Can. J. Math., Vol. XXV, No. 2, 1973, pp. 345–352

QF − 1 RINGS OF GLOBAL DIMENSION ≦ 2

CLAUS MICHAEL RINGEL

R. M. Thrall [**10**] introduced QF − 1, QF − 2 and QF − 3 rings as generalizations of quasi-Frobenius rings. (For definitions, see section 1. It should be noted that all rings considered are assumed to be left and right artinian.) He proved that QF − 2 rings are QF − 3 and asked whether all QF − 1 rings are QF − 2, or, at least, QF − 3. In [**9**] we have shown that QF − 1 rings are very similar to QF − 3 rings. On the other hand, K. Morita [**6**] gave two examples of QF − 1 rings, one of them not QF − 2 and therefore not QF − 3, the other one QF − 3, but not QF − 2. The global dimension of the latter ring is 2, and the following theorem shows that under this assumption a QF − 1 ring must always be QF − 3.

THEOREM. *A* QF − 1 *ring of left global dimension* ≦2 *is a* QF − 3 *ring.*

In order to classify finite dimensional algebras, T. Nakayama [**8**] defined the dominant dimension dom dim R of a ring R. Since dom dim $R \geqq 1$ if and only if R is a QF − 3 ring, and, in this case, dom dim $R \geqq 2$ if and only if the minimal faithful left R-module is balanced, we may reformulate the theorem as follows: a QF − 1 ring R of left global dimension ≦2 has dom dim $R \geqq 2$. It was proved by K. R. Fuller [**4**] that for a ring R with dom dim $R \geqq 2$, every faithful module which is either projective or injective has to be balanced. Naturally, the question arises whether it is possible to characterize those rings R of left global dimension ≦2 which have dom dim ≧ 2 by the fact that certain faithful R-modules are balanced. This question seems to be interesting in view of the importance of the class of rings of global dimension ≦2 and dominant dimension ≧2, recently demonstrated by M. Auslander [**1**].

The proof of the theorem uses besides the socle conditions of [**9**] a result concerning the right socle of a QF − 1 ring, and the methods to prove this are similar to those developed in [**9**]. The assumption in the theorem on the global dimension can be replaced by the (weaker) condition that the right socle, considered as a left module, is projective.

1. Preliminaries. Throughout the paper, R denotes a (left and right) artinian ring with unity. By an R-module we understand a unital R-module and the symbols ${}_RM$ and M_R will be used to underline the fact that M is a left or a right R-module, respectively.

Received January 10, 1972 and in revised form, March 21, 1972. This research was supported by an NRC grant.

https://doi.org/10.4153/CJM-1973-033-7 Published online by Cambridge University Press

The length of the module M will be denoted by ∂M. For every module M, Rad M is the intersection of all maximal submodules. The radical of R is by definition $\text{Rad}_R R$; it will be denoted by W. It is well-known that for an artinian ring, W is nilpotent. The submodule of M generated by all simple submodules, is called the socle, Soc M of M. Since R is artinian, we have for every left R-module, $\text{Rad}\, M = WM$ and $\text{Soc}\, M = \{m \in M | Wm = 0\}$. Considering ${}_RR$, we get the left socle $L = \text{Soc}\, {}_RR$, considering R_R, we get the right socle $J = \text{Soc}\, R_R$ of R.

If e is an idempotent, Re always will be considered as a left R-module, and the R-homomorphisms $Re \to Re'$ (where e' is another idempotent) will be identified with the elements of eRe'. Also, it should be noted that Re and Re' are isomorphic if there are elements $x \in eRe'$ and $y \in e'Re$ with $exy = e$. The ring R is called a basis ring if for orthogonal idempotents e and e', Re and Re' never are isomorphic. Basis rings can be characterized by the fact that $eR(1 - e) \subseteq W$ for every idempotent e. If R is an arbitrary artinian ring and we write

$$1 = \sum_{i,j} e_{ij}$$

with primitive and orthogonal idempotents e_{ij} such that $Re_{ij} \approx Re_{kl}$ if and only if $i = k$, then, for $E = \sum_i e_{i1}$, the ring ERE is a basis ring which is Morita equivalent to R.

