ON THE GENERALIZED HANKEL AND K TRANSFORMATIONS

E. L. Koh

(received February 17, 1969)

1. Introduction. The K transformation (also called the Meijer transformation) has been extended by Zemanian [1; 2] to a class of generalized functions, $\mathcal{K}'_{\mu,a}$. For $f \in \mathcal{K}'_{\mu,a}$, he defined the K transform of f by

(1) $$k_\mu f = \langle f(x), \sqrt{sx}\, K_\mu(sx) \rangle, \quad \operatorname{Re} s > \sigma_f = \max(a, 0).$$

In [2, Section 6.6] the following inversion theorem for the K transform of f is proven:

(2) $$f(x) = \lim_{r\to\infty} \frac{1}{\pi i} \int_{\sigma - ir}^{\sigma + ir} (k_\mu f)(s) \sqrt{sx}\, I_\mu(sx) ds$$

in the sense of weak convergence in $D'(I)$. Here, σ is any fixed real number greater than σ_f, μ is zero or a complex number with positive real part and $D'(I)$ is the space of Schwartz distributions on $I = (0, \infty)$. In the conventional sense where the K transform of a suitably restricted function $f(x)$ is given by

(3) $$k_\mu[f(x)] = \int_0^\infty f(x) \sqrt{sx}\, K_\mu(sx) dx$$

the inversion formula is also given by Equation (2) [3, page 125].

Weiss [4] suggested that, in the conventional sense, Equation (2) can be derived from the Hankel inversion theorem. It is the purpose of this paper to show this derivation in the generalized sense.

Canad. Math. Bull. vol. 12, no. 6, 1969

https://doi.org/10.4153/CMB-1969-094-2 Published online by Cambridge University Press

We first recall our definition of the generalized Hankel transformation [5]. Our notation shall be that of [5]. For a real number μ and a positive real number a, $\mathcal{J}_{\mu,a}$ is defined as the space of smooth functions $\phi(x)$ for which

$$(4) \qquad \tau_k^{\mu,a}(\phi) = \sup_{0<x<\infty} |e^{-ax} x^{-\mu-\frac{1}{2}} S_\mu^k(\phi)| < \infty, \quad k = 0, 1, 2, \ldots$$

where

$$S_\mu^k = (x^{-\mu-\frac{1}{2}} D\, x^{2\mu+1} D\, x^{-\mu-\frac{1}{2}})^k \text{ and } D = \frac{d}{dx} .$$

$\mathcal{J}_{\mu,a}$ is a Fréchet space when supplied with the topology generated by the collection of seminorms $\{\tau_k^{\mu,a}\}$. For each fixed complex number y in the strip $\Omega = \{y : |\text{Im } y| < a, \ y \neq 0$ or a negative number$\}$, $\mathcal{J}_{\mu,a}$ contains the function $\sqrt{yx}\ J_\mu(yx)$. The Hankel transformation, h_μ, is defined on the dual space $\mathcal{J}'_{\mu,a}$ as follows. Let μ be restricted to $-\frac{1}{2} \leq \mu < \infty$. Then for $f \in \mathcal{J}'_{\mu,a}$,

$$(5) \qquad (h_\mu f)(y) \triangleq \langle f(x), \ \sqrt{yx}\ J_\mu(yx) \rangle, \quad y \in \Omega .$$

If y is restricted to the positive real axis, then (5) is inverted by

$$(6) \qquad f(x) = \lim_{r\to\infty} \int_0^r (h_\mu f)(y) \sqrt{xy}\ J_\mu(xy) dy$$

in the sense of weak convergence in $D'(I)$.

2. <u>Derivation of the K inversion theorem.</u> We shall now derive the transform pair (1) and (2) from the pair (5) and (6). In order to assign a sense to the K transform of $f \in \mathcal{J}'_{\mu,a}$ we prove

LEMMA 1. <u>Let μ be restricted to $-\frac{1}{2} \leq \mu < 0$ and let $s \in \Gamma = \{s : \text{Re } s \geq 0, \ s \neq 0\}$. Then</u> $\sqrt{sx}\ K_\mu(sx) \in \mathcal{J}_{\mu,a}$.

