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LINEAR COMBINATIONS OF HARMONIC MEASURES AND QUADRATURE DOMAINS OF SIGNED MEASURES WITH SMALL SUPPORTS

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In this paper we discuss the shape of the quadrature domain of a signed measure for harmonic functions. It is known that the quadrature domain of a positive measure with small support is like a ball if the total measure is large enough. We show that, on the contrary, if the measure is not positive then the quadrature domain can be close to an arbitrary domain. This follows from a lemma concerning linear combinations of harmonic measures.

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1. Introduction

A bounded open connected set Ω in the Euclidean space $\mathbb{R}^d$ is called a *quadrature domain* of a signed measure μ for harmonic functions if $|\mu|(\mathbb{R}^d\backslash\Omega) = 0$ and if

$$\int h d\mu = \int_\Omega h dm$$

for every harmonic and integrable function h in Ω, where m denotes the Lebesgue measure in $\mathbb{R}^d$.

Assume now that the support of μ is contained in the ball B_r with radius r and centre at the origin. In a previous paper [23], the author proved that if μ is positive and its total measure is large enough, then the quadrature domain Ω of μ is star-shaped with respect to the origin and satisfies

$$B_{r(\mu)-r} \subset \Omega \subset B_{r(\mu)+r}.$$

Here $r(\mu)$ denotes the volume radius of μ, that is to say, the total measure $\mu(\mathbb{R}^d)$ equals the volume of a ball with radius $r(\mu)$. This means that the quadrature domain of a positive measure μ with small support is like a ball if the total measure of μ is large enough.

In this paper we show that if μ is not positive, then on the contrary, the quadrature domain of μ can be close to an arbitrary domain. More precisely, we prove the following theorem:

Theorem 1. *Let D be a bounded domain in $\mathbb{R}^d$ containing B_r. Then, for every $\epsilon > 0$, there exists a finite linear combination μ of one-point measures with support in B_r and a quadrature domain Ω of μ for harmonic functions such that*

$$D \subset \Omega \subset D_\epsilon,$$

where $D_\epsilon = \{x \in \mathbb{R}^d : \operatorname{dist}(x, D) < \epsilon\}$.

B. Gustafsson conjectured this theorem first and proved it in the two-dimensional case in [**12**]. He treated a bounded domain D with analytic boundary and its one-to-one conformal mapping g onto a domain Ω. He gave a necessary and sufficient condition for Ω to be a quadrature domain of a signed measure, which can be expressed as a real-valued distribution with support in a finite number of points, and constructed the mapping g satisfying $|g(z) - z| < \epsilon$ in D, where z denotes $x^1 + ix^2$ for $x = (x^1, x^2)$ in $\mathbb{R}^2$. To construct the mapping g, he applied the Mergelyan approximation theorem for a compact Riemann surface, the Schottky double of D, and the implicit function theorem.

Our proof is quite different from his and valid for any dimension. It is obtained by applying the following lemma:

Lemma 2. *Let D be a C^2 domain. Let $\omega_x = P_x\sigma$ be the harmonic measure on the boundary ∂D relative to x, where σ denotes the hypersurface measure on ∂D. Let $\{x_j\}_{j=1}^{\infty}$ be a countable set of points in D whose closure contains at least one interior point. Then, given any point y in D, P_y can be approximated uniformly on ∂D by finite linear combinations of $\{P_{x_j}\}_{j=1}^{\infty}$.*

Our lemma is based on the simple fact that if the closure of a subset of a domain contains at least one interior point, then the set is a set of uniqueness for any class of harmonic functions defined on the domain. A natural question is whether there exists a set of uniqueness whose closure is sufficiently small. We also discuss the problem in this paper. We construct examples of sets of uniqueness for some classes of harmonic functions such that they are countable and have no accumulation points in the domain. The problem is closely related to problems discussed by Bonsall, Bonsall and Walsh, and Hayman and Lyons in [**4**], [**5**], and [**15**], respectively. They discussed certain conditions on a subset of a domain which imply that the set is a set of uniqueness for certain classes of harmonic functions on the domain. Our examples satisfy their conditions.

This paper is organized as follows: We prove Theorem 1 in Section 2 by applying the lemma mentioned above. We prove the lemma and discuss its generalization in Section 3. We propose a new quadrature formula in Section 4. Two examples which satisfy conditions discussed by Bonsall, Bonsall and Walsh, and Hayman and Lyons are given in Section 5. Concluding remarks are given in the final section, Section 6.