The ring R is a QF $-$ 3 ring if R has a unique minimal faithful left R-module ${}_RX$ (that is, ${}_RX$ is faithful, and is a direct summand of every faithful left R-module). A QF $-$ 3 ring also has a unique minimal faithful right R-module. The ring R is QF $-$ 3 if and only if for every primitive idempotent e with $Je \neq 0$, the socle Le of Re is simple, and similarly for every primitive idempotent f with $fL \neq 0$, the socle fJ of fR is simple [**2**, Theorem (3.6)].

Module homomorphisms always act from the opposite side as the operators; in particular, every left R-module ${}_RM$ defines a right $\mathscr{C}$-module $M_{\mathscr{C}}$, where $\mathscr{C}$ is the centralizer of ${}_RM$. The double centralizer $\mathscr{D}$ of ${}_RM$ is the centralizer of $M_{\mathscr{C}}$, and there is a canonical ring homomorphism $R \to \mathscr{D}$. The module ${}_RM$ is called balanced if this morphism $R \to \mathscr{D}$ is surjective. If every finitely generated faithful (left or right) R-module is balanced, then R is said to be a QF $-$ 1 ring. Until now, no internal characterization of QF $-$ 1 rings seems to be known, but in [**9**] certain necessary socle conditions were proved. For the convenience of the reader and for later reference, we recall these conditions: If R is a QF $-$ 1 ring and e and f are primitive idempotents with $f(L \cap J)e \neq 0$, then

(1) either $\partial_R Je = 1$ or $\partial fL_R = 1$,

(2) we have $\partial_R Le \times \partial fJ_R \leqq 2$,

(3) $\partial_R Le = 2$ implies $Je \subseteq Le$, and

(3*) $\partial fJ_R = 2$ implies $fL \subseteq fJ$.

In particular, (2) shows that a QF $-$ 1 ring is very similar to a QF $-$ 3 ring.

If ${}_RM$ is an indecomposable module of finite length, then the centralizer $\mathscr{C}$

of M is a local ring. Consequently, all simple $\mathscr{C}$-modules are isomorphic. Moreover, the radical $\mathscr{W}$ of $\mathscr{C}$ is nilpotent, thus the radical of $M_{\mathscr{C}}$ is a proper submodule, and Soc $M_{\mathscr{C}}$ is essential in $M_{\mathscr{C}}$. If ${}_RM$ and ${}_RN$ are modules, then elements in the double centralizer of ${}_R(M \oplus N)$ can be constructed as follows: Let $\mathscr{C}$ be the centralizer of ${}_RM$, and let M' and M'' be $\mathscr{C}$-submodules of $M_{\mathscr{C}}$ such that the image of every R-homomorphism ${}_RN \to {}_RM$ is contained in M', whereas M'' is contained in the kernel of every R-homomorphism ${}_RM \to {}_RN$. Then, given a $\mathscr{C}$-homomorphism ψ of the form

$$M_{\mathscr{C}} \xrightarrow{\epsilon} M/M' \to M'' \xrightarrow{\iota} M_{\mathscr{C}}$$

(where ϵ is the canonical epimorphism, ι the inclusion), the trivial extension

$$\begin{bmatrix} \psi & 0 \\ 0 & 0 \end{bmatrix} : M \oplus N \to M \oplus N$$

of ψ belongs to the double centralizer of ${}_R(M \oplus N)$.

If, for a module M, there exists an exact sequence of R-modules

$$0 \to M \to D_1 \to D_2 \to \ldots \to D_n$$

with D_i both projective and injective, then the dominant dimension dom dim M of the module M is $\geqq n$. Now dom dim ${}_RR \geqq 1$ if and only if R is a QF $-$ 3 ring [**5**]. In this case, dom dim ${}_RR \geqq 2$ if and only if the minimal faithful left R-module is balanced [**7**]. Since the minimal faithful left R-module of a QF $-$ 3 ring is both projective and injective, all faithful left or right modules which are either projective or injective are balanced [**4**, Theorem 5]. In particular, also the minimal faithful right module is balanced, and dom dim $R_R \geqq 2$. So we simply may say that the dominant dimension of R is $\geqq 2$.

If there exists a natural number m such that for every exact sequence of left R-modules

$$0 \to K \to P_{m-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

with P_i projective for $0 \leqq i \leqq m - 1$, K is also projective, then the smallest such m is called the left global dimension of R. It is easy to see that the left global dimension of R is $\leqq 2$ if and only if the kernel of every R-homomorphism ${}_RF \to {}_RF'$, with ${}_RF$ and ${}_RF'$ both free, is projective.

