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MATCHINGS IN COUNTABLE GRAPHS

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1. Tutte [**9**] has given necessary and sufficient conditions for a finite graph to have a perfect matching. Different proofs are given by Brualdi [**1**] and Gallai [**2**; **3**]. The shortest proof of Tutte's theorem is due to Lovasz [**5**]. In another paper [**10**] Tutte extended his conditions for a perfect matching to locally finite graphs. In [**4**] Kaluza gave a condition on arbitrary graphs which is entirely different from Tutte's.

Here we present a necessary and sufficient condition for the existence of a perfect matching in a countable graph. The methods we use are developed in [**6**; **8**].

2. Let V be a set and denote by $[V]^2$ the set $\{\{a, b\} \mid a \neq b \text{ and } a, b \in V\}$. An ordered pair (V, E) is a *graph* if $E \subseteq [V]^2$. If $G = (V, E)$ is a graph the elements of V are called *vertices* and those of E are called *edges*. A *subgraph* (V', E') of a graph (V, E) is a graph G' such that $V' \subseteq V$ and $E \cap [V']^2 = E'$. A vertex s is *incident* with an edge $\{b, c\}$ if $s \in \{b, c\}$. If each vertex is incident with only a finite number of edges G is a *locally finite graph*. A *countable graph* is one for which V is a countable set. A *matching* in the graph $G = (V, E)$ is a subset M of E such that no two distinct edges of M are incident with the same vertex. The matching M is *perfect* (a 1-factor) provided each vertex of G is incident with one edge of M. $\mathscr{F}(G)$ denotes the set of all perfect matchings of G. Let ω be the set of all natural numbers. An injective sequence $(v_i)_{i<k\leqq\omega}$ of vertices is called a *path* if $\{v_i, v_{i+1}\} \in E$ for every i satisfying $i + 1 < k$. Let M be a perfect matching of G. A path $(v_i)_{i<k\leqq\omega}$ is called *M-alternating* if the edges $\{v_i, v_{i+1}\}$ alternately lie in M and in $E\setminus M$. Let G' be a subgraph of G and $M' \in \mathscr{F}(G')$. An *M'-augmenting path* P is an M'-alternating path starting at a vertex $s \in V(G)\setminus V(G')$ and having one of the following properties: 1) P is an infinite sequence or 2) P terminates in a vertex $v \in V(G)\setminus V(G')$. An M'-augmenting path P is said to be *proper* if P contains an edge of M'. A matching M is called *independent* if there is no proper M-augmenting path starting at a vertex $s \in V(G)\setminus V(M)$. A subgraph G' of G is said to be *inde-independent* if $\mathscr{F}(G') \neq \emptyset$ and every perfect matching $M' \in \mathscr{F}(G')$ is independent. G satisfies *condition* (A) if for every matching M and for every vertex $s \in V(G)\setminus V(M)$ there exists an M-augmenting path which starts at s. G satisfies *condition* (B) if for every independent subgraph G' of G and for every vertex $s \in V(G)\setminus V(G')$ there exists a vertex $v \in V(G)\setminus V(G')$ such that $\{s, v\} \in E(G)$.

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We shall prove the following theorem.

THEOREM. *A countable graph G has a perfect matching if and only if G satisfies condition* (A).

Of course this holds trivially for finite graphs. Therefore one obtains an extension to locally finite graphs by use of Rado's selection principle [**7**]: A locally finite graph G has a perfect matching if and only if for every *finite* matching M and for every vertex $s \in V(G)\setminus V(M)$ there exists an M-augmenting path which starts at s.

To prove the theorem we need several lemmas.

LEMMA 1. *Let M be a perfect matching of G and G' be an independent subgraph of G. Then there is no edge $\{s, v\} \in M$ such that $s \in V(G)\setminus V(G')$ and $v \in V(G')$.*

LEMMA 2. *A subgraph G' of G is independent if and only if there exists an independent perfect matching for G'.*

The proofs of Lemmas 1 and 2 are left to the reader.