2. Proof of Theorem 1

Since D is relatively compact in D_ϵ, we may assume that the boundary ∂D of D is sufficiently smooth. Let y be a point in D and let $G(x, y)$ be the Green function for D with pole at y. Let $\omega_y = P_y\sigma$ be the harmonic measure on ∂D relative to y. Then $P_y(x)$ can be expressed as

$$P_y(x) = k_d \frac{\partial G(x, y)}{\partial n_x}$$

for some constant k_d, where $\partial G(x, y)/\partial n_x$ denotes the inner normal derivative at x of $G(x, y)$. Since we have assumed that ∂D is sufficiently smooth, P_y is continuous and positive on ∂D.

First, we take $\delta > 0$ so that the quadrature domain of a measure $\chi_D m + \delta\sigma$ for subharmonic functions is contained in D_ϵ. Here χ_D denotes the characteristic function of D. We call a bounded domain $\Omega(v)$ a quadrature domain of a finite positive measure v for subharmonic functions if $v(\mathbb{R}^d \backslash \Omega(v)) = 0$ and if

$$\int s dv \leq \int_{\Omega(v)} s dm$$

for every subharmonic and integrable function s in $\Omega(v)$. For proof of the existence of the quadrature domain, see e.g. [**20**, Theorems 3.4 and 3.7], [**21**, Theorem 2], [**22**, Theorem 7.5] and [**13**, Theorem 2.1], and for the existence of δ mentioned above, see e.g. [**20**, Proposition 10.6]. Let y_0 be a point in D and take a constant $c_0 > 0$ so that

$$3c_0 P_{y_0}(x) \leq \delta$$

on ∂D.

Next, we apply the Vitali covering theorem. Let $B_r(y)$ be the ball with radius r and centre at y and take a sequence of mutually disjoint balls $B_{r_n}(y_n)$ in D so that $m(D \backslash \sum_{n=1}^{\infty} B_{r_n}(y_n)) = 0$. We write D as

$$\sum_{n=1}^{N} B_{r_n}(y_n) + E_N$$

and choose N so that the balayage $g\sigma$ of $\chi_{E_N} m$ from D onto ∂D satisfies

$$g(x) \leq c_0 P_{y_0}(x)$$

on ∂D. Here the balayage $g\sigma$ of $\chi_{E_N} m$ from D onto ∂D means that $g\sigma$ satisfies

$$\int_{\partial D} hg d\sigma = \int h\chi_{E_N} dm = \int_{E_N} h dm$$

for every function h which is harmonic in D and continuous on the closure of D. We can do this, because P_{y_0} is continuous and positive on ∂D, g can be expressed as

$$g(x) = \int_{E_N} P_y(x)dm(y) = k_d \int_{E_N} \frac{\partial G(x, y)}{\partial n_x} dm(y),$$

and the inner normal derivative of the Green function satisfies

$$\frac{\partial G(x, y)}{\partial n_x} \leq \frac{C}{|x-y|^{d-1}},$$

where C denotes a constant depending only on D (see e.g. [**24**, Theorem 2.3]).

Now, we apply Lemma 2. Take a countable subset $\{x_j\}_{j=1}^{\infty}$ of B_r so that its closure contains at least one interior point. Then, for each y_n, $n = 1, \ldots, N$, we find a finite linear combination $\sum_{j=1}^{j_n} c_{nj} P_{x_j}$ of $\{P_{x_j}\}_{j=1}^{\infty}$ satisfying

$$\left| \sum_{j=1}^{j_n} c_{nj} P_{x_j}(x) - c_n P_{y_n}(x) \right| < \frac{\min_{\partial D} c_0 P_{y_0}}{N+1}$$

on ∂D, where $c_n = m(B_{r_n}(y_n))$, and $\sum_{j=1}^{j_0} c_{0j} P_{x_j}$ satisfying

$$\left| \sum_{j=1}^{j_0} c_{0j} P_{x_j}(x) - 2c_0 P_{y_0}(x) \right| < \frac{\min_{\partial D} c_0 P_{y_0}}{N+1}$$