2. The aim of this section is to prove the following general result on QF $-$ 1 rings.

PROPOSITION. *Consider a* QF $-$ 1 *ring R with left socle L and right socle J. Let e and f be primitive idempotents. If y is an element of fJe which does not belong to L, and if $fL \neq 0$, then $Ry = Je$.*

Proof. Obviously, we may assume that R is a basis ring, because if the propo-

https://doi.org/10.4153/CJM-1973-033-7 Published online by Cambridge University Press

sition holds for a basis subring of R, it is also true for R. Also, we may assume that $y \in W$, since otherwise the conclusion is trivial.

Let e_1 be a primitive idempotent such that e_1 and $e_2 = e$ are either orthogonal or equal, and which satisfies $f(L \cap J)e_1 \neq 0$. Let x be a non-zero element in $f(L \cap J)e_1$. Since $xR \cap yR = 0$, the left R-module

$$_RM = (Re_1 \oplus Re_2)/R(x, y)$$

is indecomposable [**9**]. The endomorphisms of $_RM$ are induced by matrices

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

with entries $r_{ij} \in e_iRe_j$, for $1 \leqq i, j \leqq 2$, operating on $Re_1 \oplus Re_2$ from the right. If (r_{ij}) induces an endomorphism of $_RM$, then r_{21} belongs to the radical W of R. For, consider the image of (x, y) under (r_{ij}). We have

$$(xr_{11} + yr_{21}, xr_{12} + yr_{22}) = (\lambda x, \lambda y)$$

for some $\lambda \in R$. Thus $yr_{21} = \lambda x - xr_{11} \in L$, and, since $y \notin L$, we conclude that $r_{21} \in W$.

Also, if (r_{ij}) induces a nilpotent endomorphism of $_RM$, then $r_{22} \in W$. For, consider the image of $(0, y)$ under (r_{ij}). We have

$$(0, y)\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = (yr_{21}, yr_{22}) = (0, yr_{22}),$$

since $y \in J$ and $r_{21} \in W$. By induction, we get for natural n

$$(0, y)\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}^n = (0, yr_{22}^n).$$

Since, by assumption, (r_{ij}) induces a nilpotent endomorphism, there is some n with

$$(0, yr_{22}^n) = (\lambda x, \lambda y),$$

where λ can be chosen in Rf. But $\lambda x = 0$ implies $\lambda \in W$, thus λ is nilpotent. If $\lambda^m = 0$, then $yr_{22}^n = \lambda y$ yields $yr_{22}^{nm} = \lambda^m y = 0$, and consequently, r_{22} cannot be invertable in e_2Re_2.

Let $\mathscr{C}$ be the centralizer of $_RM$. It follows from the considerations above that $(0 \oplus Je_2) + R(x, y)/R(x, y)$ is contained in Soc $M_{\mathscr{C}}$. For, if $\mathscr{W}$ denotes the radical of $\mathscr{C}$, the elements of $\mathscr{W}$ can be lifted to matrices (r_{ij}) with r_{21} and r_{22} in W. Thus, for $z \in Je_2$, we have

$$(0, z)\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = (zr_{21}, zr_{22}) = (0, 0),$$

and thus $(0, z) + R(x, y) \in$ Soc $M_{\mathscr{C}}$.

Also, $(0 \oplus Je_2) + R(x, y)/R(x, y)$ belongs to the kernel of every homo-

https://doi.org/10.4153/CJM-1973-033-7 Published online by Cambridge University Press

morphism ${}_RM \to R(1 - e_1)$. For, we may lift such a morphism to

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} : Re_1 \oplus Re_2 \to R(1 - e_1)$$

with $r_i \in e_iR(1 - e_1)$, mapping (x, y) into 0. The last condition gives us the equality $xr_1 + yr_2 = 0$, thus, since $x \in J$ and $r_1 \in e_1R(1 - e_1) \subseteq W$, we get $yr_2 = 0$. This shows that not only r_1 but also r_2 belongs to W, and, as a consequence, the image of $(0, z) \in 0 \oplus Je_2$ under $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ is $zr_1 + zr_2 = 0$.