Proof. It is easily shown that

$$|e^{-ax}x^{-\mu-\frac{1}{2}}S_\mu^k \sqrt{sx}\, K_\mu(sx)| = |e^{-ax}s^{2k+\mu+\frac{1}{2}}(sx)^{-\mu}K_\mu(sx)| .$$

If we now restrict s as in the hypothesis, the series and asymptotic expansions of $K_\mu(z)$ [6, pages 5 and 86] yield

$$|(sx)^{-\mu}K_\mu(sx)| \le \begin{cases} a_1x^{-2\mu} + a_2x^{-2\mu+2} + \dots + b_0 + b_1x^2 + \dots & 0 < x \le 1 \\ cx^{-\mu-\frac{1}{2}}e^{-|s|x} & 1 < x < \infty . \end{cases}$$

Therefore, for $-\frac{1}{2} \le \mu < 0$,

$$\tau_k^{\mu,a}(\sqrt{sx}\, K_\mu(sx)) < \infty .$$

LEMMA 2. Let μ be restricted to $-\frac{1}{2} \le \mu < 0$ and let $s \in \Gamma$. Let $F(s) = k_\mu f = \langle f(x), \sqrt{sx}\, K_\mu(sx)\rangle$, $f \in \mathcal{J}'_{\mu,a}$. Then, in the sense of weak convergence in $D'(I)$,

$$f(x) = \lim_{r\to\infty} \frac{1}{\pi i}\int_{-ir}^{ir} F(s)\sqrt{sx}\, I_\mu(sx)ds .$$

Proof. Consider the transform pair (5) and (6). By virtue of Equations 7 and 15 of [6, pages 4-5], we can write (5) as

$$(7) \qquad (h_\mu f)(y) = \frac{1}{\pi}\langle f(x), e^{-i\frac{1}{2}(\mu+\frac{1}{2})\pi}\sqrt{-ixy}\, K_\mu(-ixy)\rangle$$

$$+ \frac{1}{\pi}\langle f(x), e^{i\frac{1}{2}(\mu+\frac{1}{2})\pi}\sqrt{ixy}\, K_\mu(ixy)\rangle .$$

Substituting (7) into (6) we obtain

(8) $$f(x) = \lim_{r\to\infty} \left\{ \frac{1}{\pi} \int_0^r F(-iy) e^{-i\frac{1}{2}(\mu+\frac{1}{2})\pi} \sqrt{xy}\, J_\mu(xy) dy \right.$$

$$\left. + \frac{1}{\pi} \int_0^r F(iy) e^{i\frac{1}{2}(\mu+\frac{1}{2})\pi} \sqrt{xy}\, J_\mu(xy) dy \right\} .$$

Let $-iy = s$ in the first integral and let $iy = s$ in the second integral of (8). We obtain

$$f(x) = \lim_{r\to\infty} \left\{ \frac{1}{\pi} \int_0^{-ir} F(s) e^{-i\frac{1}{2}(\mu+\frac{1}{2})\pi} (ixs)^{\frac{1}{2}} J_\mu(ixs) i ds \right.$$

$$\left. + \frac{1}{\pi} \int_0^{ir} F(s) e^{i\frac{1}{2}(\mu+\frac{1}{2})\pi} (-ixs)^{\frac{1}{2}} J_\mu(-ixs)(-ids) \right\} .$$

The lemma follows from the identities [6],

$$I_\mu(z) = e^{-\frac{1}{2} i\mu\pi} J_\mu(iz) \qquad -\pi < \arg z < \tfrac{1}{2}\pi$$

$$I_\mu(z) = e^{\frac{1}{2} i\mu\pi} J_\mu(-iz) \qquad -\tfrac{1}{2}\pi < \arg z < \pi$$

and the observation that if C is a right-hand semicircle of radius δ around the origin,

$$\frac{1}{\pi i} \int_C F(s) \sqrt{sx}\, I_\mu(sx) ds \to 0 \quad \text{as} \quad \delta \to 0$$

uniformly on compact subsets of $0 < x < \infty$.

THEOREM. Let μ be restricted to $-\frac{1}{2} \le \mu < 0$ and let $s \in \Gamma$. Let $F(s) = k_\mu f$, $f \in \mathcal{J}'_{\mu,a}$ for some $a > 0$. Then for a fixed real non-negative σ,

https://doi.org/10.4153/CMB-1969-094-2 Published online by Cambridge University Press

$$f(x) = \lim_{r\to\infty} \frac{1}{\pi i} \int_{\sigma - ir}^{\sigma+ir} F(s) \sqrt{sx}\, I_\mu(sx) ds$$

in the sense of weak convergence in $D'(I)$.

