ON A PROBLEM OF P. TURÁN ON LACUNARY INTERPOLATION*

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1. Introduction. In 1955, Suranyi and P. Turán [8] considered the problem of existence and uniqueness of interpolatory polynomials of degree $\leq 2n-1$ when their values and second derivatives are prescribed on n given nodes. Around this kind of interpolation - aptly termed (0, 2) interpolation - considerable literature has grown up since then. For more complete bibliography on this subject we refer to J. Balazs [3]. Later we considered [10] the problem of modified (0, 2) interpolation when the abscissas are the zeros of $(1-x^2)T_n(x)$, where $T_n(x)$ is the Tchebycheff polynomial of the first kind ($T_n(x) = \cos n\theta$, $x = \cos\theta$). We modified the original problem of P. Turán in the sense that we asked for polynomials $R_n(x)$ of degree $\leq 2n+1$ with the values of $R_n(x)$ being prescribed on the above abscissas, but the values of $R_n''(x)$ were to be given only on the zeros of $T_n(x)$. Later, in a lecture at Stanford in 1963, Professor P. Turán proposed the problem of finding the explicit form of the interpolatory polynomials on Tchebycheff abscissas in the "pure" (0, 2)

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interpolation which was not modified in the sense explained above. The object of this paper is to resolve this problem and to obtain the convergence properties of the interpolatory polynomials for a certain class of functions.

A comparison of our convergence theorem 3.1 with the corresponding result of Balazs and Turán [2] and G. Freud [5] on π - abscissas (zeros of $\pi_n(x) = (1-x^2)\,P'_{n-1}(x)$, $P_{n-1}(x)$ being the Legendre polynomial of degree $\leq n-1$), shows that our result is weaker than theirs. However, we can prove for the modified (0, 2) interpolation on Tchebycheff abscissas that our result is in a sense the best possible. We are not able to do the same for the "pure" (0, 2) case, but it seems plausible that this is so in view of our result for the "modified" case.

2. Explicit representation of interpolatory polynomials in pure (0, 2) case.

Let

$$x_{n+2} = -1 < x_{n+1} < x_n < \ldots < x_2 < x_1 = +1$$

be the zeros of $(1-x^2)\,T_n(x)$ and let $S_n(x)$ be the polynomial of degree $\leq 2n-1$ such that

$$(2.1) \qquad S_n(x_j) = a_j\,, \quad S''_n(x_j) = b_j\,, \quad j = 2, 3, \ldots, n+1\,.$$

Obviously $S_n(x)$ is given by

$$(2.2) \qquad S_n(x) = \sum_{i=2}^{n+1} a_i\,u_i(x) + \sum_{i=2}^{n+1} b_i\,v_i(x)\,,$$

where $u_i(x)$ and $v_i(x)$ are the fundamental polynomials of first and second kind whose explicit forms are given by

THEOREM 2.1. *For* n *even and* $i = 2, 3, \ldots, n+1$, *we have*

$$(2.3) \qquad u_i(x) = p_i\,r_1(x) + p'_i\,r_{n+2}(x) + r_i(x)$$

$$(2.4) \qquad v_i(x) = q_i\,r_1(x) + q'_i\,r_{n+2}(x) + \rho_i(x)\,,$$

where

$$(2.5)\qquad p_i + p_i' = \frac{2}{n^2\,(1-x_i^2)}$$

$$(2.6)\qquad p_i - p_i' = \frac{4\,A_i'}{(4n^2-1)(1-x_i^2)T_n'(x_i)}\quad ,$$

$$(2.7)\qquad q_i + q_i' = \frac{2(1-x_i^2)}{n^2}\quad ,$$

$$(2.8)\qquad q_i - q_i' = \frac{2\,A_i}{(4n^2-1)\;T_n'(X_i)}\quad ,$$

$$(2.9)\qquad \rho_i(x) = \frac{(1-x^2)^{1/4}\;T_n(x)}{2\,T_n'(x_i)}\left[A_i\int_{-1}^{x}\frac{T_n(t)}{(1-t^2)^{1/4}}\,dt + \int_{-1}^{x}\frac{\ell_i(t)}{(1-t^2)^{1/4}}\,dt\right],$$

$$(2.10)\qquad A_i\int_{-1}^{+1}\frac{T_n(t)}{(1-t^2)^{1/4}}\,dt + \int_{-1}^{+1}\frac{\ell_i(t)}{(1-t^2)^{1/4}}\,dt = 0\ ,$$

$$(2.10a)\qquad \ell_i(t) = \frac{T_n(t)}{(t-x_i)\;T_n'(x_i)}\quad ,\qquad i = 2, 3, \ldots, n+1\ ,$$

$$(2.11)\qquad r_1(x) = \frac{1+x}{2}\,T_n^2(x) + (1-x^2)\,T_n(x)\,T_n'(x) - B_n(x)T_n(x)\ ,$$

$$(2.12)\qquad r_{n+2}(x) = \frac{(1-x)}{2}\,T_n^2(x) - (1-x^2)T_n(x)\,T_n'(x) - B_n(x)T_n(x)\ ,$$

$$(2.13)\qquad B_n(x) = \frac{1}{2}(1-x^2)^{1/4}\int_{-1}^{x}\frac{T_n'(t)}{(1-t^2)^{1/4}}\,dt\ .$$