Since $x, y \in J$, every matrix

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \text{ with } r_{ij} \in e_iWe_j$$

induces a nilpotent endomorphism of ${}_RM$, thus $We_1 \oplus We_2/R(x, y) \subseteq M\mathscr{W}$. Moreover, if e_1 and e_2 are orthogonal, we have the equality

$$We_1 \oplus We_2/R(x, y) = M\mathscr{W}.$$

For, assume that $\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$ with $r_{ij} \in e_iRe_j$ induces an endomorphism φ of ${}_RM$; then $r_{12} \in e_1Re_2 \subseteq W$, and, if φ is nilpotent, we conclude similarly to a proof above that

$$(x, 0) \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}^n = (xr_{11}{}^n, 0),$$

and that therefore also $r_{11} \in W$. This shows that for $\varphi \in \mathscr{W}$, all r_{ij}'s belong to W, so $M\mathscr{W} \subseteq We_1 \oplus We_2/R(x, y)$.

Next, we claim that $(e_1, 0) + R(x, y)$ does not belong to $M\mathscr{W} = \text{Rad } M_{\mathscr{C}}$. This is obvious in the case where e_1 and e_2 are orthogonal. So, we only consider the case $e = e_1 = e_2$. If we assume that $(e, 0) + R(x, y)$ belongs to $M\mathscr{W}$, then, since $M\mathscr{W}$ is a proper R-submodule of ${}_RM$ also containing $We \oplus We/R(x, y)$, we have $M\mathscr{W} = Re \oplus We/R(x, y)$. Also, Soc $M_{\mathscr{C}}$ is an essential $\mathscr{C}$-submodule of M, thus $(Je \oplus Je) + R(x, y)/R(x, y)$ intersects Soc $M_{\mathscr{C}}$ nontrivially. Therefore, there is a non-zero $\mathscr{C}$-homomorphism ψ of the form

$$M_{\mathscr{C}} \xrightarrow{\epsilon} M/M\mathscr{W} \to (Je \oplus Je) + R(x, y)/R(x, y) \xrightarrow{\iota} M_{\mathscr{C}},$$

where ϵ is the canonical epimorphism, ι the embedding. The image of every R-homomorphism $R(1 - e) \to {}_RM$ is contained in $We \oplus We/R(x, y) \subseteq M\mathscr{W}$, since we may lift such a morphism to

$$R(1 - e) \xrightarrow{(r_1, r_2)} Re \oplus Re$$

with $r_i \in (1 - e)Re \subseteq W$. On the other side, $(Je \oplus Je) + R(x, y)/R(x, y)$ is contained in the kernel of every morphism ${}_RM \to R(1 - e)$. Thus the trivial extension ψ' of ψ to ${}_RM \oplus R(1 - e)$ belongs to the double centralizer of

${}_RM \oplus R(1-e)$. But this morphism ψ' vanishes on $M\mathscr{W} \oplus R(1-e)$ which is a faithful module since Re is embeddable in $(Re \oplus We)/R(x, y) = M\mathscr{W}$. This shows that ψ' cannot be induced by multiplication, a contradiction. So we have shown that $(e, 0) + R(x, y)$ cannot belong to $M\mathscr{W}$.

There is a $\mathscr{C}$-submodule M' of M which contains $M\mathscr{W}$ and also the images of all R-homomorphisms $R(1-e_1) \to {}_RM$, but which does not contain the element $(e_1, 0) + R(x, y)$. For, in the case where e_1 and e_2 are orthogonal, choose $M' = (We_1 \oplus Re_2)/R(x, y)$. Since all matrices $\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$ which induce endomorphisms of ${}_RM$ satisfy $r_{12}, r_{21} \in W$, we see that M' is actually a $\mathscr{C}$-submodule. Obviously, $M' \supseteq M\mathscr{W} = We_1 \oplus We_2/R(x, y)$, and given an R-homomorphism $R(1-e_1) \to {}_RM$, we may lift it to

$$R(1-e_1) \xrightarrow{(r_1, r_2)} Re_1 \oplus Re_2$$

with $r_i \in (1-e_1)Re_i$. But $r_1 \in (1-e_1)Re_1 \subseteq W$, thus the image of (r_1, r_2) is contained in $We_1 \oplus Re_2$. Secondly, consider the case $e_1 = e_2$. In this case, let $M' = M\mathscr{W}$. Since every R-homomorphism $R(1-e_1) \to {}_RM$ again can be lifted to (r_1, r_2) where now both r_1 and r_2 belong to $(1-e_1)Re_1 \subseteq W$, the image of $R(1-e_1) \to {}_RM$ has to be contained in

$$We_1 \oplus We_2/R(x, y) \subseteq M\mathscr{W} = M'.$$

So we see that also in the second case M' satisfies all conditions.