Proof. The case where $\sigma = 0$ is contained in Lemma 2. We therefore let σ be a fixed number greater than zero. We shall show that in the sense of weak convergence in $D'(I)$,

$$\lim_{r\to\infty} \frac{1}{\pi i} \int_{ir}^{\sigma+ir} F(s) \sqrt{sx}\, I_\mu(sx) ds = 0 . \tag{9}$$

Let $\phi(x) \in D(I)$. Then we wish to show that

$$\left\langle \frac{1}{\pi i} \int_{ir}^{\sigma+ir} \left\langle f(t), \sqrt{st}\, K_\mu(st) \right\rangle \sqrt{sx}\, I_\mu(sx) ds , \phi(x) \right\rangle \tag{10}$$

converges to zero as $r \to \infty$. Since $\phi \in D(I)$ and by the smoothness of the integrand, it follows that (10) is an iterated integral on (s, x) having a continuous integrand and a finite domain of integration. Thus, (10) becomes

$$\frac{1}{\pi i} \int_{ir}^{\sigma+ir} \left\langle f(t), \sqrt{st}\, K_\mu(st) \right\rangle \int_0^\infty \phi(x) \sqrt{sx}\, I_\mu(sx) dx ds$$

after we change the order of integration. By an argument based on Riemann sums for the integral $\int_{ir}^{\sigma+ir} \cdots ds$, the last expression can be written as

$$\left\langle f(t), \frac{1}{\pi i} \int_{ir}^{\sigma+ir} \sqrt{st}\, K_\mu(st) \int_0^\infty \phi(x) \sqrt{sx}\, I_\mu(sx) dx ds \right\rangle . \tag{11}$$

https://doi.org/10.4153/CMB-1969-094-2 Published online by Cambridge University Press

To show that (11) converges to zero, we shall show that the testing function in (11) converges in $\mathcal{J}_{\mu, a}$ to zero. Let

$$U_r(t) = \frac{1}{\pi i} \int_{ir}^{\sigma+ir} \sqrt{st}\, K_\mu(st) \int_0^\infty \phi(x) \sqrt{sx}\, I_\mu(sx) dx ds .$$

Interchanging the order of integration and carrying the operator $S^k_{\mu, t}$ under the integral signs, we have

$$S^k_{\mu, t} U_r(t) = \frac{1}{\pi i} \int_0^\infty \phi(x) \int_{ir}^{\sigma+ir} S^k_{\mu, t} (\sqrt{st}\, K_\mu(st)) \sqrt{sx}\, I_\mu(sx) ds dx$$

$$= \frac{1}{\pi i} \int_0^\infty \phi(x) \int_{ir}^{\sigma+ir} (\sqrt{st}\, K_\mu(st)) S^k_{\mu, x} (\sqrt{sx}\, I_\mu(sx)) ds dx$$

$$= \int_0^\infty \phi(x)\, S^k_{\mu, x} V_r(x, t) dx$$

where $V_r(x, t) = \frac{1}{\pi i} \int_{ir}^{\sigma+ir} \sqrt{st}\, K_\mu(st) \sqrt{sx}\, I_\mu(sx) ds$.

By successive integrations by parts and because ϕ is smooth with compact support on $(0, \infty)$,

(12) $$S^k_{\mu, t} U_r(t) = \int_0^\infty V_r(x, t)\, S^k_{\mu, x} \phi(x) dx .$$

We now evaluate $V_r(x, t)$ by [6, page 90]

$$\int s I_\mu(sx) K_\mu(st) ds = \frac{s}{x^2 - t^2} [x I_{\mu+1}(sx) K_\mu(st) + t K_{\mu+1}(st) I_\mu(sx)]$$

https://doi.org/10.4153/CMB-1969-094-2 Published online by Cambridge University Press

and break up (12) into integrals I_1, I_2, and I_3 with integration on $0 \le x < t - \delta$, $t - \delta \le x < t + \delta$ and $t + \delta < x < \infty$, respectively.