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Also for $2 \le i \le n+1$, we have

$$(2.14) \qquad r_i(x) = \frac{(1-x^2)\,\ell_i^2(x)}{1-x_i^2} + \frac{(1-x^2)^{1/4}\,T_n(x)}{(1-x_i^2)\,T_n'(x_i)}$$

$$\left[A_i' \int_{-1}^{x} \frac{T_n(t)}{(1-t^2)^{1/4}}\,dt + \frac{(2-x_i^2)}{2(1-x_i^2)} \int_{-1}^{x} \frac{\ell_i(t)}{(1-t^2)^{1/4}}\,dt + \int_{-1}^{x} \frac{\lambda_i(t)}{(1-t^2)^{1/4}}\,dt\right]$$

$$(2.15) \qquad \lambda_i(t) = \frac{x_i\ell_i(t) - 2(1-t^2)\ell_i'(t)}{2(t-x_i)}, \quad \ell_i(t) = \frac{T_n(t)}{(t-x_i)T_n'(x_i)}, \quad i=2,3,\ldots,n+1,$$

$$(2.16) \qquad A_i' \int_{-1}^{+1} \frac{T_n(t)}{(1-t^2)^{1/4}} + \frac{(2-x_i^2)}{2(1-x_i^2)} \int_{-1}^{+1} \frac{\ell_i(t)}{(1-t^2)^{1/4}}\,dt$$

$$+ \int_{-1}^{+1} \frac{\lambda_i(t)}{(1-t^2)^{1/4}}\,dt = 0\,.$$

Proof. The polynomials $r_i(x)$ and $\rho_i(x)$ are the fundamentals in the "modified" (0, 2) interpolation and formulae (2.14) and (2.9), giving their explicit forms, have been obtained in our earlier work [10]. The values of p_i, p_i', q_i and q_i' as stated above follow from the observation that $r_1(x)$, $r_{n+2}(x)$, $\rho_i(x)$ and $r_i(x)$ are polynomials of degree $\le 2n+1$ in x while $u_i(x)$ and $v_i(x)$ are polynomials in x of degree $\le 2n-1$. Hence equating to zero the coefficient of x^{2n+1} and x^{2n} on the right in (2.3) and (2.4) we get (2.5) - (2.8).

3. Convergence Theorems. Let us consider the sequence of points

$$(3.1) \qquad 1 = x_{1n} > x_{2n} > \ldots > x_{n+1,n} > x_{n+2,n} = -1\,,$$

where $\{x_{kn}\}$ stand for the zeros of $(1-x^2)T_n(x)$. Then forming the interpolatory polynomials for each even n, we shall write the fundamental functions as $r_{kn}(x)$, $\rho_{kn}(x)$. Let $f(x)$ be defined and continuous for $[-1, +1]$; we consider the sequence of polynomials

$$(3.2) \qquad R_n(x,f) = \sum_{k=1}^{n+2} f(x_{kn})\, r_{kn}(x) + \sum_{k=2}^{n+1} \delta_{kn}\rho_{kn}(x)$$

with arbitrary numbers δ_{kn}. We shall prove the following

THEOREM 3.1. *Let* $f(x)$ *have a continuous derivative of order* 1 *in* $[-1, +1]$ *and let* $f'(x) \in \mathrm{Lip}\,\alpha$, $\alpha > 1/2$. *If*

$$(3.3) \qquad |\delta_{kn}| \le \frac{\epsilon_n n^{1/2}}{(1-x_{kn}^2)}, \quad k = 2, 3, \ldots, n+1,$$

with

$$(3.4) \qquad \mathrm{Lt}_{n\to\infty}\, \epsilon_n = 0,$$

then the sequence $R_n(x,f)$ *converges to* $f(x)$ *uniformly in* $[-1, +1]$. *The class* $\mathrm{Lip}\,\alpha$, $\alpha > 1/2$ *cannot be replaced by* $\mathrm{Lip}\,1/2$ *even if all* δ_{kn} *are zero.*

THEOREM 3.2 (Pure (0, 2) case). *Let* $f(x)$ *have a continuous derivative of order* 1 *in* $[-1, +1]$, *and let* $f'(x) \in \mathrm{Lip}\,\alpha$, $\alpha > 1/2$. *If*

$$(3.5) \qquad \beta_{in} = \frac{\epsilon_n n^{1/2}}{1-x_{in}^2}, \quad i=2, 3, \ldots, n+1,$$

with

$$(3.6) \qquad \mathrm{Lt}_{n\to\infty}\, \epsilon_n = 0,$$

then the sequence $S_n(x,f)$ *converges uniformly to* $f(x)$ *in*

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[-1 , +1] , <u>where</u>

$$(3.7) \qquad S_n(x,f) = \sum_{i=2}^{n+1} f(x_{in})\, u_{in}(x) + \sum_{i=2}^{n+1} \beta_{in}\, v_{in}(x) .$$

For the proof of these two convergence theorems we shall need the estimates of the fundamental polynomials. From now onwards, for the sake of typographical convenience, we shall write x_i for x_{in}, $\ell_i(x)$ for $\ell_{in}(x)$ and so on. The proofs of these convergence theorems are outlined in § 9 .

4. We shall need the following lemmas.

LEMMA 4.1. <u>If</u> $x = \cos\theta$, <u>we have</u>

$$(4.1) \qquad \int_{-1}^{x} \frac{T_{2r}(t)}{(1-t^2)^{1/4}}\, dt = - \frac{\Gamma(r-\frac{1}{4})}{\Gamma(r+\frac{5}{4})} \left[\frac{\Gamma(\frac{5}{4})}{4\Gamma(\frac{3}{4})} \int_{-1}^{x} \frac{dt}{(1-t^2)^{1/4}} + (1-x^2)^{3/4} \sum_{j=0}^{r-1} \frac{\Gamma(j+\frac{5}{4})}{\Gamma(j+\frac{3}{4})} T_{2j+1}(x) \right] ,$$

$$(4.2) \qquad \int_{-1}^{x} \frac{T_{2r-1}(t)}{(1-t^2)^{1/4}}\, dt = \frac{-(1-x^2)^{3/4}\,\Gamma(r-\frac{3}{4})}{\Gamma(r+\frac{3}{4})} \left[\frac{\Gamma(\frac{3}{4})}{2\Gamma(\frac{1}{4})} + \sum_{j=1}^{r-1} \frac{\Gamma(j+\frac{3}{4})}{\Gamma(j+\frac{1}{4})} T_{2j}(x) \right] ,$$