Also, there is a $\mathscr{C}$-submodule M'' of $M_{\mathscr{C}}$ contained in Soc $M_{\mathscr{C}}$ and in the kernel of every R-homomorphism ${}_RM \to R(1-e_1)$, and containing

$$(0 \oplus Je_2) + R(x, y)/R(x, y).$$

For, we simply may take the intersection of Soc $M_{\mathscr{C}}$ and the kernels of all maps ${}_RM \to R(1-e_1)$.

By construction, M/M' and M'' both are semisimple $\mathscr{C}$-modules. Given $z \in Je_2$, there is a $\mathscr{C}$-homomorphism ψ of the form

$$M_{\mathscr{C}} \xrightarrow{\epsilon} M/M' \to M'' \xrightarrow{\iota} M_{\mathscr{C}}$$

(where again ϵ denotes the canonical epimorphism, ι the embedding) mapping $(e_1, 0) + R(x, y)$ onto the element $(0, z) + R(x, y)$. Since the image of every morphism $R(1-e_1) \to {}_RM$ is contained in M' and the kernel of every morphism ${}_RM \to R(1-e_1)$ contains M'', the trivial extension of ψ to ${}_RM \oplus R(1-e_1)$ belongs to the double centralizer of ${}_RM \oplus R(1-e_1)$. Using the fact that R is a QF $-$ 1 ring, we find an element $\rho \in R$ which induces this extension. In particular, we have

$$\rho(e_1, 0) - (0, z) \in R(x, y).$$

Thus $z \in Ry$, as we wanted to prove.

3. The main theorem. The result of the previous section can be considered as a forth socle condition for QF − 1 rings. Using these socle conditions we can show

THEOREM. *Let R be a* QF − 1 *ring and assume that the right socle J of R, considered as a left module, is projective. Then R is a* QF − 3 *ring.*

Proof. Obviously, we may assume that R is two-sided-indecomposable, i.e. that there are not two two-sided non-zero ideals I_1 and I_2 with $R = I_1 \oplus I_2$. Let e and f be primitive idempotents with $f(L \cap J)e \neq 0$. Then according to the second socle condition

$$\partial_R Le \times \partial f J_R \leqq 2.$$

We have to show that in our case the product actually is equal to 1. So, assume $\partial_R Le = 2$ and consider first the case $Le \subseteq Je$. The third socle condition implies $Le = Je$. Since Je is a projective left R-module, and Je is properly contained in Re, we find a non-zero idempotent e' such that e and e' are orthogonal, Re' is isomorphic to a direct summand of Je, and $fLe' \neq 0$. Then $fL \supseteq f(L \cap J)e \oplus fLe'$ and therefore $\partial fL_R > 1$, a contradiction to the first socle condition. If $Le \nsubseteq Je$, take a primitive idempotent f' and an element $x = f'xe \in Le \backslash Je$. Let e' be a primitive idempotent and $w = we' \in W$ with $0 \neq xw \in L \cap J$. Then $\partial f'L_R > 1$, thus, using the fact that $f'(L \cap J)e \neq 0$ the first socle condition implies $\partial_R Je' = 1$. As a consequence, $Rxw = Je'$ is projective and since it is isomorphic to Rf'/Wf', we conclude $Wf' = 0$, thus f' belongs to L. But since $x \in f'Le\backslash J$ and $Je \neq 0$, we may apply the Proposition of section 2 to the opposite ring of R in order to conclude that $xR = f'L$, and therefore we find $\rho \in R$ with $f' = x\rho = f'x\rho$. Right multiplication by x gives an isomorphism $Rf' \to Re$. But obviously $Re \nsubseteq L$, whereas $Rf' \subseteq L$. This contradiction proves that $\partial_R Le = 1$.