on ∂D. It follows that

$$\begin{aligned} \sum_{n=1}^{N} c_n P_{y_n}(x) + 2c_0 P_{y_0}(x) - \min_{\partial D} c_0 P_{y_0} &< \sum_{n=0}^{N} \sum_{j=1}^{j_n} c_{nj} P_{x_j}(x) \\ &< \sum_{n=1}^{N} c_n P_{y_n}(x) + 2c_0 P_{y_0}(x) + \min_{\partial D} c_0 P_{y_0} \end{aligned}$$

on ∂D, and so $f(x) = \sum_{n=0}^{N} \sum_{j=1}^{j_n} c_{n_j} P_{x_j}(x)$ satisfies

$$\left(\sum_{n=1}^{N} c_n P_{y_n} + c_0 P_{y_0} \right)(x) < f(x) < \left(\sum_{n=1}^{N} c_n P_{y_n} + 3c_0 P_{y_0} \right)(x)$$

on ∂D. Let $\mu = \sum_{n=0}^{N} \sum_{j=1}^{j_n} c_{nj} \delta_{x_j}$, where δ_{x_j} denotes the unit one-point measure at x_j. We note that $(f - (\sum_{n=1}^{N} c_n P_{y_n} + g))(x) > (c_0 P_{y_0} - g)(x) \geq 0$, and hence $f - (\sum c_n P_{y_n} + g)$ is positive on ∂D. Let Ω be a quadrature domain of $\chi_D m + (f - (\sum c_n P_{y_n} + g))\sigma$ for subharmonic functions. Then Ω contains the closure of D and it is a quadrature domain of μ for harmonic functions. On the other hand, since $(f - (\sum c_n P_{y_n} + g))(x) \leq (3c_0 P_{y_0} - g)(x) \leq 3c_0 P_{y_0}(x) \leq \delta$ on ∂D, we obtain $\Omega \subset D_\epsilon$. This completes the proof of Theorem 1.

3. Lemma 2 and its generalization

First, we define a set of uniqueness for a class of harmonic functions. A subset E of a domain D is called a *set of uniqueness for a class H of harmonic functions defined in D* if a function in H vanishes on E, then it vanishes identically in D. If the closure of E contains at least one interior point, then E is a set of uniqueness for any class of harmonic functions.

Next, let a be a fixed point in a domain D and let ω_a be the harmonic measure on ∂D relative to a. We denote by $L^\infty(\partial D, \omega_a)$ the class of essentially bounded functions on ∂D relative to ω_a. Let H_g be the Dirichlet solution for a resolutive boundary function g in the sense of the Perron-Wiener-Brelot method (see e.g. [**16**, Chapter 8]), and set $H_{L^\infty(\partial D, \omega_a)} = \{H_g : g \in L^\infty(\partial D, \omega_a)\}$.

Now, we prove the following proposition:

Proposition 3. *Let D be a bounded domain in $\mathbb{R}^d$ and let E be a subset of D. Let φ_x denote the Radon-Nikodym derivative of ω_x with respect to ω_a and let L be the linear space of finite linear combinations of functions in $\{\varphi_x : x \in E\}$. Then L is dense in the whole space $L^1(\partial D, \omega_a)$ of ω_a-integrable functions on ∂D if and only if E is a set of uniqueness for $H_{L^\infty(\partial D, \omega_a)}$.*

Proof. Since ω_a is finite, $L^\infty(\partial D, \omega_a)$ is isometrically isomorphic to the dual space of $L^1(\partial D, \omega_a)$. For $g \in L^\infty(\partial D, \omega_a)$, it follows that

$$H_g(x) = \int g d\omega_x = \int g \varphi_x d\omega_a.$$

Hence g vanishes on L if and only if $H_g(x) = 0$ on E. It is known that H_g vanishes identically in D if and only if g is equal to zero a.e. on ∂D relative to ω_a (see e.g. [**6**, Hilfssatz 3.1]). Hence, by the Hahn-Banach theorem, we see that L is dense in $L^1(\partial D, \omega_a)$ if and only if E is a set of uniqueness for $H_{L^\infty(\partial D, \omega_a)}$. This completes the proof of the proposition.