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$$(4.3)\qquad \int_{-1}^{x} \frac{T'_{2r}(t)}{(1-t^2)^{1/4}}\, dt = -\frac{2r\Gamma(r+\frac{1}{4})}{\Gamma(r+\frac{3}{4})} \sum_{j=1}^{r-1} \frac{\Gamma(j+\frac{3}{4})}{\Gamma(j+\frac{5}{4})} \sin(2j+1)\theta\, (1-x^2)^{1/4}$$

$$(4.4)\qquad \lambda_i(t) = \frac{n^2-1}{2n} + \frac{1}{n} \sum_{r=1}^{n-1} (n^2 - r^2 - 1)\, T_r(x_i)\, T_r(t) ,$$

where $\lambda_i(t)$ is defined in (2.15). For the proof of this Lemma see [10]. Lemma 4.1 leads us to formulate

LEMMA 4.2. The following estimates are valid:

$$(4.5)\qquad \left| \int_{-1}^{x} \frac{T_p(t)}{(1-t^2)^{1/4}}\, dt \right| \leq \frac{2}{p} \text{ for } -1 \leq x \leq +1 ,$$

$$(4.6)\qquad \left| (1-x^2)^{1/4} \int_{-1}^{x} \frac{\ell_k(t)}{(1-t^2)^{1/4}}\, dt \right| \leq \frac{41}{n} \text{ for } -1 \leq x \leq +1 ,$$

$$k = 2, 3, \dots, n+1 ,$$

$$(4.7)\qquad 0 < \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}}\, dt \leq \frac{4}{n} , \quad k = 2, 3, \dots, n+1 ,$$

$$(4.8)\qquad \left| (1-x^2)^{1/4} \int_{-1}^{x} \frac{T'_n(t)}{(1-t^2)^{1/4}}\, dt \right| \leq \frac{n\Gamma(\frac{n}{2}+\frac{1}{4})}{\Gamma(\frac{n}{2}+\frac{3}{4})} \leq n^{1/2}$$

for $-1 \leq x \leq +1$.

Proof. We shall prove (4.5) only for p even; for p odd the proof follows similarly from (4.2).

Setting

$$I(x) = \int_{-1}^{x} (1-t^2)^{-1/4}\, dt$$

which is continuous and ≥ 0 in $[-1, +1]$ and from the well known estimate [Natanson [6] page 329]

$$(4.9) \qquad \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} = O(n^{\beta}), \quad n \text{ integer}, \ \alpha > -1, \ \beta > -1,$$

we have from (4.1) and Abel's inequality

$$\left| \int_{-1}^{x} \frac{T_{2r}(t)}{(1-t^2)^{1/4}}\, dt \right| \leq \frac{\Gamma(r-\frac{1}{4})}{\Gamma(r+\frac{5}{4})} \left[\frac{I(x)}{16} + \frac{\Gamma(r+\frac{1}{4})}{\Gamma(r-\frac{1}{4})} \right] \leq \frac{2}{r},$$

because from Abel's inequality, we have

$$\left| \sum_{i=1}^{r-1} \frac{\Gamma(i+\frac{5}{4})}{\Gamma(i+\frac{3}{4})} \cos(2i+1)\theta \sin\theta \right|$$

$$\leq \frac{\Gamma(r+\frac{1}{4})}{\Gamma(r-\frac{1}{4})} \max_{1 \leq p \leq r-1} \left| \sum_{i=1}^{p} \cos(2i+1)\theta \sin\theta \right|$$

$$\leq \frac{\Gamma(r+\frac{1}{4})}{\Gamma(r-\frac{1}{4})}.$$

This proves (4.5). From a well known result of L. Fejer

$$(4.10) \qquad \ell_k(x) = \frac{1}{n} + \frac{2}{n} \sum_{r=1}^{n-1} T_r(x_k)\, T_r(x), \qquad \cos\theta = x,$$

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we have

$$(1-x^2)^{1/4} \int_{-1}^{x} \frac{\ell_k(t)}{(1-t^2)^{1/4}}\, dt = \frac{1}{n}(1-x^2)^{1/4}\, I(x) + \frac{2}{n} S_1 + \frac{2}{n} S_2 ,$$

where

$$S_1 = \sum_{r=1}^{\frac{n}{2}-1} T_{2r}(x_k)\,(1-x^2)^{1/4} \int_{-1}^{x} \frac{T_{2r}(t)}{(1-t^2)^{1/4}}\, dt ,$$

$$S_2 = \sum_{r=1}^{n/2} T_{2r-1}(x_k)(1-x^2)^{1/4} \int_{-1}^{x} \frac{T_{2r-1}(t)}{(1-t^2)^{1/4}}\, dt .$$

In order to prove (4.6), it is enough to show that $|S_1| \leq 10$, and $|S_2| \leq 10$. For this, we need the following easily verified identities:

$$(4.11) \qquad \sum_{i=0}^{\frac{n}{2}-1} \frac{\Gamma(i+\frac{5}{4})}{\Gamma(i+\frac{3}{4})} \cos\,(2i+1)\,\theta\, \sin\,\theta$$

$$= \frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{2\Gamma(\frac{n}{2}-\frac{1}{4})} - \frac{1}{4} \sum_{i=1}^{\frac{n}{2}-1} \frac{\Gamma(i+\frac{1}{4})}{\Gamma(i+\frac{3}{4})} \sin\,2i\,\theta ,$$

$$(4.12) \qquad \frac{2\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} = \sum_{j=1}^{\infty} \frac{\Gamma(j-\frac{1}{4})}{\Gamma(j+\frac{5}{4})} .$$

Further we need the following estimate which is an immediate consequence of Abel's inequality.

$$(4.13)\qquad \left| \sum_{j=1}^{\frac{n}{2}-1} \frac{\Gamma(j+\frac{1}{4})}{\Gamma(j+\frac{3}{4})} \sin 2j\,\theta \sin\theta \right| \le \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} \max_{1\le p\le \frac{n}{2}} \left| \sum_{j=1}^{p} \sin 2j\theta \sin\theta \right|$$

