-1)n(2t-1))} = 0$. ∎

We round out this section by comparing the symbolic squares and ordinary squares of ideals of six or more points in multiplicity 1 generic position.

Proposition 3.2.2 *Let $I = I(Z)$ with $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of 6 points in multiplicity 1 generic position. Then $I^2 \neq I^{(2)}$.*

Proof Since $2Z$ imposes at most $6\binom{2+1}{2} = 18$ conditions on forms of bidegree $(3,4)$, we see that $\dim((I^{(2)})_{(3,4)}) \geq 2$. Thus $\alpha(I^{(2)}) \leq 7$, but, using the fact that I is multiplicity 1 generic, we compute that $\alpha(I) = 4$ so $\alpha(I^2) = 8$, and hence $I^2 \subsetneq I^{(2)}$. ∎

To extend this result to 7 or more points, we require [31, Theorem 1]. We state only the part we need.

Lemma 3.2.3 *Let $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points in multiplicity 1 generic position, with defining ideal $I = I(Z)$. If $(i, j) \notin \{(2, s-1), (s-1, 2)\}$, then*

$$\dim(I^{(2)})_{(i,j)} = \max\{0, (i+1)(j+1) - 3s\}.$$

We now proceed to the case of 7 or more points.

Theorem 3.2.4 *Let $I = I(Z)$ with $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of $s = |Z| \geq 7$ points in multiplicity 1 generic position. Then $I^2 \neq I^{(2)}$.*

Proof Let $I = I(Z)$. To show that $I^2 \neq I^{(2)}$, we find a bidegree (i, j), where $(I^2)_{(i,j)} \neq (I^{(2)})_{(i,j)}$, which we verify by showing that the two graded pieces have different dimensions.

We divide s by 2 and by 3 to write s as $s = 2q_1 + r_1$ and $s = 3q_2 + r_2$, where $0 \leq r_1 \leq 1$ and $0 \leq r_2 \leq 2$. Because Z is in multiplicity 1 generic position,

$$H_Z(1, q_1) = \min\{2(q_1+1), 2q_1 + r_1\} = 2q_1 + r_1,$$
$$H_Z(2, q_2) = \min\{3(q_2+1), 3q_2 + r_2\} = 3q_2 + r_2.$$

It then follows from the Hilbert function that

$$\dim(I_{(1,q_1)}) = 2(q_1+1) - H_Z(1, q_1) = 2 - r_1$$
$$\dim(I_{(2,q_2)}) = 3(q_2+1) - H_Z(2, q_2) = 3 - r_2.$$

We will use this information, and Lemma 3.2.3 to compare the ideals I^2 and $I^{(2)}$ in bidegree $(3, q_1 + q_2)$. We require two claims.

Claim 1 $\dim((I^2)_{(3,q_1+q_2)}) \leq (2 - r_1)(3 - r_2)$.

Proof of Claim 1 We first note that

$$(I^2)_{(3,q_1+q_2)} = \sum_{\substack{0 \leq a,b,c,d \\ a+c=3,\ b+d=q_1+q_2}} I_{(a,b)} I_{(c,d)}.$$

The claim will follow if we show that for $(a, b) \notin \{(1, q_1), (2, q_2)\}$, we have $I_{(a,b)} I_{(c,d)} = 0$. This would then show that $(I^2)_{(3,q_1+q_2)} = I_{(1,q_1)} I_{(2,q_2)}$, and thus

$$\dim\big((I^2)_{(3,q_1+q_2)}\big) \leq \dim(I_{(1,q_1)}) \dim(I_{(2,q_2)}) = (2 - r_1)(3 - r_2).$$

If $a = 0$, then $I_{(a,b)} = 0$, since Z is in multiplicity 1 generic position and $0 \leq b \leq q_1 + q_2 \leq s - 1$. Likewise, $I_{(c,d)} = 0$ if $c = 0$.

If $a = 1$ and $b \neq q_1$, then there are two cases. If $b < q_1$, then $I_{(a,b)} = 0$, since $H_Z(a, b) = \min\{(a+1)(b+1), 2q_1 + r_1\} = (a+1)(b+1)$. On the other hand, if $b > q_1$, then $I_{(c,d)} = 0$, since $c = 2$ and $d = q_1 + q_2 - b < q_2$, so $(c+1)(d+1) \leq 3q_2 \leq 3q_2 + r_2$, whence $H_Z(c, d) = (c+1)(d+1)$. Likewise, $I_{(a,b)} I_{(c,d)} = 0$ if $c = 1$ and $d \neq q_1$.

Finally, if $a \geq 2$, then $c \leq 1$, so the same arguments apply. ∎

Claim 2 $\dim(I^{(2)})_{(3,q_1+q_2)} = q_2 + 4 - 2r_1 - r_2$.