To prove Lemma 2, take a fixed point a in D and a relatively compact subdomain D' of D so that D' contains a, y, and an interior point of the closure of $\{x_j\}_{j=1}^\infty$. Let ω'_a be the harmonic measure on $\partial D'$ relative to a, and let $f_g \sigma$ be the balayage of $g\omega'_a$ with g in $L^1(\partial D', \omega'_a)$ from D onto ∂D. Then f_g is continuous on ∂D and, for every $\epsilon > 0$, there is $\delta > 0$ such that $\int |g| d\omega'_a < \delta$ implies $\sup_{\partial D} |f_g| < \epsilon$, because f_g can be expressed as

$$f_g(x) = k_d \int_{\partial D'} \frac{\partial G(x, y)}{\partial n_x} g(y) d\omega'_a(y),$$

here $G(x, y)$ denotes the Green function for D with pole at y. Since D is a C^2 domain, $\{\partial G(x, y)/\partial n_x : y \in \partial D'\}$ is uniformly bounded and, for each $y \in \partial D'$, $\partial G(x, y)/\partial n_x$ is continuous and positive on ∂D.

Now, we apply Proposition 3 to D' and $\varphi'_y \in L^1(\partial D', \omega'_a)$, where φ'_y denotes the Radon-Nikodym derivative of ω'_y with respect to ω'_a. Then we find, for $\delta > 0$, a finite linear combination $\sum c_j \varphi'_{x_j}$ such that $\int |\sum c_j \varphi'_{x_j} - \varphi'_y| d\omega'_a < \delta$. Since the balayage of $\omega'_x = \varphi'_x \omega'_a$ from D onto ∂D equals $P_x \sigma$, that is to say, since $f_{\varphi'_x} = P_x$, it follows that $\sup_{\partial D} |\sum c_j P_{x_j} - P_y| < \epsilon$. This completes the proof of Lemma 2.

It is known, as the Brelot theorem, that a boundary function g is resolutive if and only if it is ω_a-integrable, and in which case, it follows that

$$H_g(x) = \int g d\omega_x$$

for every x in D. Therefore Proposition 3 can be generalized and discussed in the frame of compactification of domains and in axiomatic potential theory (see e.g. [**6**] and [**7**]). However, we confine ourselves to a discussion on concrete domains in $\mathbb{R}^d$ and prove a stronger version of Lemma 2.

We introduce a more general domain than a C^2 domain, namely, a Lyapunov-Dini domain discussed by K.-O. Widman. To ensure the continuity of the gradient of the Green function around the boundary, we assume further that the Dini function $\epsilon(t)$ satisfies the following additional condition: $\epsilon(t)/t^\gamma$ is decreasing for some constant γ with $0 < \gamma < 1$ (see [**24**, Theorem 2.4]). If $d = 2$, then the boundary of a Lyapunov-Dini domain consists of a finite number of disjoint Dini-smooth Jordan curves, and we can assert the continuity of the gradient of the Green function without the additional condition. For a proof, see e.g. [**19**, Theorem 3.5]. Every $C^{1,\alpha}$ domain with $0 < \alpha < 1$ is a Lyapunov-Dini domain and we can take kt^α with some constant k as the Dini function $\epsilon(t)$. For the definition of $C^{1,\alpha}$ domains, see e.g. [**11**, Definition in 6.2].

From the proof of Lemma 2 combined with the argument in Section 2, we see that, for a Lyapunov-Dini domain, the Radon-Nikodym derivative f of the balayage $f\sigma$ of $\chi_D m$ from D onto ∂D with respect to σ is positive and continuous on ∂D.

Now, we apply the Martin theory on positive harmonic functions. It asserts that, for every positive harmonic function h in D, there exists a unique finite positive measure ν on the Martin minimal boundary such that

$$h(x) = H_\nu(x) \equiv \int K(y, x) d\nu(y),$$

where $K(y, x)$ denotes the Martin kernel function for D with pole at y. If D is a Lyapunov-Dini domain, then the minimal boundary coincides with the Euclidean boundary ∂D. We normalize the kernel function as $K(y, a) = 1$ for a fixed point a. It is expressed as

$$K(y, x) = \frac{\dfrac{\partial G(y, x)}{\partial n_y}}{\dfrac{\partial G(y, a)}{\partial n_y}} = \frac{P_x(y)}{P_a(y)} = \varphi_x(y)$$

for y on ∂D, because grad $G(y, x)$ is continuous and does not vanish as a vector-valued function of y around ∂D, and $P_x(y)$ can be expressed as

$$P_x(y) = k_d \frac{\partial G(y, x)}{\partial n_y}.$$

Let $M(\partial D)$ be the class of signed measures on ∂D and set $H_{M(\partial D)} = \{H_\nu : \nu \in M(\partial D)\}$. Then $H_{M(\partial D)}$ is the class of differences of positive harmonic functions in the Lyapunov-Dini domain D.