From (4.12) it follows that

$$(4.14)\qquad \left| \sum_{r=1}^{\frac{n}{2}-1} \frac{\Gamma(r-\frac{1}{4})}{\Gamma(r+\frac{5}{4})} \cos 2r\,\theta_k \right| \le 2 .$$

Now using (4.11), (4.13), (4.14) and (4.1) we have

$$|S_1| \le \frac{\Gamma(\frac{5}{4})}{2\Gamma(\frac{3}{4})} + \frac{1}{2} + \left| \sin\theta \sum_{r=1}^{\frac{n}{2}-1} \frac{\sin 2r\,\theta \cos 2r\,\theta_k}{2(r+\frac{1}{4})} \right| ,$$

$$\le 1 + \left| \sum_{r=1}^{\frac{n}{2}-1} \frac{\sin 2r\,(\theta+\theta_k) + \sin 2r\,(\theta-\theta_k)}{4r+1} \right| .$$

Since

$$\frac{1}{4r} - \frac{1}{4r+1} = \frac{1}{4r(4r+1)}$$

and

$$\left| \sum_{j=1}^{n} \frac{\sin j\,\theta}{j} \right| \le 2\pi^{1/2} ,$$

we have $|S_1| \le 10$. The proof that $|S_2| \le 10$ is analogous. This completes the proof of (4.6). The proof of (4.7) follows from the identity (4.11) and from the result

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$$(4.16)\qquad \int_{-1}^{+1} \frac{\ell_k(t)\,dt}{(1-t^2)^{1/4}} = \frac{1}{2n}(\pi)^{1/2}\left[\frac{2\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} - \sum_{j=1}^{\frac{n}{2}-1} \frac{\Gamma(j-\frac{1}{4})}{\Gamma(j+\frac{5}{4})} \cos 2j\,\theta_k\right]$$

$$= \frac{1}{2n}(\pi)^{1/2}\left[\sum_{j=1}^{\frac{n}{2}-1} \frac{\Gamma(j-\frac{1}{4})}{\Gamma(j+\frac{5}{4})}(1-\cos 2j\theta_k) + \sum_{j=\frac{n}{2}}^{\infty} \frac{\Gamma(j-\frac{1}{4})}{\Gamma(j+\frac{5}{4})}\right] > 0 ,$$

which in turn, is a consequence of (4.1), (4.2) and (4.10). An equivalent expression for the integral in (4.16) shall be useful later, and is easy to verify; it is

$$(4.17)\qquad \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}}\,dt$$

$$= \frac{2\pi^{1/2}}{n}\left[\frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}\sin^2\theta_k + \sum_{j=1}^{\frac{n}{2}-1} \frac{\Gamma(j+\frac{3}{4})}{\Gamma(j+\frac{5}{4})} \sin(2j+1)\theta_k \sin\theta_k\right] .$$

Lastly, the proof of (4.8) follows on using Abel's inequality in (4.3).

5. Estimates for the Polynomials $\rho_{kn}(x)$.

The following Lemma gives us the estimates of the fundamental polynomials of the second kind in modified (0,2) interpolation.

LEMMA 5.1. *For* $-1 \leq x \leq +1$ *and for* $k = 2,3,\ldots,n+1$ *we have* (n even)

$$(5.1)\qquad |\rho_k(x)| \leq (1-x_k^2)^{1/2}\left[n^{-1/2}\int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}} + \frac{41}{n^2}\right] ,$$

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$$(5.2)\qquad \sum_{k=2}^{n+1} |\rho_k(x)| \leq \frac{45}{\sqrt{n}} \quad .$$

Inequality (5.2) is best possible in the sense that if $d_n = \cos \chi_n$, $\chi_n = \frac{\pi}{2} - \frac{\pi}{4n}$, we have

$$(5.3)\qquad \sum_{k=2}^{n+1} |\rho_k(d_n)| \geq \frac{c}{\sqrt{n}}, \quad n \geq n_o \quad .$$

Proof. Using (4.7) we see that (5.2) follows at once from (5.1). From formula (4.1) and (4.9) we have

$$(5.4)\qquad \left| \int_{-1}^{+1} \frac{T_n(t)}{(1-t^2)^{1/4}}\, dt \right| \leq \frac{1}{2}\, n^{-3/2} \quad ,$$

so that (5.1) follows at once from (2.9), (2.10), (4.5), (4.6) and (4.7). In order to prove (5.3) we first need the following inequality:

$$(5.5)\qquad \left| (1-d_n^2)^{1/4} \int_{-1}^{d_n} \frac{T_n(t)}{(1-t^2)^{1/4}}\, dt \right| \geq \frac{1}{32n} \quad \text{for} \quad n \geq n_o \quad .$$

In order to prove (5.5), we observe from (4.1) and (4.9) that for this purpose it suffices to show that

$$(5.6)\qquad \sin^2 \chi_n \sum_{j=0}^{\frac{n}{2}-1} \frac{\Gamma(j+\frac{5}{4})}{\Gamma(j+\frac{3}{4})} \cos(2j+1)\,\chi_n > \frac{n^{1/2}}{16} \quad .$$

Now from the identity (4.11), we see that

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$$\sin^2\chi_n \sum_{j=0}^{\frac{n}{2}-1} \frac{\Gamma(j+\frac{5}{4})}{\Gamma(j+\frac{3}{4})} \cos (2j+1)\chi_n$$

$$\geq \frac{\sin\chi_n}{2}\left[\frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{\Gamma(\frac{n}{2}-\frac{1}{4})} \sin n\chi_n - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}-1} \frac{\Gamma(j+\frac{1}{4})}{\Gamma(j+\frac{3}{4})} \sin 2j\,\chi_n\right] .$$

Since $|\sin n\chi_n| = \frac{1}{2}$, $\sin \chi_n = \cos \frac{\pi}{4n} \geq 2^{1/2}$, we have (5.6) on using (4.13), from which follows (5.5).