Secondly, assume $\partial fJ_R = 2$. If $fJ \subseteq fL$, then according to the first socle condition we have $\partial_R Je = 1$ for every primitive idempotent e with $fJe \neq 0$. Thus fJ is a direct summand of ${}_RJ$, and therefore also projective. This yields that Rf is of length 1, that is $f \in L$. But the socle condition (3*) implies $fL \subseteq fJ$, thus $Rf \subseteq L \cap J$. Since R is assumed to be two-sided-indecomposable, we have $R = RfR$, and R is semisimple; but then $\partial fR_R = 1$, a contradiction. Next, assume $fJ \nsubseteq fL$, and take a primitive idempotent e' and an element $y = fye' \in fJe' \backslash L$. By the result of section 2, $Ry = Je'$, since we assume $f(L \cap J)e \neq 0$. Now, if Je' is a proper submodule of Re', then using the fact that Je' is projective and local, we find a primitive idempotent e'', orthogonal to e', with $Je' \approx Re''$. If f' is a primitive idempotent with $f'(L \cap J)e' \neq 0$, then also $f'Le'' \neq 0$, thus $\partial f'L_R > 1$. But since $Je' \nsubseteq L$, we also have $\partial_R Je' > 1$. Together with $f'(L \cap J)e'$ this gives a contradiction to the first socle condition. So, we have to assume that $Je' = Re'$. Since $Ry = Je'$ and $y = fye'$, we may assume $e' = f$. Now $Rf \subseteq J$, and $f \notin L$, thus no simple left ideal can be isomorphic to Rf/Wf. But this is a contradiction to $fL \neq 0$, and therefore we have shown $\partial fJ_R = 1$.

COROLLARY. *A* QF − 1 *ring of left global dimension* $\leqq 2$ *is a* QF − 3 *ring.*

Proof. Let R be a QF − 1 ring of left global dimension $\leqq 2$. If $w_1, \ldots, w_n$ are generators of W_R, consider the maps

$$\varphi : {}_RR \rightarrow \bigoplus_{i=1}^{n} {}_RR$$

with $1\varphi = (w_1, \ldots, w_n)$. Then the right socle J of R is just the kernel of φ, so ${}_RJ$ has to be projective.

4. Remarks. If we consider the class of rings of left global dimension $\leqq 2$, we asked in the introduction for a characterization of those rings R with dom dim $R \geqq 2$. The following example shows that not all rings of global dimension $\leqq 2$ and dominant dimension $\geqq 2$ are QF − 1 rings.

Let R be a generalized uniserial ring with the Kupisch series

$$1, 2, 2, 3, 2.$$

Then, according to [**3**], R is not a QF − 1 ring, but since R is generalized uniserial and coincides with its complete ring of left quotients, dom dim $R \geqq 2$. Also, the global dimension of R is 2.

On the other side, the QF − 1 rings of global dimension $\leqq 2$ are not all of dominant dimension $\geqq 3$, as Morita's second example in [**6**] shows. It can easily be seen that the dominant dimension of this algebra is precisely 2.

REFERENCES

1. M. Auslander, *Representation dimension of Artin algebras* (Queen Mary College Math. Notes, London).
2. K. R. Fuller, *The structure of* QF − 3 *rings*, Trans. Amer. Math. Soc. *134* (1968), 343–354.
3. ——— *Generalized uniserial rings and their Kupisch series*, Math. Z. *106* (1968), 248–260.
4. ——— *Double centralizers of injectives and projectives over artinian rings*, Illinois J. Math. *14* (1970), 658–664.
5. J. P. Jans, *Projective injective modules*, Pacific J. Math. *9* (1959), 1103–1108.
6. K. Morita, *On algebras for which every faithful representation is its own second commutator*, Math. Z. *69* (1958), 429–434.
7. K. Morita and H. Tachikawa, QF − 3 *rings* (to appear).
8. T. Nakayama, *On algebras with complete homology*, Abh. Math. Sem. Univ. Hamburg *22* (1958), 300–307.
9. C. M. Ringel, *Socle conditions for* QF − 1 *rings* (to appear in Pacific J. Math.).
10. R. M. Thrall, *Some generalizations of quasi-Frobenius algebras*, Trans. Amer. Math. Soc. *64* (1948), 173–183.

Mathematisches Institut der Universität,
Tübingen, Germany;
Carleton University,
Ottawa, Ontario