Proposition 4. *Let D be a Lyapunov-Dini domain and let E be a subset of D. Let P_x be as in Lemma 2. Then every continuous function on ∂D can be approximated uniformly on ∂D by finite linear combinations of functions in $\{P_x : x \in E\}$ if and only if E is a set of uniqueness for $H_{M(\partial D)}$.*

Proof. We regard $M(\partial D)$ as the dual space of the linear space $C(\partial D)$ of continuous functions on ∂D. Let L be the linear space of finite linear combinations of functions in $\{P_x : x \in E\}$. For $\xi \in M(\partial D)$, set

$$\nu = \frac{\partial G(y, a)}{\partial n_y} \xi.$$

Then

$$H_\nu(x) = \int K(y, x) d\nu(y) = \int \frac{\partial G(y, x)}{\partial n_y} d\xi(y) = \frac{1}{k_d} \int P_x(y) d\xi(y).$$

Hence, by the Hahn-Banach theorem, we see that L is dense in $C(\partial D)$ if and only if E is a set of uniqueness for $H_{M(\partial D)}$. This completes the proof of the proposition.

4. A new quadrature formula

Lemma 2 asserts that ω_y can be approximated by finite linear combinations of $\{\omega_{x_j}\}_{j=1}^{\infty}$. This does not imply that the sequence of corresponding finite linear combinations of $\{\delta_{x_j}\}_{j=1}^{\infty}$ converges to some measure in D, where δ_{x_j} denotes the unit one-point measure at x_j. For example, let $\{x_j\}_{j=1}^{\infty}$ be contained in B_r and assume that the closure $\overline{B_r}$ of B_r is contained in D. Let y be a point in D but not on $\overline{B_r}$. Assume further that a sequence of linear combinations $\sum_{j=1}^{k} c_{kj}\omega_{x_j}$ of $\{\omega_{x_j}\}_{j=1}^{\infty}$ converges strongly to ω_y and the sequence of the corresponding linear combinations $\sum_{j=1}^{k} c_{kj}\delta_{x_j}$ of $\{\delta_{x_j}\}_{j=1}^{\infty}$ converges to a signed measure μ in the weak* topology. Then, for every harmonic function h on $\overline{D}$, we obtain

$$\int h d\mu = \lim_k \int hd\left(\sum_{j=1}^{k} c_{kj}\delta_{x_j}\right) = \lim_k \int hd\left(\sum_{j=1}^{k} c_{kj}\omega_{x_j}\right) = \int hd\omega_y = h(y).$$

On the other hand, for every $\epsilon > 0$, there is a harmonic homogeneous polynomial h such that $|h(x)| < \epsilon$ on $\overline{B}_r$ and $h(y) = 1$. Since the support of μ is contained in $\overline{B_r}$, this contradicts the equation above.

Thus Theorem 1 does not imply that every bounded domain is the quadrature domain of a signed measure with support in B_r. It asserts that any domain can be approximated by quadrature domains of finite linear combinations of one-point measures with supports in B_r. It gives a new quadrature formula for harmonic functions: Let D be a bounded domain in $\mathbb{R}^d$ containing $\overline{B_r}$. For every $\delta > 0$, we can find a finite number of points $\{x_j\}_{j=1}^{k}$ in B_r and real numbers $\{c_j\}_{j=1}^{k}$ such that

$$\left|\sum_{j=1}^{k} c_j h(x_j) - \int_D hdm\right| < M\delta$$

for every harmonic function h satisfying $|h(x)| < M$ in D. The proof is given by just taking a relatively compact subdomain D' of D so that $B_r \subset D'$ and $m(D \backslash D') < \delta$, and apply Theorem 1 replacing D and ϵ with D' and $\operatorname{dist}(D', \partial D)$, respectively. We can take r as small as we want.

5. Examples of sets of uniqueness which have no accumulation points in the domain

As we remarked at the beginning of Section 3, if the closure of a subset E of a domain D contains an interior point, then E is a set of uniqueness for any class of harmonic functions. In this section, we give examples of sets of uniqueness which cluster toward ∂D, but which have no accumulation points in D. Our examples are countable subsets of D which satisfy Condition (i) or (ii) below.