Now, from (2.9), (2.10), (5.5) and (4.7) we obtain

$$|\rho_{kn}(d_n)| \geq \frac{(1-d_n^2)^{1/4}|T_n(d_n)|}{2T_n'(x_{kn})}\left[|A_k\int_{-1}^{d_n} \frac{T_n(t)dt}{(1-t^2)^{1/4}}| - |\int_{-1}^{d_n} \frac{\ell_k(t)dt}{(1-t^2)^{1/4}}|\right],$$

(5.7)

$$\geq \frac{2^{-3/2}}{|T_n'(x_{kn})|}\left[\frac{\Gamma(\frac{n}{2}+\frac{5}{4})}{\Gamma(\frac{n}{2}-\frac{1}{4})}\frac{1}{32n}\int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}}\,dt - \frac{45}{n}\right],$$

for $n > n_o$. From the above, it then follows that in order to prove (5.3) it is enough to prove that

(5.8) $$I = \sum_{k=2}^{n+1} \frac{1}{|T_n'(x_k)|}\int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}}\,dt \geq \frac{4}{n\pi}\,\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} .$$

This inequality follows on using (4.16), since the left side of (5.8) now becomes

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$$= \frac{\pi^{1/2}}{2n^2}\left[\sum_{k=2}^{n+1} \sin\theta_k \left[2\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} - \sum_{r=1}^{\frac{n}{2}-1} \frac{\Gamma(r-\frac{1}{4})}{\Gamma(r+\frac{5}{4})} \cos 2r\,\theta_k\right]\right].$$

Interchanging the order of summation in the above and observing that

$$(5.9) \qquad \sum_{k=2}^{n+1} \sin(2r+1)\,\theta_k = \operatorname{cosec}(2r+1)\frac{\pi}{2n} ,$$

we have

$$I = A + B$$

where

$$A = \frac{1}{n^2}\,\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\operatorname{cosec}\frac{\pi}{2n}$$

$$B = \frac{1}{(2n^2)}\sum_{r=1}^{\frac{n}{2}-1}\frac{\Gamma(r-\frac{1}{4})}{\Gamma(r+\frac{5}{4})}\left[\operatorname{cosec}(2r-1)\frac{\pi}{2n} - \operatorname{cosec}(2r+1)\frac{\pi}{2n}\right].$$

This shows that $B \geq 0$ since for $1 \leq r \leq \frac{n}{2} - 1$, we have $0 < (2r \pm 1)\frac{\pi}{2n} \leq \frac{\pi}{2}$, and $\sin\theta$ is increasing for $0 \leq \theta \leq \frac{\pi}{2}$. Using the inequality $\sin\frac{\pi}{2n} \leq \frac{\pi}{2n}$, we have

$$I \geq \frac{4\Gamma(\frac{3}{4})}{\pi n \Gamma(\frac{5}{4})}$$

which proves (5.8) and this completes the proof of (5.3). We shall

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need in future the following inequalities:

$$(5.10) \qquad (1-x_k^2)^{1/2} \leq \frac{(2k-1)\pi}{2n} < \frac{k\pi}{n}, \quad k = 2, 3, \ldots, \frac{n}{2}+1 ,$$

$$(5.11) \qquad (1-x_k^2)^{1/2} \leq \frac{(n-k)\pi}{n}, \quad k = \frac{n}{2}+2, \ldots, n+1 .$$

6. Estimates for the polynomials $r_k(x)$.

In order to obtain the estimates of the fundamental polynomials of the first kind we shall need the following lemmas.

LEMMA 6.1. For $-1 \leq x \leq +1$,

$$(6.1) \qquad \sum_{k=2}^{n+1} \frac{(1-x^2)}{(1-x_k^2)} \ell_k^2(x) \leq 8 .$$

Proof. Since

$$1 - x^2 = 1 - x_k^2 + (x - x_k)^2 - 2x(x - x_k),$$

we have

$$\sum_{k=2}^{n+1} \frac{(1-x^2)\ell_k^2(x)}{1 - x_k^2}$$

$$\leq \sum_{k=2}^{n+1} \ell_k^2(x) + \frac{T_n^2(x)}{n^2} \sum_{k=2}^{n+1} 1 + \frac{2}{n}|x| \sum_{k=2}^{n+1} \frac{|\ell_k(x)|}{(1-x_k^2)^{1/2}} .$$

Using Schwarz inequality for the last sum on the right hand side and the inequality due to Fejer, viz.

$$(6.2) \qquad \sum_{k=2}^{n+1} \ell_k^2(x) \le 2 ,$$

we get (6.1).

LEMMA 6.2. <u>For</u> $-1 \le x \le +1$ <u>and for</u> $k = 2, 3, \ldots, n+1$,

$$(6.3) \qquad |J_k(x)| \equiv \left| (1 - x^2)^{1/4} \int_{-1}^{x} \frac{\lambda_k(t)}{(1-t^2)^{1/4}} dt \right| \le 23n .$$

<u>Also for</u> $k = 2, 3, \ldots, n+1$,

$$(6.4) \qquad \left| \int_{-1}^{+1} \frac{\lambda_k(t)}{(1-t^2)^{1/4}} dt - \frac{(n^2-1)}{2} \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}} dt \right| \le 2n^{1/2}$$

<u>Proof.</u> From the formula (4.4) and (4.6) we have

$$|J_k(x)| \le \frac{(n^2-1)}{2} \left| (1-x^2)^{1/4} \int_{-1}^{x} \frac{\ell_k(t)}{(1-t^2)^{1/4}} dt \right|$$

$$+ \left| \frac{1}{n} \sum_{r=1}^{n-1} r^2 (1-x^2)^{1/4} T_r(x_k) \int_{-1}^{x} \frac{T_r(t)}{(1-t^2)^{1/4}} dt \right|$$

$$\le 21n + 2n = 23n ,$$

whence the formula (6.3) follows on using (4.5) and (4.6). Proof of (6.4) follows on (4.5) and (4.6).