Proof of Claim 2 By Lemma 3.2.3, we have

$$\dim(I^{(2)})_{(3,q_1+q_2)} = \max\{0, 4(q_1 + q_2 + 1) - 3s\}.$$

By the definition of q_1 and q_2, we have $s \leq 2q_1 + 1$ and $s \leq 3q_2 + 2$. So

$$\begin{aligned} 4(q_1 + q_2 + 1) - 3s &= 4q_1 + 4q_2 + 4 - 3s \\ &= (2q_1 + 1) + (2q_1 + 1) + (3q_2 + 2) + q_2 - 3s \geq 0. \end{aligned}$$

Thus $\dim(I^{(2)})_{(3,q_1+q_2)} = 4(q_1 + q_2 + 1) - 3s$. Now we get

$$4(q_1 + q_2 + 1) - 3s = 4q_1 + 4q_2 + 4 - 2(2q_1 + r_1) - (3q_2 + r_2) = q_2 + 4 - 2r_1 - r_2$$

by using the fact that $s = 2q_1 + r_1$ and $s = 3q_2 + r_2$. ■

To complete the proof, it suffices to show that

$$\dim(I^{(2)})_{(3,q_1+q_2)} = q_2 + 4 - 2r_1 - r_2 > (2 - r_1)(3 - r_2) \geq \dim(I^2)_{(3,q_1+q_2)}.$$

But $q_2 + 4 - 2r_1 - r_2 > (2 - r_1)(3 - r_2)$ is equivalent to $q_2 - 1 > (r_1 - 1)(r_2 - 1)$. The maximum value of $(r_1 - 1)(r_2 - 1)$ is 1, and it occurs only for $r_1 = r_2 = 0$, whereas $q_2 - 1 > 1$ unless $s = 7$ or 8, and in both of these cases we have $q_1 - 1 = 1 \geq 0 \geq (r_1 - 1)(r_2 - 1)$. ■

Remark 3.2.5 We cannot use the above proof for the case $s = 6$, because $q_2 + 4 - 2r_1 - r_2 = (2 - r_1)(3 - r_2)$ when $s = 6$ but the proof needs $q_2 + 4 - 2r_1 - r_2 > (2 - r_1)(3 - r_2)$.

Now, we are able to prove the second main result of this paper.

Proof of Theorem 1.2 That $I^{(m)} = I^m$ for all $m \geq 1$ for s general points for $s = 1, 2, 3, 5$, follows from Theorems 2.1.2, 3.1.1, 3.1.2, and 3.1.4, respectively. That $I^{(m)} \neq I^m$ for some m for all other s follows for $s = 4$ by Theorem 3.2.1 (apply the theorem with $t = 2$), for $s = 6$ by Proposition 3.2.2 and for $s > 6$ by Theorem 3.2.4. ■

Acknowledgments We would like to thank Irena Swanson for answering some of our questions. This work was facilitated by the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada. Computer experiments carried out on `CoCoA` [5] and *Macaulay2* [13] were very helpful in guiding our research.

References

[1] C. Bocci and B. Harbourne, *Comparing powers and symbolic power of ideals.* J. Algebraic Geom. **19**(2010), no. 3, 399–417. http://dx.doi.org/10.1090/S1056-3911-09-00530-X

[2] ———, *The resurgence of ideals of points and the containment problem.* Proc. Amer. Math. Soc. **138**(2010), no. 4, 1175–1190. http://dx.doi.org/10.1090/S0002-9939-09-10108-9

[3] C. Bocci, S. Cooper, and B. Harbourne, *Containment results for ideals of various configurations of points in* $\mathbb{P}^N$. arxiv:1109.1884v1.

[4] G. V. Chudnovsky, *Singular points on complex hypersurfaces and multidimensional Schwarz lemma.* In: Seminar on Number Theory, Paris 1979–80, Progr. Math., 12, Birkhäuser, Boston, MA, 1981, pp. 29–69.

[5] CoCoATeam, *CoCoA: a system for doing computations in Commutative Algebra.* http://cocoa.dima.unige.it.

[6] W. Bruns, B. Ichim, and C. Söger, *Normaliz. Algorithms for rational cones and affine monoids.* http://www.math.uos.de/normaliz.

[7] L. Ein, R. Lazarsfeld, and K. E. Smith, *Uniform bounds and symbolic powers on smooth varieties.* Invent. Math. **144**(2001), no. 2, 241–252. http://dx.doi.org/10.1007/s002220100121

[8] H. Esnault and E. Viehweg, *Sur une minoration du degré d'hypersurfaces s'annulant en certains points.* Math. Ann. **263**(1983), no. 1, 75–86. http://dx.doi.org/10.1007/BF01457085

[9] S. Fitchett, *Maps of linear systems on blow-ups of the projective plane.* J. Pure Appl. Algebra **156**(2001), no. 1, 1–14. http://dx.doi.org/10.1016/S0022-4049(99)00115-2

[10] A. Geramita, B. Harbourne, and J. Migliore, *Classifying Hilbert functions of fat point subschemes in* $\mathbb{P}^2$. Collect. Math. **60**(2009), no. 2, 159–192. http://dx.doi.org/10.1007/BF03191208

[11] A. Gimigliano, B. Harbourne, and M. Idà, *Betti numbers for fat point ideals in the plane: a geometric approach.* Trans. Amer. Math. Soc. **361**(2009), no. 2, 1103–1127. http://dx.doi.org/10.1090/S0002-9947-08-04599-6

[12] S. Giuffrida, R. Maggioni, and A. Ragusa, *On the postulation of* 0*-dimensional subschemes on a smooth quadric.* Pacific J. Math. **155**(1992), no. 2, 251–282.

