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ON COMONOTONE APPROXIMATION

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ABSTRACT. Jackson type theorems are obtained for the comonotone approximation of piecewise monotone functions by polynomials.

1. **Introduction.** The problem of approximating a piecewise monotone function by polynomials that are comonotone with it has received considerable attention in the last decade. Recently Iliev [6] and Newman [8] independently have shown that the rate of approximation of such a function, f, by comonotone polynomials of degree $\le n$ is not worse than $C\omega(f, 1/n)$, where C is an absolute constant depending only on the number of changes of monotonicity of f. This is the Jackson rate of approximation for functions which are merely continuous.

The purpose of this note is to show that when the piecewise monotone function f has a continuous derivative the error in comonotone approximation satisfies the corresponding higher order Jackson estimate. We prove

THEOREM. *Let f be continuously differentiable in $[-1, 1]$ and change monotonicity s times, $1 \le s < \infty$, in that interval. Then for each $n \ge 1$ there exists a polynomial p_n of degree $\le n$ which is comonotone with f on $[-1, 1]$ and such that*

$$\|f - p_n\| \le \frac{C(s)}{n}\, \omega(f', 1/n) \tag{1}$$

where $C(s)$ is a constant depending only on s.

Here $\|\cdot\|$ is the uniform norm. Combining this result with smoothing arguments (see [7]) yields another proof of the result of Iliev [6] and Newman [8].

In 1977 DeVore [5] showed that a monotone function with r continuous derivatives can be approximated by monotone polynomials of degree $\le n$ with error $0(n^{-r}\omega(f^{(r)}, 1/n))$. See also DeVore [4] for the analogous result for splines, and Chui, Smith and Ward [2] and Beatson [1] for different proofs of DeVore's spline result. Whether a similar estimate holds for comonotone approximation by polynomials is at the moment, an open problem. However,

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such an estimate has been obtained for comonotone approximation by splines by Leviatan and Mhaskar [7].

In what follows Π_n denotes the set of polynomials of degree $\le n$. $C_1, C_2, \ldots$ will denote constants which do not depend on f, n, or s.

2. **Proof of the theorem.** The following lemma estimates the error in comonotone approximation of f in terms of the error in comonotone approximation of the "flipped" function $\hat{f}$ which has one less change in monotonicity.

LEMMA. *There exists a constant C_1 with the following property: Let $f \in C^1[-1, 1]$ be a piecewise monotone function with $s(\ge 1)$ changes of monotonicity in all, one of which is at zero where f has a zero. Define*

$$\hat{f}(x) = \begin{cases} f(x), & x \ge 0 \\ -f(x), & x < 0 \end{cases}$$

and suppose that for some $n \ge 1$ and some $\varepsilon > \omega(\hat{f}', 1/n)$ there exists a polynomial $p_n \in \Pi_n$ comonotone with $\hat{f}$ such that

$$\|\hat{f} - p_n\| \le \varepsilon/n, \qquad \|\hat{f}' - p_n'\| \le \varepsilon. \tag{2}$$

Then there exists a polynomial $P_{2n} \in \Pi_{2n}$ comonotone with f satisfying

$$\|f - P_{2n}\| \le C_1\varepsilon/n, \qquad \|f' - P_{2n}'\| \le C_1\varepsilon. \tag{3}$$

Proof. $\hat{f}$ has one less change of monotonicity than f. Also for $0 < |x| < k/n$, $k \ge 1$,

$$|\hat{f}'(x)| \le \omega(\hat{f}', |x|) \le k\omega(\hat{f}', 1/n) \tag{4}$$

since $\hat{f}(0) = 0$ and since $\hat{f}(0) = 0$,

$$|\hat{f}(x)| = |x|\,|\hat{f}'(\Theta x)| \le \frac{k^2}{n}\,\omega\left(\hat{f}, \frac{1}{n}\right). \tag{5}$$

We follow DeVore [5, pp. 908–909] in constructing for each $n \ge 1$ the approximation to $\operatorname{sgn}(x)$, $q_n(x) = C_n \int_0^x (T_m(t)/t)^4\,dt$, where m is the largest odd integer so that $q_n \in \Pi_n$ and C_n is chosen so that $q_n(1) = 1$. q_n is odd, monotone increasing and satisfies

$$\begin{aligned} &|\operatorname{sgn}(x) - q_n(x)| \le C_3\,|nx|^{-3}, \qquad x \in [-1, 0) \cup (0, 1], \\ &|\operatorname{sgn}(x) - q_n(x)| \le 1, \qquad -1 \le x \le 1. \end{aligned} \tag{6}$$

Since $\hat{f}(0) = 0$ we may assume $p_n(0) = 0$ replacing ε by 2ε in the first inequality of (2). Define $P_{2n}(x) = \int_0^x p_n'(t)q_n(t)\,dt$. Then P_{2n} is comonotone with

f. Also

$$
\begin{aligned}
f(x)-P_{2n}(x) &= \int_0^x [\hat{f}'(t)-p_n'(t)]\,\mathrm{sgn}(t)\,dt+\int_0^x p_n'(t)[\mathrm{sgn}\,(t)-q_n(t)]\,dt \\
&= (\hat{f}(x)-p_n(x))\,\mathrm{sgn}(x)+\int_0^x p_n'(t)[\mathrm{sgn}(t)-q_n(t)]\,dt.
\end{aligned}
\tag{7}
$$