We first explain the Conditions (i) and (ii). If a sequence of the linear combinations $\sum_{j=1}^{k} c_{kj}\delta_{x_j}$ converges strongly to a signed measure, then the measure is of the form $\sum_{j=1}^{\infty} c_j\delta_{x_j}$ with $\sum_{j=1}^{\infty} |c_j| < +\infty$. Hence, if we take account of the strong convergence of $\sum_{j=1}^{k} c_{kj}\delta_{x_j}$, it is natural to take a countable set $E = \{x_j\}_{j=1}^{\infty}$ and consider a continuous linear operator T from l^1 into $L^1(\partial D, \omega_a)$ defined by

$$Tc(y) = \sum_{j=1}^{\infty} c_j \varphi_{x_j}(y).$$

Here l^1 denotes the space of absolutely convergent series $c = \{c_j\}_{j=1}^{\infty}$ of real numbers, and φ_x denotes the Radon-Nikodym derivative of ω_x with respect to ω_a. The Hahn-Banach theorem assets that the image $T(l^1)$ of T is dense in $L^1(\partial D, \omega_a)$ if and only if the kernel of the dual operator T' of T is equal to $\{0\}$. This is nothing less than Proposition 3 in the case that E is a countable set.

The operator was treated by F. F. Bonsall, in the case of the unit disk in $\mathbb{R}^2$, with complex numbers c_j (see [**4**, Theorem 2]). From the closed range theorem, it follows that $T(l^1) = L^1(\partial D, \omega_a)$ if and only if T' has a continuous inverse on the image of T', that is to say, there is a constant C such that

$$\sup_{x \in D} |h(x)| \leq C \sup_{x \in E} |h(x)| \tag{i}$$

for every h in $H_{L^\infty(\partial D, \omega_a)}$.

A different, but closely related, problem is to find the condition that every positive lower-semicontinuous function on ∂D can be expressed as the sum mentioned above with nonnegative coefficients c_j. Here the word "positive" means that the function takes a positive value at every point on ∂D. The problem was discussed recently by many authors (see e.g. [**5**, **15**, **14**, **8**, **9**, **10** and **1**]).

Assume that φ_x is continuous on ∂D for every x in D. Let E be a subset of D. Let L^+ be the class of finite sums with positive coefficients of functions in $\{\varphi_x : x \in E\}$, and let $C^+(\partial D)$ be the class of positive continuous functions on ∂D. Then, as shown in [**5**, Lemma 11] and [**15**, Theorem 1], every positive lower-semicontinuous function on ∂D can be expressed as the sum with positive coefficients of functions in $\{\varphi_x : x \in E\}$ if and only if L^+ is dense in $C^+(\partial D)$ in the topology of uniform convergence. If we assume further that the Martin minimal boundary coincides with the Euclidean boundary ∂D and the Martin kernel function $K(y, x)$ (which is normalized as $K(y, a) = 1$) can be expressed as $\varphi_x(y)$ on ∂D for x in D, then we obtain, by applying the Hahn-Banach theorem to the convex set L^+, that L^+ is dense in $C^+(\partial D)$ if and only if

$$h(x) \leq 0 \quad \text{on } E \quad \text{implies} \quad h(x) \leq 0 \quad \text{in } D \tag{ii}$$

for every h in $H_{M(\partial D)}$. This is equivalent to the following conditions: $\sup_D h(x) = \sup_E h(x)$ for every h in $H_{M(\partial D)}$.

Every Lyapunov-Dini domain satisfies the conditions for the Martin Boundary mentioned above. The class of domains which satisfy the conditions is much wider than that of Lyapunov-Dini domains. For $d = 2$, every domain which is surrounded by a finite number of disjoint Jordan curves satisfies the conditions. For $d \geq 3$, the situation is quite different in this respect. Every Lipschitz domain satisfies the conditions (see [**17**]). There are non-Lipschitz domains which satisfy the conditions (see [**18**] and [**3**]).

Now, we construct our examples. If $d = 1$, then a subset E of a bounded open interval (a, b) which contains two different points is a set of uniqueness, and satisfies Condition (i) as well. It satisfies Condition (ii) if and only if both a and b are its accumulation points. In what follows, we assume that $d \geq 2$.