[13] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry.* http://www.math.uiuc.edu/Macaulay2/.

[14] E. Guardo, *Fat point schemes on a smooth quadric.* J. Pure Appl. Algebra **162**(2001), no. 2–3, 183–208. http://dx.doi.org/10.1016/S0022-4049(00)00123-7

[15] E. Guardo and B. Harbourne, *Resolutions of ideals of six fat points in* $\mathbb{P}^2$. J. Algebra **318**(2007), no. 2, 619–640. http://dx.doi.org/10.1016/j.jalgebra.2007.09.018

[16] E. Guardo, B. Harbourne, and A. Van Tuyl, *Asymptotic resurgences for ideals of positive dimensional subschemes of projective space.* arxiv:1202.4370v1.

[17] ———, *Fat lines in* $\mathbb{P}^3$*: powers versus symbolic powers.* arxiv:1208.5221v1.

[18] E. Guardo and A. Van Tuyl, *Fat points in* $\mathbb{P}^1 \times \mathbb{P}^1$ *and their Hilbert functions.* Canad. J. Math. **56**(2004), no. 4, 716–741. http://dx.doi.org/10.4153/CJM-2004-033-0

[19] B. Harbourne, *Birational models of rational surfaces.* J. Algebra **190**(1997), 145–162. http://dx.doi.org/10.1006/jabr.1996.6899

[20] ———, *Free Resolutions of Fat Point Ideals on* $\mathbf{P}^2$. J. Pure Appl. Algebra **125**(1998), no. 1–3, 213–234. http://dx.doi.org/10.1016/S0022-4049(96)00126-0

[21] B. Harbourne and C. Huneke, *Are symbolic powers highly evolved?* J. Ramanujan Math. Soc., to appear. arxiv:1103.5809v1.

[22] M. Hochster, *Criteria for the equality of ordinary and symbolic powers of primes.* Math. Z. **133**(1973), 53–65. http://dx.doi.org/10.1007/BF01226242

[23] M. Hochster and C. Huneke, *Comparison of symbolic and ordinary powers of ideals.* Invent. Math. **147**(2002), no. 2, 349–369. http://dx.doi.org/10.1007/s002220100176

[24] A. Li and I. Swanson, *Symbolic powers of radical ideals.* Rocky Mountain J. Math. **36**(2006), no. 3, 997–1009. http://dx.doi.org/10.1216/rmjm/1181069441

[25] L. Marino, *Conductor and separating degrees for sets of points in* $\mathbb{P}^r$ *and in* $\mathbb{P}^1 \times \mathbb{P}^1$. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **9**(2006), no. 2, 397–421.

[26] D. Mumford, *Varieties defined by quadratic equations.* In: Questions on algebraic varieties (C.I.M.E., III Ciclo, Varenna, 1969) Edizioni Cremonese, Rome, 1970, pp. 30–100.

[27] K. Schwede, *A canonical linear system associated to adjoint divisors in characteristic* $p > 0$. arxiv:1107.3833v1.

[28] J. Sidman and A. Van Tuyl, *Multigraded regularity: syzygies and fat points.* Beiträge Algebra Geom. **47**(2006), no. 1, 67–87.

[29] H. Skoda, *Estimations L^2 pour l'opérateur $\hat{\partial}$ et applications arithmétiques.* In: Journée sur les Fonctions Analytiques (Toulouse, 1976), Lecture Notes in Math., 578, Springer, Berlin, 1977, pp. 314–323.
[30] A. Van Tuyl, *The defining ideal of a set of points in multi-projective space.* J. London Math. Soc. (2) **72**(2005), no. 1, 73–90. http://dx.doi.org/10.1112/S0024610705006459
[31] ———, *An appendix to a paper of Catalisano, Geramita, Gimigliano: The Hilbert function of generic sets of* 2*-fat points in* $\mathbb{P}^1 \times \mathbb{P}^1$. In: Projective Varieties with Unexpected Properties. de Gruyter, Berlin, 2005, pp. 109–112.
[32] M. Waldschmidt, *Propriétés arithmétiques de fonctions de plusieurs variables. II.* In: Séminaire Pierre Lelong (Analyse), 1975–76, Lecture Notes Math. 578, Springer, Berlin, 1977, pp. 108–135.
[33] ———, *Nombres transcendants et groupes algébriques,* Astérisque 69/70, Sociéte Mathématiqué de France, Paris, 1979.
[34] O. Zariski and P. Samuel, *Commutative algebra. Vol. II.* The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, NJ-Toronto-London-New York, 1960.

Dipartimento di Matematica e Informatica, Viale A. Doria, 6 - 95100 - Catania, Italy
e-mail: guardo@dmi.unict.it

Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588-0130, USA
e-mail: bharbour@math.unl.edu

Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON P7B 5E1
e-mail: avantuyl@lakeheadu.ca