We shall show that $N_r(t) = e^{-at} t^{-\mu - \frac{1}{2}} I_2(t)$ can be made arbitrarily small uniformly on $0 < t < \infty$, $1 < r < \infty$ by choosing δ small enough. For convenience, let $\psi(x) = S^k_{\mu, x} \phi(x)$ and let $\operatorname{supp} \phi(x) = [A, B] \subset (0, \infty)$. If $t + \delta \le A$ or $t - \delta \ge B$, then $\psi(x) = 0$ and $N_r(t) \equiv 0$. Thus we only have to consider the interval $A - \delta < t < B + \delta$. Using asymptotic expressions for $I_\mu(z)$ and $K_\mu(z)$ we have, upon simplifying,

$$(13) \quad N_r(t) = e^{-at} t^{-\mu-\frac{1}{2}} \int_{t-\delta}^{t+\delta} \frac{\psi(x)}{2\pi i} \left\{ \frac{e^{\sigma(x-t)}-1}{x-t} e^{ir(x-t)} \right.$$

$$\left. - \frac{i(e^{-\sigma(x+t)}-1)}{x+t} e^{-ir(x+t)+i\mu\pi} \right\} \cdot (1 + O(|r|^{-1}))^2 dx .$$

The integrand in (13) is uniformly bounded on the domain $\Lambda = \{(x, t): A < x < B, \ A - \delta < t < B + \delta\}$ for all $r > 1$. We shall show this for the term involving $(x - t)^{-1}$ as it is quite clear for the other terms. Indeed,

$$\frac{e^{\sigma(x-t)}-1}{x-t} = \sigma + \frac{\sigma^2}{2!}(x - t) + \frac{\sigma^3}{3!}(x - t)^2 + \dots$$

and therefore remains bounded for all $(x, t) \in \Lambda$. Hence, given an $\varepsilon > 0$, we can choose δ so small that $|N_r(t)| < \varepsilon$ for all $r > 0$, $0 < t < \infty$, and $A < x < B$.

The uniform convergence to zero of the expressions $e^{-at} t^{-\mu-\frac{1}{2}} I_1$ and $e^{-at} t^{-\mu-\frac{1}{2}} I_3$ is verified in a similar manner. Thus (11) converges to zero for any choice of $\phi(x) \in D(I)$ as $r \to \infty$.

An analogous proof shows that

$$(14) \qquad \lim_{r \to \infty} \frac{1}{\pi i} \int_{\sigma - ir}^{-ir} F(s) \sqrt{sx}\, I_\mu(sx) ds = 0 .$$

https://doi.org/10.4153/CMB-1969-094-2 Published online by Cambridge University Press

The theorem now follows from Equations (9), (14) and Lemma 2.

3. Remarks. (i) It must be pointed out that the inversion theorem as proven here is valid for μ restricted to $-\frac{1}{2} \leq \mu < 0$. This is a departure from the theorem given in [2], where μ is zero or a complex number such that $0 < \mathrm{Re}\, \mu < \infty$. Furthermore, the path of integration in our case may be taken along any line through or to the right of the imaginary axis whereas the σ in [2] has to be a positive number. These differences, which are not critical, result from the different topologies assigned to the fundamental spaces $\mathcal{J}_{\mu, a}$ and $\mathcal{K}_{u, a}$.

(ii) In [5], an I-transformation of generalized functions was developed and an inversion theorem was stated without proof. This theorem may be proven in essentially the same steps as employed here.

Acknowledgement. The author is indebted to Professor A.H. Zemanian for suggesting the problem and for his criticisms. This work is partially supported by the National Research Council of Canada under Grant No. A7184.

REFERENCES

1. A.H. Zemanian, A distributional K transformation. J. Soc. Ind. Appl. Math. 14 (1966) 1350-1365.

2. A.H. Zemanian, Generalized integral transformations. (Interscience, New York, 1969).

3. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Tables of integral transforms, Vol. II. (McGraw-Hill, New York, 1954).

4. L. Weiss, On the foundation of transfer function analysis. Int. J. Eng. Sc. 2 (1964) 343-365.

5. E. L. Koh and A.H. Zemanian, The complex Hankel and I-transformations of generalized functions. J. Soc. Ind. Appl. Math. 16 (1968) 945-957.

6. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher transcendental functions, Vol. II. (McGraw-Hill, New York, 1953).

University of Saskatchewan
Regina