LEMMA 6.3. The following estimates are valid for $k = 2, 3, \ldots, n+1$ (n even) .

$$(6.5) \qquad \frac{8\Gamma(\frac{7}{4})}{\pi^{1/2}\,\Gamma(\frac{5}{4})} \le \sum_{k=2}^{n+1} \frac{1}{(\sin \theta_k)} \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}} dt \le 12 ,$$

$$(6.6)\qquad 0 < \sum_{k=2}^{n+1} \frac{1}{(\sin\theta_k)^3} \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}}\, dt \le 8\, n^{3/2} ,$$

$$(6.7)\qquad \frac{n^{5/2}}{16} < \sum_{k=2}^{n+1} \left| \frac{A_k'}{(1-x_k^2)T_n'(x_k)} \right| \le 15 n^{5/2} ,$$

where A_k' is given by (2.16).

<u>Proof.</u> Using the left inequality in (4.7), (4.17) and (5.9) we have that the left side of (6.5)

$$= \frac{2(\pi)^{1/2}}{n}\left[2\,\frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}\operatorname{cosec}\frac{\pi}{2n} + \sum_{k=1}^{\frac{n}{2}-1}\frac{\Gamma(k+\frac{3}{4})}{\Gamma(k+\frac{5}{4})}\operatorname{cosec}\frac{(2k+1)\pi}{2n}\right]$$

$$\le \frac{2(\pi)^{1/2}}{n}\left[2\,\frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}\, n + \sum_{k=1}^{\frac{n}{2}-1}\frac{\Gamma(k+\frac{3}{4})}{\Gamma(k+\frac{5}{4})}\,\frac{n}{(2k+1)}\right] \le 6(\pi)^{1/2} < 12 ,$$

where we have used the estimates (4.9) and the inequalities $\operatorname{cosec}\frac{(2k+1)\pi}{2n} \le \frac{n}{2k+1}$ for $0 \le k \le \frac{n}{2} - 1$.

The proof of the left inequality in (6.5) is clear on using (4.17). In order to prove (6.6) we proceed as above and use (4.17) so that the left side of (6.6) is

$$\le \frac{2\,\pi^{1/2}}{n} \sum_{k=2}^{n+1}\left\{2\,\frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}\,\frac{1}{\sin\theta_k} + \sum_{j=1}^{\frac{n}{2}-1}\frac{\Gamma(j+\frac{3}{4})}{\Gamma(j+\frac{5}{4})}\,\frac{1}{\sin^2\theta_k}\right\}$$

$$\le 4n^{-1/2} \sum_{k=2}^{n+1} \frac{1}{\sin^2\theta_k} \le 8n^{3/2} .$$

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Here we have used (4.9), (5.10) which proved (6.6). Now we first prove the right side of (6.7) and observe that

$$\frac{1}{|(1-x_k^2)\, T_n'(x_k)|} = \frac{1}{n \sin \theta_k}, \quad k = 2,3, \ldots, n+1 .$$

Now using (2.16) and (6.4), we have

$$|A_k'| \leq n^{3/2} \left[\left(\frac{(2-x_k^2)}{2(1-x_k^2)} + \frac{(n^2-1)}{2} \right) \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}} \, dt + 2n^{1/2} \right] .$$

The right hand side of (6.7) follows on using (6.5) and

$$\sum_{k=2}^{n+1} \operatorname{cosec} \theta_k \leq 2n \log n .$$

To prove the lower inequality in (6.7) we have from (2.16) and (6.4)

$$|A_k'| \geq \frac{1}{4} n^{3/2} \left[\left| \int_{-1}^{+1} \frac{\lambda_k(t)}{(1-t^2)^{1/4}} \, dt \right| - \frac{(2-x_k^2)}{2(1-x_k^2)} \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}} \right],$$

$$\geq \frac{1}{4} n^{3/2} \left[\frac{(n^2-1)}{2} \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}} \, dt - 2n^{1/2} \right.$$

$$\left. - \frac{1}{\sin^2 \theta_k} \int_{-1}^{+1} \frac{\ell_k(t)}{(1-t^2)^{1/4}} \, dt \right] .$$

Hence, using (6.5) and (6.6) we get the required result. This completes the proof of the Lemma 6.3.

LEMMA 6.4. For $-1 \leq x \leq +1$, we have (for n even)

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$$(6.8) \qquad \sum_{k=1}^{n+2} |r_{kn}(x)| \leq 140\, n^{3/2} .$$

The inequality (6.8) is best possible in the sense that if $d_n = \cos \chi_n$, $\chi_n = \frac{\pi}{2} - \frac{\pi}{4n}$ we have

$$(6.9) \qquad \sum_{k=1}^{n+2} |r_{kn}(d_n)| > \frac{1}{2^{10}}\, n^{3/2} , \qquad \text{for } n \geq n_o .$$

Proof. We shall first need the estimates

$$(6.10) \qquad |r_{1n}(x)| \leq 3n , \quad |r_{n+2,\,n}(x)| \leq 3n , \quad -1 \leq x \leq +1 .$$

These estimates follow immediately from (2.11), (4.8), (2.12) and the observation

$$|(1-x^2)\, T_n'(x)| \leq n .$$

On using (6.7), (6.1), (6.3), (4.5), (5.10) and (6.10), the result (6.8) follows immediately. The inequality (6.9) can be proved on the same lines as the proof given for (5.3). Here we have to use (6.3), (6.7), (6.1), (6.2), (4.6), (4.5) and (5.10).

7. Pure (0, 2) case:

Estimates of the fundamental polynomials of the second kind in pure (0, 2) case.