Let ρ be a positive function defined in D such that $\rho(x) \leq \delta(x)$ in D, where $\delta(x) = \text{dist}(x, \partial D)$. Let $\{D_n\}_{n=1}^{\infty}$ be an exhaustion of D. We call a subset E of D a $\{\rho, \{D_n\}\}$-*set* if $E \cap B_{\rho(x)}(x)$ is not empty for every x on ∂D_n and for every n. If E is a $\{\rho, \{D_n\}\}$-set, then a countable dense subset of E is also a $\{\rho, \{D_n\}\}$-set. Hence we may assume that E is countable; moreover, if ρ is lower-semicontinuous, we can choose a subset of E so that it is a $\{\rho, \{D_n\}\}$-set and it has no accumulation points in D. Let $\kappa_d = 2(d/(d-1))\sigma_{d-1}/\sigma_d = (2/\sqrt{\pi})\Gamma((d/2)+1)/\Gamma((d+1)/2)$, where σ_d denotes the total surface measure of the unit hypersphere in $\mathbb{R}^d$. It follows that $1 < \kappa_d < \sqrt{d}$.

We proceed to show that if k satisfies $0 < k < 1/(\kappa_d + 1)$ and if a countable set $E = \{x_j\}_{j=1}^{\infty}$ is a $\{k\delta, \{D_n\}\}$-set, then E satisfies (i). We first note that

$$|\text{grad } h|(y) \leq \frac{\kappa_d M}{R}$$

holds if a harmonic function h satisfies $|h(x)| \leq M$ in $B_R(y)$. The constant κ_d is the best for inequality (see e.g. [**2**, Theorem 6.18]). Next, we set $M_D = \sup_D |h(x)|$ and $M_E = \sup_E |h(x)|$, and apply the inequality above to points y on the line segment joining $x \in \partial D_n$ and $x_j \in E \cap B_{k\delta(x)}(x)$. Since $\delta(y) \geq (1-k)\delta(x)$, we obtain

$$|h(x_j) - h(x)| = \left| \int_0^1 \frac{dh(x + t(x_j - x))}{dt} dt \right| \leq \frac{\kappa_d M_D k}{1-k},$$

and so $|h(x)| \leq \kappa_d M_D k/(1-k) + M_E$ in D_n. Hence $M_D \leq ((1-k)/(1-(\kappa_d+1)k))M_E$, and E satisfies (i).

If we take $\rho(x)$ so that $\rho(x)/\delta(x)$ tends to zero as x tends toward ∂D, then every $\{\rho, \{D_n\}\}$-set satisfies Condition (ii). In fact, let $h = h^+ - h^-$ with positive harmonic functions h^+ and h^- in D satisfy $h(x) \leq 0$ on a $\{\rho, \{D_n\}\}$-set $E = \{x_j\}_{j=1}^{\infty}$. For every $\epsilon > 0$, we take D_n so that $\rho(x)/\delta(x) \leq \epsilon$ on ∂D_n. We apply the Harnack inequality to positive harmonic functions in $B_{\delta(x)}(x)$ with x on ∂D_n. Then, for $x_j \in E \cap B_{\rho(x)}(x)$, it follows that

$$(1 - k(\epsilon))h^+(x) \leq h^+(x_j) \leq h^-(x_j) \leq (1 + k(\epsilon))h^-(x),$$

where $k(\epsilon)$ is a positive function of ϵ which tends to zero as ϵ tends to zero. Hence

$$h^+(x) \leq \frac{1 + k(\epsilon)}{1 - k(\epsilon)} h^-(x)$$

holds in D_n. By letting ϵ tend to zero, we obtain $h^+(x) \leq h^-(x)$ in D. Thus E satisfies (ii).

6. Concluding remarks

In Section 5, we constructed examples of sets of uniqueness for some classes of harmonic functions. They satisfy Condition (i) or (ii) discussed by Bonsall, Bonsall and Walsh, and Hayman and Lyons. The set of uniqueness which was treated in the proof of our Theorem 1 is contained in a small ball, and so it satisfies neither Condition (i) nor (ii). This reveals that Conditions (i) and (ii) are too strong to prove Theorem 1. We need the fact that, for any given small ball contained in a domain and any class of harmonic functions defined in the domain, there is a set of uniqueness for the class which is contained in the ball.

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