Let $\eta=\mathrm{sgn}(x)/n$. If $0<|x|\le i/n$, $i\ge 1$, then using (2), (4), (6) and that $\hat{f}'(0)=0$ we have

$$
\begin{aligned}
\left|\int_0^x p_n'(t)[\mathrm{sgn}(t)-q_n(t)]\,dt\right| &\le \sum_{k=0}^{i-1}\left|\int_{k\eta}^{(k+1)\eta} p_n'(t)[\mathrm{sgn}(t)-q_n(t)]\,dt\right| \\
&\le \frac{1}{n}[\omega(\hat{f}',1/n)+\varepsilon]+\frac{1}{n}\sum_{k=1}^{i-1}[(k+1)\omega(\hat{f}',1/n)+\varepsilon]C_3k^{-3}\le C_4\varepsilon/n.
\end{aligned}
$$

Estimating the other term in (7) using (2) we have the first inequality in (3). The proof of the second inequality in (3) is similar and will be omitted.

Proof of the theorem. For small n, say $n<N(s)$ the theorem is trivial since if $f'(\alpha)=0$

$$
|f(x)-f(\alpha)|\le|x-\alpha|\,|f'(\zeta)|\le 2\omega(f',2)\le D(s)\frac{1}{n}\,\omega(f',1/n).
$$

We prove the theorem for large n by induction on s the number of changes of monotonicity. We consider the proposition that there exist constants $C(s)$ and $N(s)$, such that for any piecewise monotone function $f\in C^1[-1,1]$ with $s\ge 0$ changes of monotonicity, such that f' has zeros in $[-1,1]$ (this assumption is needed only when $s=0$), and any $n\ge N(s)$, there exists a $p_n\in\Pi_n$ comonotone with f satisfying

$$
\|f-p_n\|\le C(s)\frac{1}{n}\,\omega(f',1/n),\qquad \|f'-p_n'\|\le C(s)\omega(f',1/n).
$$

The proposition is true for $s=0$ as can be seen by using the following construction of DeVore [3, p. 341]. Assume without loss of generality that f vanishes at one of the zeros of f' (subtract a constant from f and add it to the approximation if necessary), and extend f by linear functions to $[-3,3]$ preserving the modulus of continuity of f'. Note that

$$
\max\{\|f\|_{[-3,3]},\|f'\|_{[-3,3]}\}\le C_5 n\omega(f',1/n) \text{ and}
$$

define

$$
\Lambda_n(f,x)=\int_{-2}^{2}\lambda_n(x-t)f(t)\,dt+a_nx
$$

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where $\{\lambda_n\}$ is a suitable sequence of positive polynomial kernels (see DeVore [3]) with

$$\int_{-4}^{4} \lambda_n(t)\,dt = 1, \qquad \int_{-4}^{4} \lambda_n(t)t^2dt = 0(n^{-2}) \quad \text{and} \quad \|\lambda_n\|_{[-4,4]\setminus[-1,1]} = 0(n^{-2}).$$

Then

$$\Lambda_n'(f, x) = \int_{-2}^{2} \lambda_n(x-t)f'(t)dt + \lambda_n(x+2)f(-2) - \lambda_n(x-2)f(2) + a_n.$$

where

$$|\lambda_n(x+2)f(-2) - \lambda_n(x-2)f(2)| \le C_6 n^{-1}\omega\left(f', \frac{1}{n}\right), \qquad -1 \le x \le 1.$$

It is clear that $\Lambda_n(f)$ will have the same constant monotonicity as f, and the desired approximation properties, provided a_n is properly chosen from $\pm C_6 n^{-1}\omega(f', 1/n)$.

Assuming now that the proposition is true for $s-1$ we will prove it for s. Given f with $s(\ge 1)$ changes of monotonicity extend its definition to $[-3, 3]$ as before. Since f has at least one turning point $f'(\alpha) = 0$ for some $-1 < \alpha < 1$. Working with the interval, I, of length 4 centered at α (obviously $I \subseteq [-3, 3]$) we see that a change of variable $y = \frac{1}{2}(x-\alpha)$ yields a function $g(y) = f(x)$ defined for $-1 \le y \le 1$ that has a turning point at zero and $\omega(g', \delta) \le 4\omega(f', \delta)$. We assume w.l.o.g. that $g(0) = 0$ and flip g at zero to get $\hat{g}$. Note that $\omega(\hat{g}', \delta) \le 2\omega(g', \delta)$ and that $\hat{g}$ has $s-1$ changes of monotonicity. Using the lemma and the inductive hypothesis there exists a sequence $\{h_n\}_{n=2N(s-1)}^{\infty}$ of comonotone approximations to g. Inverting the change of variable the sequence $\{p_n(x)\}$, $p_n(x) = h_n(y)$ verifies the proposition for s.

REMARK. It is possible to use similar arguments to obtain Jackson type theorems for coconvex approximation of functions with only one inflection point.

Note added April 21, 1982. In correspondence with G. Iliev we have learned that he has independently obtained similar estimates for functions which have a single change of monotonicity. He has also obtained a result for functions with a Lipschitz second derivative. See G. L. Iliev and M. Trifonova, *Partially monotone approximation of differentiable functions*, preprint.

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