LEMMA 7.1. For $-1 \leq x \leq +1$, $n = 2, 4, 6, \ldots$ we have

$$(7.1) \qquad |v_i(x)| \leq 2\, n^{-1/2} (1-x_i^2)^{1/2} \left[(1-x_i^2)^{1/2}\, n^{-1} + \int_{-1}^{+1} \frac{\ell_i(t)}{(1-t^2)^{1/4}}\, dt \right]$$

$$\leq 10\, (1-x_i^2)^{1/2}\, n^{-3/2} ,$$

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and

$$(7.2) \qquad \sum_{i=2}^{n+1} |v_i(x)| \le 10\, n^{-1/2} .$$

The inequality (7.2) is best possible in the sense that

$$(7.3) \qquad \sum_{i=2}^{n+1} |v_i(0)| > \frac{\Gamma(\frac{3}{4})}{10\Gamma(\frac{5}{4})} n^{-1/2} , \qquad n \ge n_3 .$$

Proof. First we observe that (7.2) is an immediate consequence of (7.1). Using (2.7), (2.8), (2.11), (2.12) we have

$$q_i r_{1n}(x) + q_i' r_{n+2,n}(x)$$

$$= (1-x_i^2)\, n^{-2} \left[T_n^2(x) - T_n(x)(1-x^2)^{1/4} \int_{-1}^{x} \frac{T_n'(t)}{(1-t^2)^{1/4}} dt \right]$$

$$+ \frac{A_i}{(4n^2-1)T_n'(x_i)} [\, xT_n^2(x) + 2(1-x^2)\, T_n(x)\, T_n'(x) \,]$$

whence, on using (2.10), (4.7), (4.8), (5.4) and the observation

$$(7.4) \qquad |(1-x^2)\, T_n'(x)| \le n ,$$

we have

$$|q_i\, r_{1,n}(x) + q_i'\, r_{n+2,n}(x)|$$

$$\le (1-x_i^2) n^{-1/2} \left[2n^{-1} + \frac{1}{(1-x_i^2)^{1/2}} \int_{-1}^{+1} \frac{\ell_i(t)}{(1-t^2)^{1/4}} dt \right] .$$

Now combining this result with the known estimate of $\rho_i(x)$ we get the required result as stated in (7.1). In order to prove (7.3) we note that from (2.11) and (2.12) we get

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$$(7.5)\qquad r_{1n}(0) = r_{n+2,n}(0) = \frac{1}{2}\left[1 - (-1)^{n/2}\int_{-1}^{0}\frac{T_n'(t)}{(1-t^2)^{1/4}}\,dt\right] .$$

Further, from (2.9) and (2.10) we get

$$\rho_i(0) = \frac{(-1)^{n/2}}{2\,T_n'(x_i)}\left[-\frac{1}{2}\int_{-1}^{+1}\frac{\ell_i(t)}{(1-t^2)^{1/4}}\,dt + \int_{-1}^{0}\frac{\ell_i(t)}{(1-t^2)^{1/4}}\,dt\right] ,$$

whence from (4.7) and similar results we get

$$(7.6)\qquad |\rho_i(0)| \le \frac{2}{n^2} \qquad i = 2, 3, \ldots, n+1 .$$

Also from (4.3) we have

$$(7.7)\qquad \int_{-1}^{0}\frac{T_n'(t)}{(1-t^2)^{1/4}}\,dt = n\,\frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{\Gamma(\frac{n}{2}+\frac{3}{4})}\sum_{r=0}^{\frac{n}{2}-1}(-1)^r\,\frac{\Gamma(r+\frac{3}{4})}{\Gamma(r+\frac{5}{4})} .$$

Since the terms in the summation on the right are monotonically decreasing and alternately positive and negative, the sum on the right is greater than $\frac{2}{5}\,\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}$, hence we have

$$(7.8)\qquad \left|\int_{-1}^{0}\frac{T_n'(t)}{(1-t^2)^{1/4}}\,dt\right| \ge \frac{1}{10}\,n^{1/2}\,\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} .$$

Hence, using (2.4), (7.6), (7.8), we have

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$$|v_i(0)| \geq (1-x_i^2) \frac{1}{n^2} \left[\left| \int_{-1}^{0} \frac{T_n'(t)dt}{(1-t^2)^{1/4}} \right| - 1 \right] - |\rho_i(0)|$$

$$\sum_{i=2}^{n+1} |v_i(0)| \geq \frac{1}{10} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} n^{-1/2} \quad \text{for} \quad n \geq n_2 .$$

This completes the proof of the above lemma.

8. Estimates of the fundamental polynomials of the first kind.

Now we shall be able to obtain the estimates of the fundamental polynomials of the first kind.

LEMMA 8.1. *For* $-1 \leq x \leq +1$ *we have*

$$(8.1) \qquad \sum_{i=2}^{n+1} |u_i(x)| \leq 142 n \log n .$$

Proof. From (2.3), (2.5), (2.6) and using (2.11) - (2.16) we get

$$(8.2) \qquad u_i(x) = \frac{1}{n^2} \frac{1}{(1-x_i^2)} T_n^2(x) + \frac{(1-x^2)\ell_i^2(x)}{1-x_i^2}$$

$$+ F_1(x) + F_2(x) + F_3(x) + F_4(x) ,$$

where

$$F_1(x) = \frac{(2-x_i^2)(1-x^2)^{1/4} T_n(x)}{2(1-x_i^2)^2 T_n'(x_i)} \int_{-1}^{x} \frac{\ell_i(t)}{(1-t^2)^{1/4}} dt ;$$

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using (4.6) we have

$$(8.3) \qquad |F_1(x)| \le (1-x_i^2)^{-3/2} \quad n^{-2} \qquad\qquad i = 2, 3, \ldots, n+1 .$$

$F_2(x)$ is given by the expression

$$F_2(x) = \frac{(1-x^2)^{1/4} \quad T_n(x)}{(1-x_i^2) \; T_n'(x_i)} \int_{-1}^{x} \frac{\lambda_i(t)}{(1-t^2)^{1/4}} \, dt \; ;$$

using (6.3) we have

$$(8.4) \qquad |F_2(x)| \le 23(1-x_i^2)^{-1/2} \quad , \qquad\qquad i = 2, 3, \ldots, n+1 .$$