LEMMA 3. *Every graph G has a maximal independent subgraph.*

Proof. Let $\mathscr{C}$ be a chain of independent subgraphs of G. Let $(C_\alpha)_{\alpha<\beta}$ be a subchain of $\mathscr{C}$ such that $\bigcup_{\alpha<\beta}C_\alpha = \bigcup \mathscr{C}$ where α, β are ordinals. Choose $M_\alpha \in \mathscr{F}(C_\alpha)$ for every $\alpha < \beta$. By transfinite recursion we define a (transfinite) sequence $(M_\alpha^*)_{\alpha<\beta}$ of matchings. Let $M_0^* = M_0$ and assuming $M_\sigma^* \in \mathscr{F}(C_\sigma)$ to be defined for $\sigma < \alpha < \beta$, define

$$M_\alpha^* = (M_\alpha\setminus(M_\alpha \cap E(C_\sigma))) \cup M_\sigma^* \quad \text{for } \alpha = \sigma + 1, \quad \text{and}$$

$$M_\alpha^* = \bigcup_{\sigma<\alpha} M_\sigma^* \quad \text{for limit ordinal } \alpha.$$

Then $\bigcup_{\alpha<\beta} M_\alpha^*$ is a perfect matching of $\bigcup_{\alpha<\beta} C_\alpha = \bigcup \mathscr{C}$. By Lemma 2 the subgraph $\bigcup \mathscr{C}$ is independent. Thus the result follows by Zorn's Lemma.

LEMMA 4. *Let M be a matching of G and s be a vertex of G. Let $\mathscr{P}$ be the set of all M-alternating paths $(v_i)_{i<k\leqq\omega}$ such that $v_0 = s$ and $v_1 = b$ in case $\{s, b\} \in M$ and let $\mathscr{P}$ be the set of all proper M-alternating paths starting at s in case $s \in V(G)\setminus V(M)$. If every path of $\mathscr{P}$ terminates in an edge of M then $M^* = M \cap [V(\bigcup \mathscr{P})]^2$ is an independent matching.*

Proof. Note that every path of $\mathscr{P}$ is finite. Assume that there exists a vertex $v \in V(G)\setminus V(M^*)$ and a proper M^*-augmenting path $(v_i)_{i<k\leqq\omega}$ which starts at v. By assumption $v \neq s$. Let $\{c_1, c_2\} \in M^*$ be such that $v_1 \in \{c_1, c_2\}$. Let $(d_r)_{r<l\leqq\omega}$ be a path of $\mathscr{P}$ which contains the edge $\{c_1, c_2\}$ and let r_0 be the smallest $r < \omega$ for which there is an $i < \omega$ such that $v_i = d_r$. Denote by i_0 the number $i < \omega$ satisfying $v_i = d_{r_0}$. Distinguish two cases.

Case 1. $\{v_{i_0}, v_{i_0+1}\} \in M^*$. Let $c_n = d_n$ if $n \leqq r_0$ and $c_n = v_{(n-r_0)+i_0}$ if $r_0 < n < r_0 + (k - i_0)$. Then $(c_n)_{n<r_0+(k-i_0)}$ is an M^*-alternating path which

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starts at s and does not terminate in an edge of M. This contradicts the assumption.

Case 2. $\{v_{i_0}, v_{i_0+1}\} \notin M^*$. Let $c_n = d_n$ if $n \leqq r_0$ and $c_n = v_{i_0-(n-r_0)}$ if $r_0 < n \leqq r_0 + i_0$. Then $(c_n)_{n \leqq r_0+i_0}$ is an M^*-alternating path which starts at s and terminates in v. Thus $v \in V(M^*)$ which is impossible.

LEMMA 5. *A graph G satisfies condition* (A) *if and only if G satisfies condition* (B).

Proof. We only prove the nontrivial implication. Assume that G does not satisfy condition (A). Then there exists a matching M and a vertex $s \in V(G) \backslash V(M)$ such that there is no M-augmenting path which starts at s. Let $\mathscr{P}$ be the set of all proper M-alternating paths which start at s. By Lemma 4, $M^* = M \cap [V(\bigcup \mathscr{P})]^2$ is an independent matching and therefore G does not satisfy condition (B).