For $F_3(x)$ we have

$$F_3(x) = - \frac{(1-x^2)^{1/4} \quad T_n(x)}{n^2(1-x_i^2)} \int_{-1}^{x} \frac{T_n'(t)}{(1-t^2)^{1/4}} \, dt \; ;$$

on using (4.8) we get

$$(8.5) \qquad |F_3(x)| \le \frac{n^{-3/2}}{(1-x_i^2)} \quad .$$

Lastly

$$F_4(x) = \frac{A_i' \quad T_n(x)}{(4n^2-1)(1-x_i^2) \quad T_n'(x_i)} \; [I_1] \quad ,$$

where

$$I_1 = (4n^2-1)(1-x^2)^{1/4} \int_{-1}^{x} \frac{T_n(t)}{(1-t^2)^{1/4}} \, dt + 4n \sin n\theta \sin\theta + 2x\, T_n(x) \; .$$

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Therefore using (4.1) and Abel's inequality we get

$$I_1 = 2T_{n-1}(x) - 4\frac{\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(\frac{n}{2}+\frac{1}{4})}$$

$$\left[\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}(1-x^2)^{1/4}\int_{-1}^{x}\frac{dt}{(1-t^2)^{1/4}} - \sum_{j=1}^{n/2}\frac{\Gamma(j+\frac{1}{4})}{\Gamma(j+\frac{3}{4})}\sin 2j\theta \sin\theta\right].$$

Since

$$\frac{\Gamma(\frac{n}{2}+\frac{3}{4})}{\Gamma(\frac{n}{2}+\frac{1}{4})} < 2n^{1/2}$$

and

$$\left|\sum_{j=1}^{\frac{n}{2}-1}\frac{\Gamma(j+\frac{1}{4})}{\Gamma(j+\frac{3}{4})}\sin 2j\theta\sin\theta\right|$$

$$\leq \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})}\max_{1\leq p\leq\frac{n}{2}-1}\left|\sum_{j=1}^{p}\sin 2j\theta\sin\theta\right|,$$

we have

$$|I_1| \leq 2 + 8n^{1/2}\cdot 3 \leq 26n^{1/2} \quad .$$

Thus we obtain

$$(8.6) \qquad |F_4(x)| \leq \left|\frac{A_i' \; 26\, n^{1/2}}{(4n^2-1)(1-x_i^2)\, T_n'(x_i)}\right| \quad .$$

Hence from (8.2) on using (8.3) - (8.6) together with (6.7),

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and (6.1) we get at once (8.1).

In order to prove our main convergence theorem we require a lemma on approximating polynomials.

LEMMA (8.2). Let $f(x)$ have a continuous derivative in $[-1, +1]$ and let $f'(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$; then there exists a sequence of polynomial $\varphi_n(x)$ of degree at most n such that

$$(8.7) \qquad |f(x) - \varphi_n(x)| \le C\, n^{-1-\alpha}\, [(1-x^2)^{1+\alpha/2} + n^{-1-\alpha}],$$

where C is a constant independent of both x and n. Further, we have

$$(8.8) \qquad |\varphi_n''(x)| \le C_1\, n^{1-\alpha} \min [(1-x^2)^{\frac{\alpha-1}{2}}\; n^{1-\alpha}] \text{ for } -1 \le x \le 1 .$$

The existence of $\varphi_n(x)$ and the inequality (8.7) is due to Dzydayk [4] and (8.8) follows by using the same method as given in Lemma 1 in G. Freud [5].

9. Proof of Theorem 3.2. By the above Lemma there exists a sequence of polynomial $\varphi_n(x)$ which satisfies (8.7) and (8.8). Then by the uniqueness theorem we have

$$S_n(x) = \sum_{i=2}^{n+1} f(x_i)\, u_i(x) + \sum_{i=2}^{n+1} \beta_i\, v_i(x),$$

$$\varphi_n(x) = \sum_{i=2}^{n+1} \varphi(x_i)\, u_i(x) + \sum_{i=2}^{n+1} \varphi_n''(x_1)\, v_i(x).$$

Therefore

$$|S_n(f,x) - f(x)| = |S_n(x,f) - \varphi_n(x) + \varphi_n(x) - f(x)|$$

$$\le |S_n(x,f) - \varphi_n(x)| + |\varphi_n(x) - f(x)|$$

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$$\leq |S_n(x, f) - \varphi_n(x)| + o(1) \text{ by } (8.7)$$

$$\leq \sum_{i=2}^{n+1} |f(x_i) - \varphi_n(x_i)| |u_i(x)| + \sum_{i=2}^{n+1} |\beta_i| |v_i(x)|$$

$$+ \sum_{i=2}^{n+1} |\varphi_n''(x_i) v_i(x)| + o(1)$$

$$= I_2 + I_3 + I_4 + o(1) .$$

But by using (8.7) and (8.1) we get

$$|I_2| = \sum_{i=2}^{n+1} |f(x_i) - \varphi_n(x_i)| |u_i(x)| \leq 2C n^{-1-\alpha} \sum_{i=2}^{n+1} |u_i(x)|$$

$$\leq \frac{284 \ C \ n \log n}{n^{1+\alpha}} = o(1) .$$

For I_3 we have on using (7.1), (3.5), (6.5) and (3.6)

$$|I_3| = \sum_{i=2}^{n+1} \frac{\epsilon_n n^{1/2}}{(1-x_i^2)} \left[2 (1-x_i^2) n^{-3/2} + 2 (1-x_i^2)^{1/2} n^{-1/2} \int_{-1}^{+1} \frac{\ell_i(t)}{(1-t^2)} dt \right]$$

$$= o(1) .$$

Lastly on using (8.8) and (7.1) we get

$$|I_4| \leq 10 \sum_{i=2}^{n+1} n^{-3/2} (1-x_{in}^2)^{1/2} n^{1-\alpha} (1-x_i^2)^{\alpha-1/2}$$

$$= o(1) , \quad \text{for } \alpha > \frac{1}{2} .$$

Thus we obtain that $|S_n(f, x) - f(x)| = o(1)$.

This completes the proof of the theorem 3.2. The proof of

Theorem 3.1 is on the same lines as above and is left out. That Theorem 3.1 is best possible can be proved on the same lines as in the paper of Balázs and P. Turán [2] on using (5.3) and (6.9).

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