LEMMA 6. *If G satisfies condition* (B) *and has no nonempty independent subgraph then $G\backslash v$ satisfies condition* (B) *for every* $v \in V(G)$.

Proof. Assume that there is a vertex v such that $G^* = G\backslash v$ does not satisfy condition (B). Thus there exists an independent subgraph G' of G^* and a vertex $s \in V(G^*) \backslash V(G')$ such that $b \in V(G')$ for every edge $\{s, b\} \in E(G^*)$. By Lemma 5 there is a matching $M' \in \mathscr{F}(G')$ and a proper M'-augmenting path P which starts at s and terminates at v. Since the edges in P can be switched there is a perfect matching M_1 of $G_1 = G' \cup \{s, v\}$. Let v be incident with $\{v, b\} \in M_1$ and $\mathscr{P}$ be the set of all M_1-alternating paths $(v_i)_{i<k\leqq\omega}$ such that $v_0 = v$ and $v_1 = b$. Assume that there exists a path $(v_i)_{i<k\leqq\omega}$ of $\mathscr{P}$ which does not terminate in an edge of M_1. This means that $G' \cup \{s\}$ or $G' \cup \{s, v_{k-1}\}$ possesses a perfect matching which is absurd. By Lemma 4 the graph $\bigcup \mathscr{P}$ is a nonempty independent subgraph of G, which contradicts the assumption.

LEMMA 7. *Let G be a graph satisfying condition* (B) *and having no nonempty independent subgraph. Let $s \in V(G)$, G' be a maximal independent subgraph of $G\backslash s$ and $M' \in \mathscr{F}(G')$. Assume that $(v_i)_{i<k\leqq\omega}$ is an M'-augmenting path which starts at s. Then*

$$G^* = G\backslash(\{s\} \cup \{v_i | i < k \leqq \omega\} \cup V(G'))$$

satisfies condition (B).

Proof. By Lemma 6, $G_1 = G\backslash s$ satisfies condition (B). We have to prove that $G_2 = G_1 \backslash V(G')$ also satisfies condition (B). By the maximality of G' the graph G_2 has no nonempty independent subgraph. Therefore it suffices to prove that G_2 has no isolated vertex. This is obvious since G_1 satisfies condition (B). If $k = \omega$ then $G^* = G_2$ and the lemma is proved. If $k < \omega$ then $G^* = G_2 \backslash v_{k-1}$ and the result follows by Lemma 6.

Proof of the Theorem. Clearly (A) is a necessary condition for a graph to have a perfect matching. Assume that G is a countable graph satisfying (A).

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We have to show that G has a perfect matching. It will be enough to show that, if v is any vertex, then there is a matching M of G such that $v \in V(M)$ and $H = G \backslash V(M)$ also satisfies condition (A).

Let G_0 be a maximal independent subgraph of G which exists by Lemma 3. Choose $M_0 \in \mathscr{F}(G_0)$ and let $G_1 = G \backslash V(G_0)$. Then G_1 has no nonempty independent subgraph and satisfies (B). If $v \in V(G_0)$ we are done. Suppose $v \notin V(G_0)$. Let G_2 be a maximal independent subgraph of $G_1 \backslash v$ and let $M_2 \in \mathscr{F}(G_2)$. By Lemma 5 there is an M_2-augmenting path $(v_i)_{i<k\leqq\omega}$ in G_1 which starts at $v_0 = v$. Therefore $\{v_i | i < k\} \cup V(G_2)$ has a perfect matching M'. Then $M = M_0 \cup M'$ is a matching of G which contains v and $H = G \backslash V(M)$ satisfies (B), and hence condition (A), by Lemmas 7 and 5.

COROLLARY 8. *For each countable graph $G = (V, E)$ there is a maximal subset $V' \subseteq V$ which has a perfect matching.*

It should be mentioned that the condition (A) is not sufficient for a non-denumerable graph to have a perfect matching. To see this consider the complete bipartite graph $K(\aleph_0, m)$, where m is any cardinal greater than $\aleph_0$.

Since every component of a locally finite graph G is countable the Schröder-Bernstein Theorem is a corollary of our theorem.

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